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SOME GEOMETRIES ASSOCIATED WITH PARABOLIC REPRESENTATIONS OF GROUPS OF LIE TYPE

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Introduction. Suppose (P, Δ) is an undirected graph without loops or multiple edges. We will denote by $\Delta(x)$ the vertices adjacent to x and $x^\perp = \{x\} \cup \Delta(x)$. Let (G, P) be a transitive permutation representation of a group G in a set P, and Δ be a non-trivial self-paired (i.e. symmetric) orbit for the action of G on $P \times P$. We identify Δ with the set of all two subsets $\{x, y\}$ with (x, y) in Δ. Then we have a graph (P, Δ) with $G \leqq \mathrm{Aut}(P, \Delta)$, transitive on both P and Δ. For x, y an adjacent pair of points we define the (*singular*) *line* xy on x and y by

$$xy = \bigcap_{z \in x^\perp \cap y^\perp} z^\perp.$$

It is well known (c.f. [**4**]) that G is transitive on lines, a line is a clique (i.e. complete subgraph) and if $u \neq v$ are on xy, then $xy = uv$. As a result all lines have the same cardinality and also satisfy:

If z is a point not on xy and z is adjacent to at least two points of xy, then z is adjacent to every point of xy.

If we let L be the set of all such lines we get an *incidence structure* (P, L) (by this we mean a set of points and a collection of distinguished subsets called lines) with $G \leqq \mathrm{Aut}(P, L)$, transitive on both P and L.

Generally, (P, L) is trivial in the sense that lines only carry two points. This will certainly be the case if $G_x^{\Delta(x)}$ is primitive on $\Delta(x)$ since $xy - \{x\}$ is a block of imprimitivity for the action of $G_x^{\Delta(x)}$ on $\Delta(x)$. Thus the representations of McL and HiS as rank three groups are examples of representations where the associated incidence structures are trivial. The representation of M_{22} as a rank three group acting on the seventy seven blocks of the Steiner system $S(3, 6, 22)$ also affords trivial structures. However, in this representation a point stabilizer is isomorphic to a semi-direct product $Z_2{}^4 A_6$, it is faithful on both suborbits, and has a set of imprimivity on one of the suborbits. Therefore the imprimitivity of $G_x^{\Delta(x)}$ on $\Delta(x)$ is not sufficient for the existence of thick (i.e. with more than two points) lines.

We give some non-trivial examples:

(1) Let $G \simeq \Sigma_{3k}$, the symmetric group on $3k$ letters, with k at least two. Let P be the set of all k-subsets of the $3k$ letters and Δ the set of pairs of non-

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intersecting k-subsets. The line on such a pair consists of the pair together with the complement of their union.

Remark. When $k = 2$ this is just the generalized quadrangle associated with $Sp_4(2) \simeq \Sigma_6$.

(2) Let $G \simeq M_{24}$, the Mathieu group on 24 letters. Let P be the set of blocks of the Steiner system $S(5, 8, 24)$ and Δ the set of pairs of non-intersecting blocks. G is rank four in this representation and the line on such a pair consists of the trio containing the pair.

(3) Let V be a finite dimensional vector space equipped with a nondegenerate orthogonal, alternate or hermitian form f such that the index of f is at least two. Let $G = O(f)$, the group of invariants of f, so that G is an orthogonal, symplectic or unitary group. Let P be the set of singular points (i.e. isotropic one-dimensional subspaces of V), and Δ the pairs of orthogonal points in P. Then G is transitive on P and Δ, and the line on a pair from Δ consists of the points in the two space spanned by the pair. These incidence structures are examples of prepolar spaces which are classified in [**1**].

(4) Let $PSL_n(q) \leqq G \leqq P\Gamma L_n(q)$, n at least four. Let P_k be the collection of all k-subspaces of $PG(n - 1, q)$ with k at least one and not greater than $(n - 1)/2$, and let Δ be the set of pairs of k-subspaces intersecting in a $k - 1$-subspace. Then the line on such a pair (x, y) consists of the $q + 1$ k-subspaces incident with the flag $x \cap y \subset \langle x, y \rangle$.

In Examples (3) and (4) the groups considered are groups of Lie type, and the permutation representations are parabolic representations, that is, equivalent to the action of the group on the coset space of G modulo some parabolic subgroup. If we consider other parabolic representations of these groups or of other groups of Lie type we can give many more examples of thick incidence structures afforded by groups. This suggests that their existence is a "Lie" property, that every parabolic representation of a group of Lie type affords thick incidence structure(s). This is so. It is our objective to prove.

THEOREM 1. *Let G be a group of Lie type, Φ the associated root system, Π a fundamental base for Φ, $\delta \subset \Pi$, $K = G_\delta$ the parabolic subgroup corresponding to δ, $P = G/K$ the coset space of G modulo K. Then there is a set $\{\Delta_\alpha : \alpha \in \Pi - \delta\}$ of distinguished, non-trivial, self-paired orbitals such that lines with respect to Δ_α carry greater than two points for each α in $\Pi - \delta$. Moreover the stabilizer of any line is another parabolic subgroup.*

The existence of these suborbits is proved in Section 1. In Section 2 we determine the actual lines and their stabilizers, then in Section 3 we determine some properties of these Lie Incidence structures.

1. Proof of Theorem 1. We begin with some notation. Let G be a group of

Lie type and let (B, N) be a (B, N)-pair for G. Then $G = \langle B, N\rangle$, $H = B \cap N \lhd N$ and $W = N/H$ is a finite group generated by a distinguished set of involutions $\{r_i : 1 \leqq i \leqq l\}$. These and other properties are well known. For more details refer to [**2**, **5**]. Elements of W are cosets of H, and so for w in W, wB, BwB are unambiguous. We remark that w is a distinguished generator if and only if $B \cup BwB$ is a group.

Associated with G and W is a root system Φ and for each r_i a fundamental root α_i. Set $\Pi = \{\alpha_i : 1 \leqq i \leqq l\}$. Then we may consider $\Phi \subseteq E^l$ with Π as base and the r_i act on E as reflections by

$$r_i(v) = v - 2(v, \alpha_i)/(\alpha_i, \alpha_i) \cdot \alpha_i = v - \langle v, \alpha_i\rangle\alpha_i.$$

There is a unique element w_0 in W such that w_0 takes Φ^+ to Φ^-. For α_i in Π, let $X_{\alpha_i} = B \cap r_i w_0 B w_0^{-1} r_i$. For any α in Π there is a w in W and α_i in Π with $w\alpha_i = \alpha$. Then set $X = X_{\alpha_i}{}^{w}$. These are well-defined. For $\psi \subset \Phi$, let $\psi^+ = \{\alpha \in \psi : \alpha > 0\}$, $\psi^- = \psi - \psi^+$. For $\delta \subset \Pi$, let $R_\delta = \{\alpha : \operatorname{spt}(\alpha) \subsetneqq \delta\}$, $S_\delta = \Phi - R_\delta$. We also set $W_\delta = \langle r_i : \alpha_i \in \delta\rangle$ and then the parabolic subgroup G_δ corresponding to W is $BW_\delta B$. We also have $G_\delta = V_\delta L_\delta H$ where $V_\delta = \langle X_\alpha : \alpha \in R_\delta{}^+\rangle$, $L_\delta = \langle X_\alpha : \alpha \in S_\delta\rangle$.

We will prove Theorem 1 by showing for each α_i in $\Pi - \delta$ that the orbital Δ_i defined by $yK \in \Delta_i(xK)$ if and only if $x^{-1}y \in Kr_iK$ is self-paired and yields (singular) lines with greater than two points. For convenience we drop the subscript and write Δ in place of Δ_i.

LEMMA 1.1. *Δ is self-paired.*

Proof. We may appeal to the correspondence between $G_\theta - G_\gamma$ double cosets and $W_\theta - W_\gamma$ double cosets for $\theta, \gamma \subset \Pi$. It then suffices to show the orbital of the representation $(W, W/W_\delta)$ given by $wW_\delta \in \Delta'(vW_\delta)$ if and only if $v^{-1}w \in W_\delta r_i W_\delta$ is self-paired. But since r_i is an involution this is immediate.

We must now show that (singular) lines with respect to Δ have greater than two points. By transitivity on lines it suffices to show the line through K and r_iK contains greater than two points. Let $t = \{K\} \cup \{xr_iK : x \in X_{\alpha_i}\}$. We show that the line on K and r_iK contains t.

Notation. For $\alpha \in \Pi$, $a(\alpha) = \{\beta \in \Pi - \{\alpha\} : \langle\alpha, \beta\rangle \neq 0\}$. Let $A = G_\theta \cap G_\gamma$ where $\theta = \Pi - a(\alpha_i)$ and $\gamma = \delta \cup \{\alpha_i\}$. It is not difficult to see that A stabilizes t and is transitive (in fact two-transitive) on it.

Remark. $BW_\delta\langle r_i\rangle W_\delta B = KAK = BW_\delta(W_\theta \cap W_\gamma)W_\delta B$.

LEMMA 1.2. *Let $x \in X_{\alpha_i}$ and $y \in BW_\delta\langle r_i\rangle W_\delta B \cap xr_iBW_\delta\langle r_i\rangle W_\delta B$. If we express y as b_1wb_2 with b_1, b_2 in B and w in W, then w is in $r_iW_\delta\langle r_i\rangle W_\delta \cap W_\delta\langle r_i\rangle W_\delta$.*

Proof. Since $y \in BW_\delta\langle r_i\rangle W_\delta B$, clearly w is in $W_\delta\langle r_i\rangle W_\delta$. Also y in $Br_iBW_\delta\langle r_i\rangle W_\delta B$ implies we can express y as $b_1'r_ib_2'w'b_3'$, with b_1', b_2', b_3' in B and w' in $W_\delta\langle r_i\rangle W_\delta$. Then y is in $Br_iw'B \cup Bw'B$ and y is in $Br_iw'B$ unless

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$l(r_iw') < l(w')$. If y is in $Br_iw'B$ then $w = r_iw'$ is in $W_\delta\langle r_i\rangle W_\delta \cap r_iW_\delta\langle r_i\rangle W_\delta$. So we may assume $l(r_iw') < l(w')$, and so in particular w' is not in W_δ. If we consider an expression for w' of the form $w_1 \ldots w_kr_iw_{k+1} \ldots w_n$ with w_j in W_δ, fundamental reflections for $1 \leqq j \leqq n$, and of minimal length, then $l(w') = n + 1$. Now since $l(r_iw') < l(w')$ one of the following must occur:

(a) there is a $j \leqq k$ such that
$w' = r_iw_1 \ldots w_{j-1}w_{j+1} \ldots w_kr_iw_{k+1} \ldots w_n$;

(b) there is a $j \geqq k + 1$ such that
$w' = r_iw_1 \ldots w_kr_iw_{k+1} \ldots w_{j-1}w_{j+1} \ldots w_n$; or

(c) $w' = r_1w_1 \ldots w_kw_{k+1} \ldots w_n$.

In each case we have that w' is in $r_iW_\delta\langle r_i\rangle W_\delta \cap W_\delta\langle r_i\rangle W_\delta$, and so is w.

Let $S = r_iW_\delta r_iW_\delta \cap W_\delta r_iW_\delta$. We have that $BW_\delta\langle r_i\rangle W_\delta B \cap xr_iW_\delta\langle r_i\rangle W_\delta B \subseteq BSB \cup B\langle r_i\rangle W_\delta B$ for each x in X_{α_i}. We next show

$$(1.1)\quad S = \bigcup_{\substack{\alpha_j \in a(\alpha_i) \cap \delta \\ |\alpha_j| = |\alpha_i|}} (W_\theta \cap W_\gamma)r_jr_iW.$$

Denote the left hand side expression by U. We have that $wW_\delta \in \Delta'(W_\delta) \cap \Delta'(r_iW_\delta)$ if and only if w is in S. Moreover $W_\theta \cap W_\gamma$ fixes and is transitive on $\{W_\delta, r_iW\}$. Therefore, S is a union of $W_\theta \cap W_\gamma$ -- W_δ double cosets. We first show the inclusion $U \subseteq S$. By the above we must show the double coset representatives r_jr_i with α_j in $a(\alpha_i) \cap \delta$, $|\alpha_j| = |\alpha_i|$, are in S. But $r_jr_i = r_jr_i1$ is in $W_\delta r_iW_\delta$. Since $|\alpha_j| = |\alpha_i|$, $(r_jr_i)^3 = 1$. So $r_jr_i = r_ir_jr_ir_j \in r_iW_\delta r_iW_\delta$ and r_jr_i is in S. We now show the opposite inclusion.

Let (*) be the relation

$$(*)\qquad w \text{ is in } S,\ l(w) \leqq l(w') \text{ for every } w' \text{ in } (W_\theta \cap W_\gamma)wW_\delta.$$

If w satisfies (*), then $l(w) \leqq l(w')$ for all w' in wW_δ and so w is a distinguished coset representative for $(W, W/W_\delta)$ and therefore $Q(w) \subseteq R_\delta^+$, where in general for an element v of W, $Q(v)$ is the set of positive roots made negative by v.

LEMMA 1.3. *$l(w) \geqq 2$ if w satisfies (*).*

Proof. Since w is in $W_\delta r_iW_\delta$, if $l(w) = 1$, then w must be a fundamental reflection for a root in $\delta \cup \{\alpha_i\}$. But then w is in $(W_\theta \cap W_\gamma)W_\delta$, but 1 is in $(W_\theta \cap W_\gamma)W_\delta$ and so $l(w)$ is not minimal among elements of $(W_\theta \cap W_\gamma)wW_\delta$. Therefore if $l(w) \leqq 2$, and w satisfies (*) then $w = 1$. But 1 is not in $W_\delta r_iW_\delta$, so 1 is not in S and the lemma is complete.

LEMMA 1.4. *For w in $W_\delta r_iW_\delta$, $|R_\delta^+ \cap w^{-1}(R_\delta^-)| = 1$.*

Proof. Write $w = w_1r_iw_2$ with w_1, w_2 in W_δ. Then $w_2^{-1}\alpha_i$ is in R_δ^+ and $ww_2^{-1}\alpha_i = -w_1\alpha_i$ is in R_δ^-, so $|R_\delta^+ \cap w^{-1}(R_\delta^-)| \geqq 1$. If β is in R_δ^+ and $w_1r_iw_2\beta \in R_\delta^-$, then $r_iw_2\beta \in w_1^{-1}(R_\delta^-) = R_\delta^-$ and $w_2\beta \in R_\delta^+$. Then $w_2\beta \in Q(r_i) = \{\alpha_i\}$ and so $\beta = w_2^{-1}\alpha_i$. Therefore the lemma will be true when we show: if also $w = w_3r_iw_4$ with w_3, w_4 in W, then $w_4^{-1}\alpha_i = w_2^{-1}\alpha_i$. But if $w_1r_iw_2 = w_3r_iw_4$

then $w_3^{-1}w_1r_iw_2w_4^{-1} = r_i$. Now α_i is in $Q(r_i) = Q(w_3^{-1}w_1r_iw_2w_4^{-1})$. Also $r_i\alpha_i$ is in R_δ^-. Hence $r_iw_2w_4^{-1}\alpha_i$ is in R_δ^- since $w_3^{-1}w_1$ is in W. But $w_2w_4^{-1}\alpha_i$ is in R_δ^+, so $w_2w_4^{-1}\alpha_i = \alpha_i$ and we are done.

Now assume w satisfies (*). Since r_iw is in $(W_\theta \cap W_\gamma)wW_\delta$ and r_i is a fundamental reflection, $l(r_iw) = l(w) + 1$. Then r_iw is also a distinguished coset representative for $(W, W/W_\delta)$ and $Q(r_iw) = Q(w) \cup \{w^{-1}\alpha_i\} \subseteq R^+$. Let β_1 be in $Q(w)$, β_2 be in $Q(r_iw)$ such that $w\beta_1$ and $r_iw\beta_2$ are in R_δ^-. If β is in $Q(w) - \{\beta_1, \beta_2\}$, then β is in R^+ and so by Lemma 1.4 and the choice of β, $w\beta$ and $r_iw\beta$ are not in R_δ^-. But $r_iw\beta = w\beta - \langle w\beta, \alpha_i\rangle\alpha_i$ is in Φ^- and so $\langle w\beta, \alpha_i\rangle = 0$.

LEMMA 1.5. *If w satisfies* (*), *then* $l(w) \leqq 2$.

Proof. Assume on the contrary that $l(w) = n > 2$. Write $w = w_1 \dots w_n$, a product of fundamental reflections. Since w is in $W_\delta r_i W_\delta$ we may assume each w_j is a reflection for a root in $\delta \cup \{\alpha_i\}$. First suppose $w_1 \neq r_j$ for some root α_j in $a(\alpha_i) \cap \delta$, then w_1 is in $W_\theta \cap W_\gamma$ and hence w_1w is in $(W_\theta \cap W_\gamma)w\mathrm{W}_\delta$ and $l(w_1w) = l(w_2 \dots w_n) = n - 1 < l(w)$, a contradiction. So $w_1 = r_j$, some α_j in $a(\alpha_i) \cap \delta$. Now $Q(w) = Q(w_2 \dots w_n) \cup \{(w_2 \dots w_n)^{-1}\alpha_j\}$ and $r_iw(w_2 \dots w_n)^{-1}\alpha_j = -r_i\alpha_j = -(\alpha_j - \langle\alpha_j, \alpha_i\rangle\alpha_i) \in R_\delta^-$. Hence $w\beta_2 = -\alpha_j$. We next claim $w_2 = r_i$ and $w\beta_1 = -r_j\alpha_i$. Let $w_2 = r_\beta$, $\beta \in \delta \cup \{\alpha_i\}$. We have $w(w_3 \dots w_n)^{-1}\beta = r_jw_2\beta = r_jr_\beta\beta = -r_j\beta = -(\beta - \langle\beta, \alpha_j\rangle\alpha_j) \neq -\alpha_j = w\beta_2$. If $-r_j\beta \neq w\beta_1$, then $\langle\beta - \langle\beta, \alpha_j\rangle\alpha_j, \alpha_i\rangle = 0$. But then $\langle\beta, \alpha_i\rangle - \langle\beta, \alpha_j\rangle\langle\alpha_j, \alpha_i\rangle = 0$. Since $\langle\alpha_j, \alpha_i\rangle \neq 0$, $\langle\beta, \alpha_i\rangle = 0$ if and only if $\langle\beta, \alpha_j\rangle = 0$. If $\{\beta, \alpha_i\rangle = \langle\beta, \alpha_j\rangle = 0$, then $w_2 \in W_\theta \cap W_\gamma$ and $r_jw_2 = w_2r_j$ and hence $w_2w \in (W_\theta \cap W_\gamma)wW_\delta$ and $l(w_2w) = l(w_2w_1w_2 \dots w_n) = l(w_1w_3 \dots w_n) = n - 1 < l(w)$, a contradiction. If $-r_j\beta \neq w\beta_1$ and $\langle\beta, \alpha_i\rangle\langle\beta, \alpha_j\rangle \neq 0$, then $\beta \neq \alpha_i$ and we have a loop in the Dynkin diagram (i.e. three pairwise non-orthogonal fundamental roots). This cannot be. Hence we can conclude that $-r_j\beta = -(\beta - \langle\beta, \alpha_j\rangle\alpha_j) \in R_\delta^-$. Since $\beta \in \delta \cup \{\alpha_i\}$ and $\beta - \langle\beta, \alpha_j\rangle\alpha_j \in R_\delta$, also $\beta \in \Pi - \delta$, and therefore $\beta = \alpha_i$ and the claim is proved. Therefore $w = r_jr_iw_3 \dots w_n$. Assume $w_3 = r_\beta$ with $\beta \in \delta \cup \{\alpha_i\}$. Then $(w_4 \dots w_n)^{-1}\beta \in Q(W)$ and $w(w_4 \dots w_n)^{-1}\beta = -r_jr_i\beta \neq w\beta_1, w\beta_2$. So $\langle r_jr_i\beta, \alpha_i\rangle = 0$, and $r_jr_i\beta \notin R_\delta^+$. But $r_jr_i\beta = \beta - \langle\beta, \alpha_i\rangle\alpha_i + (\langle\beta, \alpha_i\rangle\langle\alpha_i, \alpha_j\rangle - \langle\beta, \alpha_j\rangle)\alpha_j$. Therefore we must have $\langle\beta, \alpha_i\rangle = 0$, and so $r_jr_i\beta = \beta - \langle\beta, \alpha_j\rangle\alpha_j$. But since $\langle r_jr_i\beta, \alpha_i\rangle = 0$, also $\langle\beta, \alpha_j\rangle = 0$. Then $r_jr_iw_3 = w_3r_jr_i$. $w_3 \in W_\theta \cap W_\gamma$, and hence $w_3w \in (W_\theta \cap W_\gamma)wW_\delta$. Also $l(w_3w) = l(r_jr_iw_4 \dots w_n) = n - 1\langle l(w)$, a contradiction and this completes the lemma.

The next lemma will show $|\alpha_j| = |\alpha_i|$ if $r_jr_i \in S$ with $\alpha_j \in a(\alpha_i)$.

LEMMA 1.6. *If $w \in r_iW_\delta r_i \cap W_\delta r_iW_\delta$, then $w\alpha_i \notin R_\delta^-$.*

Proof. Suppose on the contrary that $w\alpha_i \in R_\delta^-$. Let $w_1, w_2, w_3 \in W_\delta$ with $w = w_1r_iw_2 = r_iw_3r_i$. $w_1r_iw_2\alpha_i \in R_\delta^-$, so $r_iw_2\alpha_i \in w^{-1}(R_\delta^-) = R_\delta^-$. $w_2\alpha_i \in R_\delta^+$, so $w_2\alpha_i \in Q(r_i) = \{\alpha_i\}$. Then $r_iw_2 = w_2r_i$. This yields $r_iw_3r_i = w_1r_iw_2 = w_1w_2r_i$ and cancelling we get $r_i = w_1w_2w_3^{-1} \in W_\delta$, a contradiction.

Now if $\alpha_j \in a(\alpha_i) \cap \delta$ and $|\alpha_j| \neq |\alpha_i|$ then $\langle\alpha_j, \alpha_i\rangle\langle\alpha_i, \alpha_j\rangle > 1$ (not necessarily

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an integer-${}^2F_4$). We have $r_j r_i \in S$ if and only if $r_i r_j r_i \in S$. Clearly $r_i r_j r_i \in r_i W_\delta r_i$. Since $r_i r_j r_i \alpha_i = (1 - \langle \alpha_i, \alpha_j \rangle. \langle \alpha_j, \alpha_i \rangle)\alpha_i + \langle \alpha_i, \alpha_j \rangle \alpha_j \in R_\delta^-$ by Lemma 1.6, $r_i r_j r_i \notin W_\delta r_i W_\delta$ and so $r_j r_i \notin S$ and the equality between S and U is established.

We collect what we have shown thus far: Lemma 1.2 implies if $yK \in K^\perp \cap (xr_iK)^\perp$ for x in X_{α_i}, then y is in $BW_\delta \langle r_i \rangle W_\delta B \cap Br_i W_\delta \langle r_i \rangle W_\delta B = B\langle r_i \rangle W_\delta B \cup BSB$. Also $y \in B\langle r_i \rangle W_\delta B$ if and only if $yK \in t$. So if $yK \in K^\perp (xr_iK)^\perp - t$, for $x \in X_{\alpha_i}$, then $y \in BSB = BUB$. We now show that $\bigcap_{yK \in t} \Delta(yK) \supseteq \{zK : z \in BSB\}$. Since clearly t is itself a clique this will imply the line on K and xr_iK contains t for any $x \in X_{\alpha_i}$.

Since t is invariant under $A = G_\theta \cap G_\gamma$ (where $\theta = \Pi - a(\alpha_i)$, $\gamma = \delta \cup \{\alpha_i\}$), so is $\bigcap_{yK \in t} \Delta(yK)$ and $Y = \{z : zK \bigcap_{yK \in t} (yK)\}$. Therefore Y is a union of $A - K$ double cosets. We already know

$$Y \subseteq BSB = BUB = \bigcup_{\substack{\alpha_j \in a(\alpha_i) \cap \delta \\ |\alpha_i| = |\alpha_j|}} Ar_j r_i K.$$

To get equality it suffices to show $r_j r_i K \subseteq Y$ where $\alpha_j \in a(\alpha_i) \cap \delta$ with $|\alpha_j| = |\alpha_i|$. And hence we must show $r_j r_i K \subseteq BW_\delta r_i W_\delta B \cap xr_i BW_\delta r_i W_\delta B$ for each $x \in X_{\alpha_i}$. Clearly $r_j r_i K \subseteq BW_\delta r_i W_\delta B$. But by taking suitable representative for r_i and r_j we can show $r_i r_j r_i r_j B \subseteq xr_i BW_\delta r_i W_\delta B$ and so $r_i r_j r_i r_j K \subseteq xr_i BW_\delta r_i W_\delta B$. But $r_i r_j r_i r_j K = r_j r_i K$ and we are done.

2. Line stabilizers. Now that we have shown that the line on K and r_iK carries greater than two points and in fact contains t, we determine the actual line on K and r_iK and its stabilizer. Denote the line on K and r_iK by T. As we shall see, except under very special circumstances, $t = T$ and the stabilizer of the line is $A = G_\theta \cap G_\gamma$, where $\theta = \Pi - a(\alpha_i)$, and $\gamma = \delta \cup \{\alpha_i\}$.

Since A takes any two points of t to another pair on t, A preserves T. Hence $\{y : yK \in t\}$ is a union of A_K-double cosets and is contained in

$$AK \cup \left[\bigcup_{\substack{\alpha_j \in a(\alpha_i) \cap \delta \\ |\alpha_j| = |\alpha_i|}} Ar_j r_i K \right].$$

There are three cases to be considered, depending on $|\{\alpha_j \in a(\alpha_i) \cap \delta : |\alpha_i| = |\alpha_j|\}|$.

Case (i). $\{\alpha_j \in a(\alpha_i) \cap \delta : |\alpha_i| = |\alpha_j|\} = \emptyset$. Then $\{y : yK \in T\} = AK$ and $t = T$.

Case (ii). $|\{\alpha_j \in a(\alpha_i) \cap \delta : |\alpha_i| = |\alpha_j|\}| \geqq 2$. It suffices to show for any α_j, $\alpha_j' \in a(\alpha_i) \cap \delta$ with $|\alpha_i| = |\alpha_j| = |\alpha_j'|$ that $r_j r_i W \notin \Delta'(r_j' r_i W)$, or equivalently $r_i r_j' r_j r_i \notin W_\delta r_i W_\delta$. But $r_i r_j' r_j r_i \in r_i W_\delta r_i$ and $r_i r_j' r_j r_i \alpha_i = -(\alpha_i + \alpha_j + \alpha_j') \in R_\delta^-$ and so by Lemma 1.6, $r_i r_j' r_j r_i \notin W_\delta r_i W_\delta$.

Case (iii). There is a unique α_j with $\langle \alpha_i, \alpha_j \rangle \neq 0$, $\alpha_j \neq \alpha_i$ and $|\alpha_j| = |\alpha_i|$.

Notation: For a set χ of fundamental roots, and $\alpha \in \chi$, $C_\chi(\alpha)$ will denote the connected component of χ containing α where the relation is generated by non-orthogonality.

Claim: Unless $C_\delta(\alpha_j)$ forms an A_n-system of roots for some n, with $|a(\alpha_j) \cap C_\delta(\alpha_j)| = 1$, then $t = T$.

Note. $W_\theta \cap W_\gamma = \langle r_i \rangle \times \langle r_k : \alpha_k \in C_\delta(\alpha_j) - \{\alpha_j\} \rangle \times Z$ with $[Z, r_j] = 1$. So $(W_\theta \cap W_\gamma) r_j r_i W_\delta = (\langle r_i \rangle \times \langle r_k : \alpha_k \in C_\delta(\alpha_j) - \{\alpha_j\} \rangle) r_j r_i W_\delta$.

To prove the claim it suffices to show, by our correspondence between parabolic double cosets in W and G that $\{W_\delta, r_i W_\delta\}$ is the line on W_δ and $r_i W_\delta$. This amounts to showing that the cosets of W_δ in $(W_\theta \cap W_\gamma) r_j r_i W_\delta$ do not form a clique with respect to Δ'. Therefore it is enough to find $w \in \langle r_k : \alpha_k \in C_\delta(\alpha_j) - \{\alpha_j\} \rangle \times \langle r_i \rangle$ such that $r_i r_j w r_j r_i \notin W_\delta \langle r_i \rangle W_\delta$. Since $r_i r_j w r_i r_j r_i r_i r_j w r_j r_i W_\delta$, must in fact find w in $\langle r_k : \alpha_k \in C_\delta(\alpha_j) - \{\alpha_j\} \rangle$. By our assumption on $C_\delta(\alpha_j)$ it must be the case that $a(\alpha_i) \cap \delta = \{\alpha_j\}$. Set $V = \langle r_k : \alpha_k \in C_\delta(\alpha_j) \cup \{\alpha_i\} \rangle$, $V_i = \langle r_k : \alpha_k \in C_\delta(\alpha_j) \rangle$ and $\bar{V} = \langle r_k : \alpha_k \in \delta - C_\delta(\alpha_j) \rangle$. Then $W_\delta \langle r_i \rangle W_\delta = V_i \langle r_i \rangle V_i \bar{V}$ and from this it suffices to find $w \in \langle r_k : \alpha_k \in C_\delta(\alpha_j) - \{\alpha_j\} \rangle$ with $r_i r_j w r_j r_i \notin V_i \langle r_i \rangle V_i$. But this assertion is the same as the assertion that V is not doubly transitive on the cosets of V_i, which is obvious from our assumption on $C_\delta(\alpha_j)$ and the parabolic representations of Weyl groups.

It now remains to consider the case where $C_\delta(\alpha_j)$ is an A_n-system of roots with α_j an end-node (i.e. when $|a(\alpha_j) \cap C(\alpha_j)| = 1$). We claim that in this case $T \supset t$, that in fact $T = \{yK : y \in AK \cup Ar_j r_i K\} = t^{A'}$ where $A' = G_{\theta'} \cap G_\gamma$, $\theta' = [\Pi - a(\alpha_i)) \cup \{\alpha_j\}$. By reductions similar to the above and the correspondence between double cosets in the Weyl group W and in G for parabolics, it suffices to show the line on W_δ and $r_i W_\delta$ is $\{W_\delta, r_i W_\delta\} W_{\theta'} \cap W_\gamma$. For this we must show that the cosets of W_δ in $(\langle r_i \rangle \times \langle r_k : \alpha_k \in C_\delta(\alpha_j) - \{\alpha_j\} \rangle) r_j r_i W_\delta$ form a clique with respect to Δ'. Letting V, V_i, be as above, then since $V_i r_i V_i \subseteq W_\delta r_i W_\delta$ it suffices to show $r_i r_j w r_j r_i \in V_i \langle r_i \rangle V_i$ for any $w \in \langle r_k : \alpha_k \in C_\delta(\alpha_j) - \{\alpha_j\} \rangle$. But from our assumption on $C_\delta(\alpha_j)$, $(V, V/V_i)$ is doubly transitive and hence $V = V_i \langle r_i \rangle V_i$ and so clearly for any $w \in \langle r_k : \alpha_k \in C_\delta(\alpha_j) - \{\alpha_j\} \rangle$, $r_i r_j w r_j r_i \in V = V_i \langle r_i \rangle V_i$ and T is as asserted.

Remark. In Case (i) and in Case (iii) where $C_\delta(\alpha_j)$ is an A_n-system of roots with $|a(\alpha_j) \cap C_\delta(\alpha_j)| = 1$ we see that the line T is a maximal clique with respect to Δ. Otherwise the number of orbits on points adjacent to each point of T is $|\{\alpha_j \in a(\alpha_i) \cap \delta : |\alpha_j| = |\alpha_i|\}|$. It is also clear that except in Case (iii) when $T \supset t$, that $G_T = G_t = A$, and in Case (iii) when $T \supset t$, $G_T = A' = G_{\theta'} \cap G_\gamma$ where $\theta' = (\Pi - a(\alpha_i)) \cup \{\alpha_j\}$, $\gamma = \delta \cup \{\alpha_i\}$.

3. Properties of Lie incidence structures. Now that we have proved the existence of non-trivial incidence structures for all of the parabolic representations of groups of Lie type and found the stabilizers of the various lines that arise we investigate some properties of these structures. First we must define the notion of subspace and singular subspace, which are applicable for any incidence structure. We will show for the Lie incidence structures many of the subspaces and all the singular subspaces are parabolic in the sense that their stabilizers are parabolic subgroups.

Definition. If (P, L) is an incidence structure, then a *subspace* is a subset $X \subseteq P$ with the property that if a line l intersects X in at least two points, then $l \subseteq X$. A subspace is *singular* if it is a clique with respect to the point graph (the point graph is the graph (P, Δ) where Δ consists of the pairs of collinear points.

LEMMA 3.1. *If $\{X_i : i \in I\}$ is a collection of subspaces, then $\bigcap_{i\in I} X_i$ is a subspace.*

Proof. Let $l \in L$ with $|l \cap \bigcap_{i\in I} X_i| \geqq 2$. Then for each $i \in I$, $|l \cap X_i| \geqq 2$ and so $l \subseteq X_i$ for every i in I and $l \subseteq \bigcap_{i\in I} X_i$.

Definition. For $X \subseteq P$, the subspace spanned by X is denoted by $\langle X \rangle$ and

$$\langle X \rangle = \bigcap_{\substack{Y \supseteq X \\ Y \text{ a subspace}}} Y.$$

This is meaningful since $P \supseteq X$ and P is a subspace.

LEMMA 3.2. *Let (P, L) be an incidence structure with the property that if $l \in L$, $z \in P - l$, and z is adjacent to two points of l then z is adjacent to every point of l. Then maximal cliques with respect to the point graph (P, Δ) of (P, L) are subspaces and hence singular.*

Proof. Let M be a maximal clique. Then $M = \bigcap_{x\in M} x^\perp$. Let $l \in L$ with $|l \cap M| \geqq 2$, and $x \in M$. If $x \in M - l$, then $|l \cap \Delta(x)| \geqq |l \cap M| \geqq 2$, and so by our hypothesis $l \subseteq \Delta(x) \subseteq x^\perp$. If $x \in l$, then $l \subseteq x^\perp$ by definition of the point graph. Hence $l \subseteq \bigcap_{x\in M} x^\perp = M$.

An immediate consequence is

COROLLARY. *If (P, L) satisfies the hypothesis of Lemma 3.2 and X is a clique, then $\langle X \rangle$ is a singular subspace.*

Proof. By Zorn's Lemma there is a maximal clique, and hence a singular subspace, containing X.

LEMMA 3.3. *Suppose (G, P) is a parabolic representation of a group of Lie type, $\Delta = \Delta_t$ a self-paired orbit corresponding to some deleted root of the fundamental system, M a singular subspace of the associated incidence structure and assume M properly contains a line. Then M is a projective space.*

Proof. By the axioms of Veblen and Young it suffices to prove for each line $l \subseteq M$ and point $x \in M - l$ that $\langle l, x \rangle$ is a projective plane and so we may assume $M = \langle l, x \rangle$. By transitivity on lines we may assume l is the line on

$K = G_\delta$ and $r_i K$ where δ is the set of roots corresponding to this representation. Since M is singular $x \in K^\perp \cap (r_i K)^\perp - l$. Then $x = yK$ for some y in $\bigcup_{\alpha_j \in a(\alpha_i), |\alpha_j| = |\alpha_i|} A r_j r_i K$. Since $G_{\theta \cap \gamma} = A$ is transitive on the cosets of K in $A r_j r_i K$ we may assume $x = r_j r_i K$ for some $\alpha_j \in a(\alpha_i) \cap \delta$ with $|\alpha_j| = |\alpha_i|$. For $\chi \subseteq \Pi$, let $a(\chi) = \{\beta \in \Pi - \chi : \langle \beta, \alpha \rangle \neq 0$ for some $\alpha \in \chi\}$. Let $A_2 = G_{\theta_2} \cap G_\gamma$, γ as before and $\theta_2 = \Pi - a(\alpha_i, \alpha_j)$. Then A_2 fixes $Z = \{z : z$ is on a line from $r_j r_i K$ to a point of $l\}$, and induces a PSL_3 on Z in one of its doubly transitive representations. Thus Z is a projective plane and hence a singular subspace. As $\{x\} \cup l \subseteq Z$, $\langle x, l \rangle \subseteq Z$. But clearly we must have $\langle x, l \rangle \supseteq Z$ so we get equality. Note that A stabilizes $Z = M$ and so the stabilizer of M contains a Borel subgroup and hence is a parabolic subgroup (c.f. Theorem 8.3.2 of [**2**]). By considering fundamental reflections for roots not in $W_{\theta_2} \cap W_\gamma$, we see that in fact $G_M = A_2$ in this case.

LEMMA 3.4. *Let $\delta \subset \Pi$, $\alpha_i \in \Pi - \delta$, $\Delta' = \Delta_i'$. Assume $\{\alpha_j \in a(\alpha_i) : |\alpha_i| = |\alpha_j|\} \neq \emptyset$ and if there is a unique α_j in this set then $C_\delta(\alpha_j)$ is not an A-system of roots with α_j an end-node. Let $\alpha_i = \beta_1, \beta_2, \ldots, \beta_n$ be a sequence of roots in $\delta \cup \{\alpha_i\}$ with $\{\beta_1, \ldots, \beta_n\}$ a naturally ordered A-system of roots. Set $v_0 = 1$, $v_{k+1} = r_{\beta_{k+1}} v_k$. Then $\bar{M} = \{v_0 W_\delta, v_1 W_\delta, \ldots, v_n W_\delta\}$ is a clique with respect to Δ'. $W_{\bar{M}}$, the stabilizer in W of $\bar{M}$, is equal to $W_\theta \cap W_\gamma$ where $\theta_n = \Pi - a(\{\beta_1, \ldots, \beta_n\})$ and $\gamma = \delta \cup \{\alpha_i\}$. Also*

$$\bigcap_{k=0}^{n} \Delta'(w_k W) = \left\{ w W_\delta : w \in \bigcup_{\substack{\beta \in a(\beta_n) \cap \delta - \{\beta_{n-1}\} \\ |\beta| = |\beta_n|}} W_{\bar{M}} r_\beta v_n W_\delta \right\}.$$

Proof. Clearly $W_{\theta_n} \cap W_\gamma$ stabilizes $\bar{M}$ and induces a Σ_{n+1} on it. As W_δ, $r_i W_\delta = v_1 W_\delta$ are adjacent and in $\bar{M}$, $\bar{M}$ is a clique. Suppose $w \notin W_\gamma$, then clearly $w \notin W_{\bar{M}}$ and also $w W_\delta \notin \bigcap_{k=0}^{n} \Delta'(v_k W)$. As everything can be calculated inside W_γ, we may assume $W_\gamma = W$ and $\delta \cup \{\alpha_i\} = \gamma = \Pi$. Therefore the representation is a maximal parabolic representation of W (but not necessarily primitive). W_{θ_n} induces the full symmetric group on $\bar{M}$, so if w is in $W_{\bar{M}} - W_{\theta_n}$ we may assume w acts as the identity on $\bar{M}$, and so is in $C_W(W_{\{\beta_1, \ldots, \beta_n\}})$. But $C_W(W_{\{\beta_1, \ldots, \beta_n\}}) = W_{\Pi - \{\beta_1, \ldots, \beta_n\} - a(\{\beta_1, \ldots, \beta_n\})} \subseteq W_{\theta_n}$. So in fact $W_{\bar{M}} \subseteq W_{\theta_n}$ and we have equality. The last part of the lemma we prove by considering the different Weyl groups and representations separately. As they are all similar we only give the details for a couple.

(a). $W = W(A_s)$. Then $\alpha_i = \beta_1$ must be an interior root from our hypotheses. Assume that α_i is the ith node in a natural ordering of the roots with $i \leqq s/2$. Then the graph may be described as follows: the vertices are the i-subsets of a set of cardinality $s + 1$. Two are adjacent if they meet in an $i - 1$ subset. Cliques come of two types—some set of i-subsets contained in an $(i + 1)$-subset, or some set of i-subsets containing a given $(i - 1)$-subset. In either case under the action of the stabilizer of the subset there is a single orbit of vertices adjacent to each one in the clique and this suffices.

(b). $W = W(B_s)$. Then $\alpha_i = \beta_1$ is either an end-node or an interior node, but is not adjacent to a node of different length. Let us consider the case that α_i is an end node. Then the graph may be described as follows: there are $2s$ vertices $x_1, x_2, \ldots, x_{2s-1}, x_{2s}$. x_{2k-1} is adjacent to all vertices except itself and x_{2k}, and similarly x_{2k} is adjacent to all other nodes except itself and x_{2k-1}. Then under the action of W there is a single class of cliques of size n, and under the stabilizer of such a clique there is a single orbit of vertices adjacent to every point of the clique, and this suffices. If α_i is an interior node say the ith node in a natural ordering of the roots then the graph can be described as follows: the vertices are the i-subsets of $x_1, x_2, \ldots, x_{2s}$ which are cliques. Two are adjacent if they meet in an $i - 1$ subset and join to a clique. Then a clique for this graph is either of the following type—a set of i-subsets contained in some clique of $x_1, \ldots, x_{2s}$ of size $i + 1$, or some set of i-subsets of $x_1, \ldots, x_{2s}$ contained in some clique of $x_1, \ldots, x_{2s}$ and containing a given $(i - 1)$-subset. In either case the stabilizer of such a clique has a single orbit on vertices adjacent to every point of the clique and this suffices.

(c). W is exceptional. Then some identifications similar to the above can be made or the result can be proved by direct calculation using distinguished coset representatives.

With Lemma 3.4 we can prove

LEMMA 3.5. *Assume the hypotheses of Lemma* 3.3. *Then G_M, the stabilizer of M, is a parabolic subgroup.*

Proof. The proof is by induction on the projective dimension of $M = n$. We first show that under the action of G, M has an image $M' = \langle v_j K : 0 \leqq j \leqq n \rangle$ where there is an A-system of roots $\alpha_i = \beta_1, \beta_2, \ldots, \beta_n$ in $\delta \cup \{\alpha_i\}$ and $v_0 = 1$, $v_{j+1} = r_{\beta_{j+1}} v_j$. Suppose we have shown this. Then we may assume $M = M'$. By looking at generators we see that $G_{\theta_n} \cap G_\gamma$ is contained in G_M, where $\theta_n = \Pi - a(\{\beta_1, \ldots, \beta_n\})$. Then G_M contains a Borel subgroup and so is parabolic (c.f. Theorem 8.3.2 of [**2**]). By considering reflections not in $W_{\theta_n} \cap W_\gamma$ it is then easy to see that in fact $G_M = G_{\theta_n} \cap G_\gamma$.

For dimension of M equal to two this was shown in Lemma 3.3. So assume dimension of $M = n > 2$. Let M_H be a hyperplane of M. By induction the lemma applies to M_H, so in fact we may assume $M_H = \langle v_j K : 0 \leqq j \leqq n - 1 \rangle$ and $G_{M_H} = G_{\theta_{n-1}} \cap G_\gamma$ where $\theta_{n-1} = \Pi - a(\{\beta_1, \ldots, \beta_{n-1}\})$. Let $xK \in M - M_H$. Then since $G_{M_H} W G = G_{M_H} W K = G$ we may assume x is in W. But then

$$xW \in \bigcap_{j=0}^{n-1} \Delta'(v_j W) = \bigcup_{\substack{\beta \in a(\beta_{n-1}) \cap \delta - \{\beta_{n-2}\} \\ |\beta| = |\beta_{n-1}|}} (W_{\theta_{n-1}} \cap W_\gamma) r_\beta v_{n-1} W_\delta$$

and so we may assume $x = r_\beta v_{n-1}$. Setting $\beta_n = \beta$ completes the lemma.

Definition. By the diagram of the incidence structure (P, L_i) with point graph (P, Δ_i), $\Delta_i = \Delta_{\alpha_i}$, arising from G, Π, δ, $\alpha_i \in \Pi - \delta$ we mean the triple (Π, δ, α_i).

We have shown that the classes (under the action of G) of singular subspaces of (P, L_i) are in one-to-one correspondence with the subsets $\chi \subseteq \delta \cup \{\alpha_i\}$ such that $\alpha_i \in \chi$ and χ is a fundamental base for an A-system of roots.

In another paper we will consider the structures associated with certain maximal parabolic representations of groups of type A_n, B_n, C_n, D_n, and E_6 and characterize them when the underlying field is finite.

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