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SHELLABILITY OF SEMIGROUP RINGS

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Abstract. The concepts of Λ-shellability of locally finite posets as well as of extendable sequentially Koszul algebras will be introduced. It will be proved that the divisor poset of a homogeneous semigroup ring is Λ-shellable if and only if the semigroup ring is extendable sequentially Koszul. Examples of extendable sequentially Koszul semigroup rings contain all monomial ASL's (algebras with straightening laws) and all second squarefree Veronese subrings.

Introduction

Let A be a homogeneous semigroup ring over a field K, i.e., A is a subring of a polynomial ring over K generated by a finite number of monomials and A is a graded algebra $A = A_0 \oplus A_1 \oplus \cdots$ with $A_0 = K$ such that the minimal system of monomial generators of A is contained in A_1. We write Σ_A for the infinite poset (partially ordered set) consisting of all monomials belonging to A, ordered by divisibility. Thus, in particular, Σ_A possesses a unique minimal element 1 ($\in K$), and is locally finite and pure, i.e., if $\alpha, \beta \in \Sigma_A$ with $\alpha < \beta$, then the closed interval $[\alpha, \beta] = \{\gamma \in \Sigma_A \,;\, \alpha \le \gamma \le \beta\}$ is finite and pure. (A finite poset P is called pure if all maximal chains (totally ordered sets) contained in P have the same cardinality.) The infinite poset Σ_A is said to be the divisor poset of A.

It is known, e.g., [11] that a homogeneous semigroup ring A over a field K is Koszul if and only if Σ_A is Cohen-Macaulay over K (i.e., for all $\alpha \in \Sigma_A$, the closed interval $[1, \alpha]$ of Σ_A is Cohen-Macaulay over K). See also [6]. (We refer the reader to, e.g., [3] and [7] for the detailed information about Cohen-Macaulay posets.) In [5], the concept of strongly Koszul algebras is introduced, and it is proved that a homogeneous semigroup ring A is strongly Koszul if and only if, for all $\alpha \in \Sigma_A$, the closed interval $[1, \alpha]$ of Σ_A is wonderful (locally semimodular), i.e., if $\beta \to u < w$, $\beta \to v < w$ in

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$[1, \alpha]$, then $u \to w'$, $v \to w'$ for some $w' \leq w$. (We write $u \to w$ if u is covered by w, i.e., $u < w$ and $u < v < w$ for no $v \in \Sigma_A$.)

In the combinatorics on finite posets, there is the hierarchy as follows:

$$\text{wonderful} \quad \Rightarrow \quad \text{shellable} \quad \Rightarrow \quad \text{Cohen-Macaulay}.$$

(A finite pure poset P is called shellable if there exists an ordering of the maximal chains $C_1, C_2, \ldots, C_s$ of P such that, if $1 \leq i < j \leq s$, then there is $k < j$ with $C_i \cap C_j \subset C_k \cap C_j$ and $\sharp(C_k \cap C_j) = \delta - 1$, where δ is the cardinality of any maximal chain of P, and where $\sharp(C_k \cap C_j)$ is the cardinality of $C_k \cap C_j$.) It is a fundamental question to investigate the class of homogeneous semigroup rings with shellable divisor posets. It seems, however, unknown if there is a Koszul semigroup rings with a nonshellable divisor poset.

A sufficient condition for a homogeneous algebra to be Koszul is that its defining ideal possesses a quadratic Gröbner basis. In [11], the shellability of divisor posets of homogeneous semigroup rings is studied from the viewpoint of noncommutative Gröbner bases, and it is shown [11, Corollary 3.6] that the divisor poset of a 'quasi-poset' semigroup ring (i.e., a homogeneous semigroup ring whose defining ideal has a quasi-poset initial ideal) is shellable. (An ideal I of the polynomial ring $K[x_1, x_2, \ldots, x_n]$ is called quasi-poset if I is generated by quadratic monomials and satisfies the following condition: If $1 \leq i \leq k \leq j \leq n$ and if $x_i x_j \in I$, then either $x_i x_k \in I$ or $x_k x_j \in I$.) It is likely that the quasi-poset semigroup rings form a rather small subclass of the class of semigroup rings having quadratic Gröbner bases. For example, the d-th squarefree Veronese subring of $K[x_1, x_2, \ldots, x_n]$, where $2 \leq d < n$, is quasi-poset if and only if either (i) $d = 2$ and $3 \leq n \leq 4$, or (ii) $d \geq 3$ and $n = d + 1$ ([8, Theorem 2.3]). See also [10].

Our original motivation of the present paper is to find a useful criterion for the divisor poset of a homogeneous semigroup ring to be shellable. We introduce the combinatorial notion of Λ-shellability of locally finite posets and present the algebraic concept of (extendable) sequentially Koszul algebras. There is the following hierarchy in the class of homogeneous semigroup rings:

$$\text{strongly Koszul} \quad \Rightarrow \quad \text{extendable sequentially Koszul} \quad \Rightarrow \quad \text{Koszul}.$$

Theorem 3.1, which is the main theorem of the present paper, guarantees that the divisor poset of a homogeneous semigroup ring is Λ-shellable if and

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only if the semigroup ring is extendable sequentially Koszul. The class of extendable sequentially Koszul semigroup rings contains all monomial ASL's (algebras with straightening laws) as well as all second squarefree Veronese subrings. See Theorems 2.4 and 2.5. In particular, divisor posets of these semigroup rings are shellable. (Note that the shellability of monomial ASL's follows from [11, Corollary 3.6] since every monomial ASL is quasi-poset, while the shellability of second squarefree Veronese subrings does not follow from [11, Corollary 3.6].) It is remarkable that there exists an extendable sequentially Koszul semigroup ring having *no* quadratic Gröbner basis. See Example 3.5 (a). We do not know if there is a quasi-poset semigroup ring which is not extendable sequentially Koszul.

§1. Λ-shellability of locally finite posets

We introduce the notion of Λ-shellability of locally finite posets (partially ordered sets) and show that all rank-selected subposets of a Λ-shellable poset are Λ-shellable.

First, we recall some fundamental materials on finite posets. See, e.g., [12] for the detailed information. Let P be a finite poset. A chain of P is a totally ordered set C of P. The length of a chain C of P is $\sharp(C) - 1$, where $\sharp(C)$ is the cardinality of C as a finite set. A pure poset is a finite poset any of whose maximal chain has the same length. If $\alpha, \beta \in P$ with $\alpha < \beta$, then the closed interval $[\alpha, \beta]$ is the subposet $\{\gamma \in P \,;\, \alpha \leq \gamma \leq \beta\}$ of P. We write $\alpha \to \beta$ with $\alpha, \beta \in P$ for the covering relation of P, i.e., $\alpha < \beta$ in P and $\alpha < \gamma < \beta$ for no $\gamma \in P$. A chain $C : \alpha_0 < \alpha_1 < \cdots < \alpha_q$ of P is called saturated if α_{i-1} is covered by α_i, i.e., $\alpha_{i-1} \to \alpha_i$ for all $1 \leq i \leq q$.

A finite pure poset P is called *shellable* if there exists a total ordering (called a *shelling*) of the maximal chains $C_1, C_2, \ldots, C_s$ of P such that, if $1 \leq i < j \leq s$, then there is $k < j$ with $C_i \cap C_j \subset C_k \cap C_j$ and $\sharp(C_k \cap C_j) = \delta - 1$, where δ is the cardinality of any maximal chain of P. See [1] and [2] for the foundations on shellability of finite posets.

A *locally finite* poset is an infinite poset Σ any of whose closed interval is finite. A locally finite poset Σ is called *pure* if every closed interval of Σ is pure.

We work with an infinite poset Σ which is locally finite and pure and which possesses a unique minimal element 1. Let $\mathcal{M}_q(\Sigma)$ denote the set of saturated chains of Σ of length q starting from 1, in other words, every chain belonging to $\mathcal{M}_q(\Sigma)$ is of the form

$$1 \to \alpha_1 \to \alpha_2 \to \cdots \to \alpha_q$$

with each $\alpha_i \in \Sigma$.

Let Ω denote a totally ordered set (say, $\mathbb{Z}$). A *chain-edge labeling* of Σ is a map

$$\lambda : \bigcup_{q=1}^{\infty} \mathcal{M}_q(\Sigma) \longrightarrow \bigcup_{q=1}^{\infty} \Omega^q,$$

which associates a saturated chain

$$C : 1 = \alpha_0 \to \alpha_1 \to \alpha_2 \to \cdots \to \alpha_q$$

belonging to $\mathcal{M}_q(\Sigma)$ with

$$\lambda(C) = \big(\lambda(C; \alpha_0 \to \alpha_1), \lambda(C; \alpha_1 \to \alpha_2), \ldots, \lambda(C; \alpha_{q-1} \to \alpha_q)\big) \in \Omega^q,$$

satisfying the following condition:

> If $p \leq q$ and if a saturated chain $C' : 1 \to \alpha_1 \to \cdots \to \alpha_p$ belonging to $\mathcal{M}_p(\Sigma)$ is a subchain of a saturated chain $C : 1 \to \alpha_1 \to \cdots \to \alpha_p \to \alpha_{p+1} \to \cdots \to \alpha_q$ belonging to $\mathcal{M}_q(\Sigma)$, then $\lambda(C'; \alpha_{i-1} \to \alpha_i) = \lambda(C; \alpha_{i-1} \to \alpha_i)$ for all $1 \leq i \leq p$.

A chain-edge labeling λ of Σ is said to be a Λ-*labeling* if the following conditions are satisfied:

(Λ-1) If $C : 1 = \alpha_0 \to \alpha_1 \to \cdots \to \alpha_q \in \mathcal{M}_q(\Sigma)$ and if $\alpha_q \to \beta$ and $\alpha_q \to \gamma$ with $\beta \neq \gamma$, then $\lambda(C \cup \{\beta\}; \alpha_q \to \beta) \neq \lambda(C \cup \{\gamma\}; \alpha_q \to \gamma)$;

(Λ-2) Let $C : 1 = \alpha_0 \to \alpha_1 \to \cdots \to \alpha_{q-1} \to \alpha_q \in \mathcal{M}_q(\Sigma)$, and let $\alpha_q \to \beta$ and $\alpha_q \to \gamma$ with $\beta \neq \gamma$. If β covers an element $\alpha' \in \Sigma$ with $\alpha_{q-1} \to \alpha'$ satisfying $\lambda((C \setminus \{\alpha_q\}) \cup \{\alpha'\}; \alpha_{q-1} \to \alpha') < \lambda(C; \alpha_{q-1} \to \alpha_q)$ and if γ covers *no* element $\alpha' \in \Sigma$ with $\alpha_{q-1} \to \alpha'$ satisfying $\lambda((C \setminus \{\alpha_q\}) \cup \{\alpha'\}; \alpha_{q-1} \to \alpha') < \lambda(C; \alpha_{q-1} \to \alpha_q)$, then $\lambda(C \cup \{\beta\}; \alpha_q \to \beta) < \lambda(C \cup \{\gamma\}; \alpha_q \to \gamma)$.

A chain-edge labeling λ of Σ is said to be a Λ-*shelling* if λ is a Λ-labeling and if, for all $\alpha \in \Sigma$, the total ordering $C_1, C_2, \ldots, C_s$ of the maximal chains of the closed interval $[1, \alpha]$ with

$$\lambda(C_1) <_{lex} \lambda(C_2) <_{lex} \cdots <_{lex} \lambda(C_s)$$

defines a shelling of $[1, \alpha]$. Here, $<_{lex}$ is the lexicographic order on Ω^q (and q is the length of any maximal chain of $[1, \alpha]$), i.e., $(a_1, \dots, a_q) <_{lex} (b_1, \dots, b_q)$ if $a_1 = b_1, \dots, a_{i-1} = b_{i-1}$ and $a_i < b_i$ for some $1 \leq i \leq q$.

An infinite poset Σ which is locally finite and pure with a unique minimal element is called Λ-*shellable* if Σ admits a Λ-shelling.

Let Σ be an infinite poset which is locally finite and pure with a unique minimal element 1. The *rank-selected* subposet of Σ of order d with $0 < d \in \mathbb{Z}$ is the subposet of Σ which consists of those elements $\alpha \in \Sigma$ such that the length of any maximal chain of the closed interval $[1, \alpha]$ belongs to $\{0, d, 2d, 3d, \dots\}$.

THEOREM 1.1. *All rank-selected subposets of a Λ-shellable poset are Λ-shellable.*

Proof. Let Σ be a Λ-shellable poset with a Λ-shelling λ, and let $\Sigma^{(d)}$ be the rank-selected subposet of Σ of order d. If $C : 1 = \alpha_0 \to \alpha_1 \to \cdots \to \alpha_q \in \mathcal{M}_q(\Sigma^{(d)})$, then we write $\bar{C}$ for the maximal chain of $[1, \alpha_q]$ in Σ with $\alpha_i \in \bar{C}$ for each i such that, for any maximal chain C' of $[1, \alpha_q]$ in Σ with $\alpha_i \in C'$ for each i, we have $\lambda(\bar{C}) \leq_{lex} \lambda(C')$. If

$$\bar{C} : \cdots \to \alpha_{i-1} \to \alpha_{i-1}^{(1)} \to \alpha_{i-1}^{(2)} \to \cdots \to \alpha_{i-1}^{(d-1)} \to \alpha_i \to \cdots,$$

then we define $\lambda^{(d)}(C; \alpha_{i-1} \to \alpha_i) \in \Omega^d$ to be

$$\big(\lambda(\bar{C}; \alpha_{i-1} \to \alpha_{i-1}^{(1)}), \lambda(\bar{C}; \alpha_{i-1}^{(1)} \to \alpha_{i-1}^{(2)}), \dots, \lambda(\bar{C}; \alpha_{i-1}^{(d-1)} \to \alpha_i)\big).$$

This technique enables us to obtain a chain-edge labeling

$$\lambda^{(d)} : \bigcup_{q=1}^{\infty} \mathcal{M}_q(\Sigma^{(d)}) \longrightarrow \bigcup_{q=1}^{\infty} (\Omega^d)^q$$

of $\Sigma^{(d)}$, which satisfies the condition (Λ-1).

Let C and C' be maximal chains of the closed interval $[1, u]$ of $\Sigma^{(d)}$ with $\lambda^{(d)}(C) <_{lex} \lambda^{(d)}(C')$. Then $\lambda(\bar{C}) <_{lex} \lambda(\bar{C}')$. Hence there exists a maximal chain C_0 of the closed interval $[1, u]$ of Σ such that $\lambda(C_0) <_{lex} \lambda(\bar{C}')$, $\bar{C} \cap \bar{C}' \subset C_0 \cap \bar{C}'$ and $\sharp(C_0 \cap \bar{C}') = \sharp(\bar{C}) - 1$. Since $\lambda(C_0) <_{lex} \lambda(\bar{C}')$, it follows that $\bar{C}' \setminus C_0 \subset \Sigma^{(d)}$. Thus the maximal chain $C'' = C_0 \cap \Sigma^{(d)}$ of $[1, u]$ of $\Sigma^{(d)}$ satisfies $\lambda^{(d)}(C'') <_{lex} \lambda^{(d)}(C')$, $C \cap C' \subset C'' \cap C'$ and $\sharp(C'' \cap C') = \sharp(C) - 1$.

Hence $\lambda^{(d)}$ turns out to be a Λ-shelling of $\Sigma^{(d)}$, provided that $\lambda^{(d)}$ satisfies the condition (Λ-2). Let $C : 1 = \alpha_0 \to \alpha_1 \to \cdots \to \alpha_{q-1} \to \alpha_q \in \mathcal{M}_q(\Sigma^{(d)})$, and let $\alpha_q \to \beta$ and $\alpha_q \to \gamma$ with $\beta \neq \gamma$ in $\Sigma^{(d)}$. Suppose that β covers an element $\alpha' \in \Sigma^{(d)}$ with $\alpha_{q-1} \to \alpha'$ satisfying $\lambda^{(d)}((C \setminus \{\alpha_q\}) \cup \{\alpha'\}; \alpha_{q-1} \to \alpha') < \lambda^{(d)}(C; \alpha_{q-1} \to \alpha_q)$ and that γ covers no element $\alpha' \in \Sigma^{(d)}$ with $\alpha_{q-1} \to \alpha'$ satisfying $\lambda^{(d)}((C \setminus \{\alpha_q\}) \cup \{\alpha'\}; \alpha_{q-1} \to \alpha') < \lambda^{(d)}(C; \alpha_{q-1} \to \alpha_q)$. Let $C_1 = C \cup \{\beta\}$, $C_2 = C \cup \{\gamma\}$ and $C_3 = (C \setminus \{\alpha_q\}) \cup \{\alpha', \beta\}$. Let $\alpha_q \to \beta'$ in $\bar{C}_1$, $\alpha_q \to \gamma'$ in $\bar{C}_2$ and $\alpha'' \to \alpha_q$ in $\bar{C}$. Since $\lambda(\bar{C}_3) <_{leq} \lambda(\bar{C}_1)$, the shellability of $[1, \beta]$ in Σ guarantees the existence of $\xi \in \Sigma^{(d)}$ with $\alpha'' \to \xi \to \beta'$ in Σ and with $\lambda((\bar{C} \setminus \{\alpha_q\}) \cup \{\xi\}; \alpha'' \to \xi) < \lambda(\bar{C}; \alpha'' \to \alpha_q)$. Since there is no $\xi \in \Sigma^{(d)}$ with $\alpha'' \to \xi \to \gamma'$ in Σ and with $\lambda((\bar{C} \setminus \{\alpha_q\}) \cup \{\xi\}; \alpha'' \to \xi) < \lambda(\bar{C}; \alpha'' \to \alpha_q)$, it follows that $\lambda(\bar{C} \cup \{\beta'\}; \alpha_q \to \beta') < \lambda(\bar{C} \cup \{\gamma'\}; \alpha_q \to \gamma')$. Hence $\lambda^{(d)}(C \cup \{\beta\}; \alpha_q \to \beta) < \lambda^{(d)}(C \cup \{\gamma\}; \alpha_q \to \gamma)$, as desired. ∎

§2. Extendable sequentially Koszul algebras

We introduce the concept of extendable sequentially Koszul algebras and show that all ASL's (algebras with straightening laws) as well as all second squarefree Veronese subrings are extendable sequentially Koszul algebras.

Let K be a field and $A = A_0 \oplus A_1 \oplus \cdots$ a homogeneous K-algebra. Let $G = \{s_1, \ldots, s_n\}$ be a minimal system of generators of A with each $s_i \in A_1$.

(1) We say that a subset $\{s_{i_1}, \ldots, s_{i_q}\}$ of G has *linear quotients* if there exists an ordered sequence

$$\mathbf{s} = (s_{j_1}, \ldots, s_{j_q})$$

of $\{s_{i_1}, \ldots, s_{i_q}\}$ such that, for all initial sequences $(s_{j_1}, \ldots, s_{j_p})$ of $\mathbf{s}$, the colon ideal

$$I(\mathbf{s}; p) = (s_{j_1}, \ldots, s_{j_{p-1}}) : s_{j_p}$$

is generated by a subset $G(\mathbf{s}; p)$ of G.

(1′) We say that a subset $\{s_{i_1}, \ldots, s_{i_q}\}$ of G has *extendable linear quotients* if there exists an ordered sequence

$$\mathbf{s} = (s_{j_1}, \ldots, s_{j_n})$$

of G with

$$\{s_{i_1}, \ldots, s_{i_q}\} = \{s_{j_1}, \ldots, s_{j_q}\}$$

such that, for all initial sequences $(s_{j_1},\dots,s_{j_p})$ of $\mathbf{s}$, the colon ideal

$$I(\mathbf{s};p)=(s_{j_1},\dots,s_{j_{p-1}}):s_{j_p}$$

is generated by a subset $G(\mathbf{s};p)$ of G.

(2) We say that a subset $\{s_{i_1},\dots,s_{i_q}\}$ of G has *linear quotients of second level* if there exists an ordered sequence $\mathbf{s}=(s_{j_1},\dots,s_{j_q})$ of $\{s_{i_1},\dots,s_{i_q}\}$ such that, for all $1<p\leq q$, the colon ideal $I(\mathbf{s};p)$ is generated by a subset $G(\mathbf{s};p)$ of G and, in addition, $G(\mathbf{s};p)$ has linear quotients.

(2′) We say that a subset $\{s_{i_1},\dots,s_{i_q}\}$ of G has *extendable linear quotients of second level* if there exists an ordered sequence $\mathbf{s}=(s_{j_1},\dots,s_{j_n})$ of G with $\{s_{i_1},\dots,s_{i_q}\}=\{s_{j_1},\dots,s_{j_q}\}$ such that, for all $1<p\leq n$, the colon ideal $I(\mathbf{s};p)$ is generated by a subset $G(\mathbf{s};p)$ of G and, in addition, $G(\mathbf{s};p)$ has extendable linear quotients.

(3) We say that a subset $\{s_{i_1},\dots,s_{i_q}\}$ of G has *linear quotients of ρ-th level* if there exists an ordered sequence $\mathbf{s}=(s_{j_1},\dots,s_{j_q})$ of $\{s_{i_1},\dots,s_{i_q}\}$ such that, for all $1<p\leq q$, the colon ideal $I(\mathbf{s};p)$ is generated by a subset $G(\mathbf{s};p)$ of G and, in addition, $G(\mathbf{s};p)$ has linear quotients of $(\rho-1)$-th level.

(3′) We say that a subset $\{s_{i_1},\dots,s_{i_q}\}$ of G has *extendable linear quotient of ρ-th level* if there exists an ordered sequence $\mathbf{s}=(s_{j_1},\dots,s_{j_n})$ of G with $\{s_{i_1},\dots,s_{i_q}\}=\{s_{j_1},\dots,s_{j_q}\}$ such that, for all $1<p\leq n$, the colon ideal $I(\mathbf{s};p)$ is generated by a subset $G(\mathbf{s};p)$ of G and, in addition, $G(\mathbf{s};p)$ has extendable linear quotients of $(\rho-1)$-th level.

DEFINITION 2.1. (a) A homogeneous K-algebra $A=A_0\oplus A_1\oplus\cdots$ is called *sequentially Koszul* (resp. *extendable sequentially Koszul*) if A admits a minimal system of generators $G=\{s_1,\dots,s_n\}$ $(\subset A_1)$ such that G has linear quotients (resp. extendable linear quotients) of ρ-th level for all $\rho\geq 2$.

(b) A homogeneous semigroup ring is called *sequentially Koszul* (resp. *extendable sequentially Koszul*) if its minimal system of monomial generators has linear quotients (resp. extendable linear quotients) of ρ-th level for all $\rho\geq 2$.

Recall from [5] that a homogeneous K-algebra $A=A_0\oplus A_1\oplus\cdots$ is called *strongly Koszul* if A admits a minimal system of generators $G=\{s_1,\dots,s_n\}$ $(\subset A_1)$ satisfying the following condition:

($*$) For all subsequences $s_{i_1},\dots,s_{i_q}$ of $(s_1,\dots,s_n)$ with $i_1<\cdots<i_q$ and for all $1<p\leq q$, the colon ideal $(s_{i_1},\dots,s_{i_{p-1}}):s_{i_p}$ is generated by a subset of G.

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Moreover, a homogeneous semigroup ring is called *strongly Koszul* if its minimal system of monomial generators $G = \{s_1, \dots, s_n\}$ satisfies the condition $(*)$ above. It follows from [5, Proposition 1.4] that a homogeneous semigroup ring with its minimal system of monomial generators $G = \{s_1, \dots, s_n\}$ is strongly Koszul if and only if the colon ideal $(s_i) : s_j$ is generated by a subset of G for all $i \neq j$. Thus, in particular, any strongly Koszul semigroup ring is extendable sequentially Koszul.

In [5, Theorem 1.2] it is proved that every strongly Koszul algebra is Koszul. The technique appearing in the proof of [5, Theorem 1.2] can be applied to sequentially Koszul algebras and we obtain

THEOREM 2.2. *A sequentially Koszul algebra is Koszul.* (*Thus, in particular, every extendable sequentially Koszul algebra is Koszul.*)

One of the most distinguished classes of extendable sequentially Koszul algebras is the class of ASL's. First of all, we recall some fundamental materials on ASL's from [4]. See also [3] and [7]. Let K be a field and $A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$ a graded algebra over $A_0 = K$. Let P be a finite poset which is a subset of A_1 and suppose that A is generated by P as an algebra over K. A product of the form $\alpha_1 \alpha_2 \cdots \alpha_q$ with each $\alpha_i \in P$ is called a *standard monomial* of A if $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_q$. Then A is an *algebra with straightening laws* (ASL for short) on P over K if

(ASL-1) The set of standard monomials of A is a basis of A as a vector space over K;

(ASL-2) If α and β in P are incomparable and if

$$\alpha\beta = \sum_i r_i \gamma_{i_1} \gamma_{i_2},$$

where $0 \neq r_i \in K$ and $\gamma_{i_1} \leq \gamma_{i_2}$, is the unique expression for $\alpha\beta$ in A as a linear combination of standard monomials guaranteed by (ASL-1), then $\gamma_{i_1} \leq \alpha$ and $\gamma_{i_1} \leq \beta$ for every i.

The relations mentioned in (ASL-2) are called the *straightening relations*. It then follows that if $w = \beta_1 \beta_2 \cdots$ is a nonstandard monomial of A and if

$$w = \sum_i r_i \gamma_{i_1} \gamma_{i_2} \cdots$$

with $0 \neq r_i \in K$ and with $\gamma_{i_1} \leq \gamma_{i_2} \leq \cdots$ is the standard monomial expression of w, then $\gamma_{i_1} < \beta_j$ for all i and j.

A *poset ideal* of a finite poset P is a subset I of P (possibly, empty) such that $\alpha \in I$ and $\beta \in P$ together with $\beta \leq \alpha$ in P imply $\beta \in I$. If A is an ASL on P over K and if I is a poset ideal of P, then the ideal $(\{\xi\ ;\ \xi \in I\})$ of A generated by I has a K-basis consisting of those standard monomials $\gamma_{i_1}\gamma_{i_2}\cdots$ with $\gamma_{i_1} \leq \gamma_{i_2} \leq \cdots$ and with $\gamma_{i_1} \in I$.

LEMMA 2.3. *Let I be a poset ideal of P and $\alpha \in P \setminus I$. Suppose that $I \cup \{\alpha\}$ is a poset ideal of P. Then, in A, we have*

$$(\{\xi\ ;\ \xi \in I\}) : \alpha = (\{\zeta \in P\ ;\ \zeta \not\geq \alpha\}).$$

Proof. First, if $\zeta \in P$ with $\zeta \not\geq \alpha$, then $\zeta\alpha$ belongs to $(\{\xi\ ;\ \xi < \alpha\})$ by (ASL-2). Thus, $\zeta\alpha \in (\{\xi\ ;\ \xi \in I\})$ since $(\{\xi\ ;\ \xi < \alpha\}) \subset (\{\xi\ ;\ \xi \in I\})$.

Second, suppose that $\delta \in A$ belongs to $(\{\xi\ ;\ \xi \in I\}) : \alpha$ and let $\delta = \sum_i r_i\gamma_{i_1}\gamma_{i_2}\cdots$ be the standard monomial expression of δ with $0 \neq r_i \in K$ and with $\gamma_{i_1} \leq \gamma_{i_2} \leq \cdots$ for each i. Let $S = \{i\ ;\ \alpha \leq \gamma_{i_1}\}$ and $T = \{i\ ;\ \alpha \not\leq \gamma_{i_1}\}$. Then, for each $i \in S$, the monomial $\alpha\gamma_{i_1}\gamma_{i_2}\cdots$ is standard with $\alpha\gamma_{i_1}\gamma_{i_2}\cdots \notin (\{\xi\ ;\ \xi \in I\})$. If $i \in T$, then every standard monomial appearing in the standard monomial expression of $\alpha\gamma_{i_1}\gamma_{i_2}\cdots$ is of the form $\eta_{i_1}\eta_{i_2}\cdots$ with $\eta_{i_1} \leq \eta_{i_2} \leq \cdots$ and with $\eta_{i_1} < \alpha$. Since I is a poset ideal of P, it follows that $\alpha\delta$ belongs to $(\{\xi\ ;\ \xi \in I\})$ if and only if $S = \emptyset$. Thus, $(\{\xi\ ;\ \xi \in I\}) : \alpha \subset (\{\zeta \in P\ ;\ \zeta \not\geq \alpha\})$. □

THEOREM 2.4. *Every ASL is extendable sequentially Koszul.*

Proof. Let $P = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a finite poset and suppose that $i < j$ if $\alpha_i < \alpha_j$. Let $A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$ be an ASL on P $(\subset A_1)$ over a field $K = A_0$. We will prove that the minimal system of generators $P = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ has extendable linear quotients of ρ-th level for all $\rho \geq 2$.

Since $\{\alpha_1, \alpha_2, \ldots, \alpha_i\}$ is a poset ideal of P for every $1 \leq i \leq n$, Lemma 2.3 guarantees that

$$(\alpha_1, \alpha_2, \ldots, \alpha_i) : \alpha_{i+1} = (\{\alpha_k\ ;\ \alpha_k \not\geq \alpha_{i+1}\})$$

for every $1 \leq i < n$. Let $\{\alpha_k\ ;\ \alpha_k \not\geq \alpha_{i+1}\} = \{\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_{q_i}}\}$ with $1 \leq j_1 < j_2 < \cdots < j_{q_i} \leq n$. We write $(\alpha_{p_1}, \alpha_{p_2}, \ldots, \alpha_{p_n})$ for the ordered sequence $(\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_{q_i}}, \alpha_{t_1}, \alpha_{t_2}, \ldots, \alpha_{t_{n-q_i}})$ of P with $t_1 < t_2 < \cdots < t_{n-q_i}$. Since $\{\alpha_{p_1}, \alpha_{p_2}, \ldots, \alpha_{p_i}\}$ is a poset ideal of P for every $1 \leq i \leq n$, it follows that $(\alpha_{p_1}, \alpha_{p_2}, \ldots, \alpha_{p_i}) : \alpha_{p_{i+1}} = (\{\alpha_k\ ;\ \alpha_k \not\geq \alpha_{p_{i+1}}\})$. Now, continuing such procedures enables us to see that P has extendable linear quotients of ρ-th level for all $\rho \geq 2$, as desired. □

We next present a class of extendable sequentially Koszul semigroup rings. Let $K[x_1, x_2, \ldots, x_n]$ be the polynomial ring in n variables over a field K and write $R_{n,2}$ for the subring of $K[x_1, x_2, \ldots, x_n]$ generated by all squarefree quadratic monomials. We call $R_{n,2}$ the *second squarefree Veronese subring* of $K[x_1, x_2, \ldots, x_n]$. It is known [13] that $R_{n,2}$ has a squarefree quadratic Gröbner basis and, in particular, is normal and Koszul.

THEOREM 2.5. *The second squarefree Veronese subring $R_{n,2}$ is extendable sequentially Koszul.*

We will prove in fact that $R_{n,2}$ is extendable sequentially Koszul with respect to the lexicographic order $x_1x_2 > x_1x_3 > \cdots > x_1x_n > x_2x_3 > \cdots$ of the generators.

In order to simplify notation we write (ij) for x_ix_j. If all integers i, j, i', j' are different, then we call $(ij), (i'j')$ a *bad pair.*

LEMMA 2.6. *If a sequence $\mathcal{L} = ((i_1j_1), \ldots, (i_mj_m))$ with $(i_1j_1) > \cdots > (i_mj_m)$ satisfies the following condition*:

> $(**)$ *For every bad pair $(ij) > (kl)$ in $\mathcal{L}$, at least one of the elements (ik), (il), (jk), (jl) belongs to $\mathcal{L}$ and is bigger than (kl),*

then $\mathcal{L}$ has linear quotients and $((i_1j_1), \ldots, (i_{q-1}j_{q-1})) : (i_qj_q)$ satisfies $(**)$ for every $2 \leq q \leq m$.

Proof. First we compute all colon ideals of the form $(i'j') : (ij)$ with $(i'j') \neq (ij)$. There are two cases to be considered:

Case 1. The number of elements of the set $\{i', j', i, j\}$ is 3. We may assume that $i' = i$. Then $(ij') : (ij) = ((j'k)\ ; k \neq j', j)$.

Case 2. $(i'j')$, (ij) is a bad pair. Then $(i'j') : (ij) = ((i'j'), (i'k)(j'k)\ ; k \neq i', j', i, j)$.

In Case 1, let $a \in (ij') : (ij)$ be a semigroup element. Then $a(ij) = b(ij')$ for some other semigroup element b. It follows that (kj') is a factor of a for some $k \neq j'$. If $k \neq j$, we are done. Assume $k = j$. Suppose all factors $(i_\ell j_\ell)$, $1 \leq \ell \leq r$, of a contain j. Then j has to appear $(r+1)$-times in b, which is a contradiction. Therefore, $a = (j'j)(i''j'') \cdots$ where $i'' \neq j \neq j''$. Since $i'' \neq j''$, we may assume $i'' \neq j'$. Then $a = (j'i'')(jj'') \cdots$. This completes the proof of Case 1.

In Case 2, let $a \in (i'j') : (ij)$ be a semigroup element which contains no factor $(i'j')$. Then $a = (i'i_1)(j'j_1) \cdots$. If $i_1 \neq j_1$, then $a = (i'j')(i_1j_1) \cdots$

which is a contradiction. Hence $i_1 = j_1$ and $a = (i'i_1)(j'i_1)\cdots$. Suppose $i_1 = i$. Then any factor of a must be (ti) with $t \neq i$, because a contains no factor $(i'j')$. Thus, we get the contradiction $a \notin (i'j') : (ij)$. This completes the proof of Case 2.

Let now $I = ((i_1j_1), \dots, (i_{q-1}j_{q-1}))$, $2 \le q \le m$, and $a \in I : (i_qj_q)$ be a semigroup generator of the colon ideal. Then $a \in (i_pj_p) : (i_qj_q)$ for some $p < q$. If (i_pj_p), (i_qj_q) is not a bad pair, then by Case 1, the element a is of degree 1 (in the algebra).

Now suppose that $(i_pj_p), (i_qj_q)$ is a bad pair. Then by Case 2, if a is not of degree 1, then $a = (i_pk)(j_pk)$. By condition $(**)$, at least one of the two factors (i_pk), (j_pk) of a belongs to $I : (i_qj_q)$. Therefore this colon ideal is linear.

It remains to show that $I : (i_qj_q)$ satisfies condition $(**)$. From the preceding arguments it follows that $I : (i_qj_q)$ is generated by all $((i_pj_p) : (i_qj_q))$ such that $(i_pj_p), (i_qj_q)$ is not a bad pair. Set $i = i_q$, $j = j_q$, assume $i < j$, and take a bad pair a, b in $I : (ij)$. Then there exist $(i_sj_s), (i_tj_t) \in I$, $i_s < j_s$, $i_t < j_t$ such that $(i_sj_s), (ij)$ and $(i_tj_t), (ij)$ are not bad pairs, and $a \in (i_sj_s) : (ij)$ and $b \in (i_tj_t) : (ij)$. Therefore a and b are of the form as described in Case 1. Altogether there are six possibilities to be considered. We treat one of them, the other cases being similar.

So let $i = i_t$ and $j = j_s$; then

$$Q_s = (i_sj) : (ij) = ((i_sk)\ ;\ k \neq i_s, i), \quad Q_t = (ij_t) : (ij) = ((kj_t)\ ;\ k \neq j, j_t).$$

Therefore $a = (i_sk_1)$ for some $k_1 \neq i_s, i$, and $b = (k_2j_t)$ for some $k_2 \neq j, j_t$.

There are four cases to be considered.

(a) $i_s < k_1$ and $k_2 < j_t$. If $(i_sk_1) > (k_2j_t)$, then $i_s < k_2$, and so $i_s < j_t$. Since $i < j_t$, it follows that $(i_sj_t) \in Q_s$ and $(i_sj_t) > (k_2j_t)$.

If $(i_sk_1) < (k_2j_t)$, then $k_2 < i_s$, and so $(k_2i_s) \in Q_s$ and $(k_2i_s) > (i_sk_1)$.

(b) $i_s < k_1$ and $k_2 > j_t$. If $(i_sk_1) > (j_tk_2)$, then $i_s < j_t < k_2$. Since $i < j_t$, it follows that $(i_sj_t) \in Q_s$ and $(i_sj_t) > (j_tk_2)$.

If $(i_sk_1) < (j_tk_2)$, then $j_t < i_s < k_1$, and so $(j_ti_s) \in Q_s$ and $(j_ti_s) > (i_sk_1)$.

(c) $i_s > k_1$ and $k_2 < j_t$. If $(k_1i_s) > (k_2j_t)$, then $k_1 < k_2$, and so $k_1 < j_t$. Since $k_1 < j$, it follows that $(k_1j_t) \in Q_t$ and $(k_1j_t) > (k_2j_t)$.

If $(k_1i_s) < (k_2j_t)$, then $k_2 < k_1$, and so $k_2 < i_s$. Since $k_2 < i$ it follows that $(k_2i_s) \in Q_s$ and $(k_2i_s) > (k_1i_s)$.

(d) $i_s > k_1$ and $k_2 > j_t$. If $(k_1i_s) > (j_tk_2)$, then $k_1 < j_t$, and since $k_1 < i_s < i < j$, it follows that $(k_1j_t) \in Q_t$ and $(k_1j_t) > (j_tk_2)$.

If $(k_1 i_s) < (j_t k_2)$, then $j_t < k_1$, and so $j_1 < i_s < i$. Hence $(j_t i_s) \in Q_s$ and $(j_t i_s) > (k_1 i_s)$.

Thus we see that in all four cases condition $(**)$ is satisfied. This completes the proof of the lemma. □

Proof of Theorem 2.5. Fix a semigroup element $(i_1 j_1)$ and consider the ideal $I_1 = ((kl)\ ;\ (kl) > (i_1 j_1))$. In what follows, we call ideals of this form generated by an *initial sequence*. By Lemma 2.6 the first quotient $J_1 = I_1 : (i_1 j_1)$ is linear and J_1 has linear quotients with respect to the lexicographic order of the generators. Let $(i_1 j_1) > (i_2 j_2) > \cdots > (i_\mu j_\mu) > \cdots$ be all the generators which are not in J_1 and set $I_2 = (J_1, (i_1 j_1), \ldots, (i_{\mu-1} j_{\mu-1}))$ with $\mu \geq 1$. We have to show that $J_2 = I_2 : (i_\mu j_\mu)$ is linear and has linear quotients. Then we consider $I_3 = (J_2, (p_1 q_1), \ldots, (p_{\nu-1} q_{\nu-1}))$, where $(p_1 q_1) > \cdots > (p_\nu q_\nu) > \cdots$ are all the generators which are not in J_2 and we have again to show that $I_3 : (p_\nu q_\nu)$ is linear and has linear quotients. We continue this procedure which, of course, is finite.

Thus, assume that $I = I_\ell$ for some $\ell \geq 1$ is linear and let (ij) be the biggest element which is not contained in I. Assume that $J = I : (ij)$ is linear and that J has linear quotients (with respect to the lexicographic order of the generators). Let $(i_1 j_1) > (i_2 j_2) > \cdots$ be all elements which are not in J, and set $L = (J, (i_1 j_1), \ldots, (i_{\mu-1} j_{\mu-1}))$ and $(pq) = (i_\mu j_\mu)$ with $\mu \geq 1$. Note first that by definition of L one has:

(a) if $(tk) > (pq)$, then $(tk) \in L$;

(b) if $(tk) < (pq)$, and $(tk) \in L$, then $(tk) \in J$.

We will show:

(1) $L : (pq)$ is linear;

(2) $L : (pq)$ has linear quotients (with respect to the lexicographic order of the generators).

In what follows we refer to Case 1 and Case 2 from the proof of Lemma 2.6, and to condition $(**)$ of the same lemma.

(1) Let $a \in L : (pq)$ be a semigroup generator of the colon ideal. Then $a \in (rs) : (pq)$ for some $(rs) \in L$. If $(rs), (pq)$ is not a bad pair, then by Case 1 the element a is of degree 1 (in the algebra).

Now suppose that $(rs), (pq)$ is a bad pair. Then by Case 2, if a is not of degree 1, then $a = (rk)(sk)$ for some $k \neq r, s, p, q$. If $(rs) > (pq)$, then $t = \min\{r, s\} < p$ and $t < q$, therefore $(tk) > (pq)$ which implies that $(tk) \in L$. Hence a factor of a is in $L : (pq)$.

It remains the case $(rs) < (pq)$. According to (b) we have $(rs) \in J$. Since by assumption J is linear, from the arguments above it follows that J

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is generated by all $(uv):(ij)$ such that $(uv) \in I$ and $(uv),(ij)$ is not a bad pair. Thus, there exists $(ml) \in I$ such that $(rs) \in (ml):(ij)$ and $(ml),(ij)$ is not a bad pair. By Case 1 we may assume that $(ml) = (ir)$. Then if $k \neq j$, we have $(rk)(ij) = (ir)(kj) \in I$, therefore $(rk) \in J$. If $k = j$ then $(rq)(ij) = (ir)(qj) \in I$, so that $(rq) \in J \subset L$. Since $(rj)(pq) = (pj)(rq)$, one obtains that $(rj) \in L:(pq)$. This completes the proof of (1).

(2) We show first the following claim:

Let $(rs) \in L:(pq)$ and assume that $(rs) \notin L$. Then we have the following:

(c) All the elements of the set $\{r,s,p,q\}$ are different.

(d) If $r < s$, then $(pr) \in L$ or $(qr) \in L$.

Since $L:(pq)$ is linear, as before it follows that $(rs) \in (ml):(pq)$ where $(ml) \in L$ and $(ml),(pq)$ is not a bad pair. By Case 1 we may assume that $(ml) = (pr)$. Then $s \neq q, r$ and $r \neq p, q, s$, and since $(pr) \in L$ and by assumption $(rs) \notin L$ one has $p \neq s$. This shows (c).

Let now $r < s$. By (c) we have $r \neq p, q$. Since $L:(pq)$ is linear, as before it follows that at least one of the elements (pr), (qr), (ps), (qs) belongs to L. If L is generated by an initial sequence, then it is clear that the claim is true. So, we can assume that (c) is true for all quotients of the form $L':(p'q') \subset L$, $L' \neq L$, which are obtained by the same construction as L.

Suppose now that $(pr) \notin L$ and $(qr) \notin L$. Then (ps) or (qs) belongs to L. We may assume that $(ps) \in L$. By (a) one has $(pr) < (pq)$, so that $q < r < s$. Therefore $(ps) < (pq)$. By (b) one obtains that $(ps) \in J$. Since $(pq) \notin L$ and $(pq) > (ps) \in J \subset L$, (ps) does not belong to any ideal contained in L which is generated by an initial sequence. Therefore there exists a quotient $L':(p'q') \subset L$ such that $(ps) \in L':(p'q')$ and $(ps) \notin L'$ for some $(p'q') > (pq)$. Since $(ps) \in J \subset L$ and $(ps) \notin L'$ one has $L' \neq L$, so that by assumption $(p'p) \in L'$ or $(q'p) \in L'$. First assume $(p'p) \in L'$. Then if $q \neq q'$, one has $(pq)(p'q') = (p'p)(q'q) \in L'$, therefore we obtain the contradiction $(pq) \in L':(p'q') \subset L$. Hence $q = q'$. But then $r \neq q'$, so that $(pr)(p'q') = (p'p)(rq') \in L'$. Hence $(pr) \in L':(p'q')$ contradicting our assumption. Suppose now that $(q'p) \in L'$. Then if $q \neq p'$, one has $(pq)(p'q') = (q'p)(qp')$, and if $q = p'$, then $(pr)(p'q') = (q'p)(p'r)$. In each case we get a contradiction which shows (d).

We will show that $L:(pq)$ satisfies condition $(**)$. This will imply that $L:(pq)$ has linear quotients, see Lemma 2.6.

If L is an ideal generated by an initial sequence, then by Lemma 2.6, $L:(pq)$ satisfies condition $(**)$, so we may assume that J satisfies $(**)$.

Let $a = (st) > b = (ml)$ be a bad pair in $L : (pq)$. Since $L : (pq)$ is linear, as we noted already, we have $a \in (uv) : (pq)$ and $b \in (u'v') : (pq)$ with $(uv), (u'v') \in L$ such that $(uv), (pq)$ and $(u'v'), (pq)$ are not bad pairs. We may assume that $(uv) = (qs)$, so that $t \neq p$. There are two cases: $(u'v') = (pl)$ and $(u'v') = (ql)$. We will consider the first one, the other one being treated similarly. So, assume $b \in (pl) : (pq)$, $(pl) \in L$. Since $(sl)(pq) = (pl)(qs) \in L$, one obtains that $(sl) \in L : (pq)$, therefore if $(sl) > b$ we are done. Hence we may assume $(sl) < b = (ml)$, i.e., $m < s$. Then since $a > b$, one obtains $t < s$, $t < m$ and $t < l$. Therefore $(tl) > b$. If $t \neq q$, then $(tl)(pq) = (pl)(tq) \in L$, so that (tl) satisfies the desired condition. So, we may assume that $t = q$. Then $a \in L$. If $m = p$ then $b \in L$ too. Since $(qp) > a > b$, by (b), a, b is a bad pair in J and by assumption J satisfies $(**)$. Therefore (ql) or (ps) belongs to $J \subset L$. Hence we may assume $m \neq p$. Then $(ms)(pq) = (qs)(mp) \in L$, so if $(ms) > b$ we are done.

Thus it remains the case $a = (qs) \in L$, $q < l < s$, $q < m < s$, and $p \neq q, s, m, l$. We have to show that $(ql) \in L : (pq)$ or $(qm) \in L : (pq)$. If $(ql) > (pq)$ or $(qm) > (pq)$ then by (a), $(ql) \in L$ or $(qm) \in L$. Therefore we can assume $(ql) < (pq)$ and $(qm) < (pq)$, i.e., $p < l$ and $p < m$. Then $a < (pq)$, so that by (b), $a \in J$. Since a is not contained in any ideal generated by an initial sequence and included in L, there is a quotient $I' : (i'j') \subset L$ such that $a \in I' : (i'j')$ and $a \notin I'$. According to (d) one has $(i'q) \in I'$ or $(j'q) \in I'$. We may assume that $(i'q) \in I'$. Since $m \neq l$, one of m, l is different from j', say $l \neq j'$. Then $(ql)(i'j') = (i'q)(lj') \in I'$, therefore $(ql) \in I' : (i'j') \subset L$. □

§3. Shellability of divisor posets

The purpose of the present section is to show that the divisor poset of a homogeneous semigroup ring is Λ-shellable if and only if the semigroup ring is extendable sequentially Koszul.

Given a homogeneous semigroup ring A over a field K, we write Σ_A for the infinite poset consisting of all monomials belonging to A, ordered by divisibility. Thus, for monomials u and v of A, we have $u \leq v$ in Σ_A if and only if $v = uw$ for some monomial w of A. Then Σ_A is locally finite and pure, and possesses a unique minimal element 1 $(\in K)$. The infinite poset Σ_A is called the *divisor poset* of A.

We now come to the main theorem of the present paper.

THEOREM 3.1. *Let A be a homogeneous semigroup ring and Σ_A its divisor poset. Then A is extendable sequentially Koszul if and only if Σ_A*

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is Λ*-shellable.*

Proof. ("only if") Let $G = \{s_1, \ldots, s_n\}$ denote the minimal system of monomial generators of A. First, since A is extendable sequentially Koszul, we can find an ordered sequence $\mathbf{s}_0$ of G which has extendable linear quotients of *any* level. Second, for each $1 \leq p \leq n$, let $\mathbf{s}_1(p)$ denote an ordered sequence of G, which has extendable linear quotients of any level, arising from the colon ideal $I(\mathbf{s}_0; p)$. Note that $\mathbf{s}_1(1) = \mathbf{s}_0$. Next, for each $1 \leq p \leq n$ and $1 \leq p' \leq n$, let $\mathbf{s}_2(p, p')$ denote an ordered sequence of G, which has extendable linear quotients of any level, arising from the colon ideal $I(\mathbf{s}_1(p); p')$. Note that $\mathbf{s}_2(p, 1) = \mathbf{s}_1(p)$. Continue these procedures, and we obtain an ordered sequence $\mathbf{s}_q(p_1, p_2, \ldots, p_q)$ of G, which has extendable linear quotients of any level, for all $q \geq 0$ and for all $(p_1, p_2, \ldots, p_q)$ with each $1 \leq p_j \leq n$. The ordered sequences $\mathbf{s}_q(p_1, p_2, \ldots, p_q)$ of G occurring in the step q are called the q-th *derived sequences* of G.

Now, for each saturated chain $1 = u_0 \to u_1 \to \cdots \to u_q$ of Σ_A of length q starting from 1, we associated a q-th derived sequence of G as follows. First, for the unique saturated chain of length 0 starting from 1, we associate the 0-th derived sequence $\mathbf{s}_0$. Second, noting that each element u of Σ_A has n covering elements $uG = \{us_1, \ldots, us_n\}$, if $\mathbf{s}_{q-1}(p_1, p_2, \ldots, p_{q-1}) = (s_{i_1}, \ldots, s_{i_n})$ is the $(q-1)$-th derived sequence associated with $1 = u_0 \to u_1 \to \cdots \to u_{q-1}$ and if $u_q = u_{q-1}s_{i_p}$, then the q-th derived sequence associated with $1 = u_0 \to u_1 \to \cdots \to u_q$ will now be the ordered sequence $\mathbf{s}_q(p_1, p_2, \ldots, p_{q-1}, p)$ of G arising from the colon ideal $(s_{i_1}, \ldots, s_{i_{p-1}}) : s_{i_p}$.

If $C : 1 = u_0 \to u_1 \to \cdots \to u_q$ is a saturated chain of Σ_A of length q starting from 1 and if $\mathbf{s}_q(p_1, p_2, \ldots, p_q)$ is the q-th derived sequence associated with C, then we define $\lambda(C) \in \mathbb{Z}^q$ to be $\lambda(C) = (p_1, p_2, \ldots, p_q)$. Then λ is a chain-edge labeling of Σ_A. We claim that the map λ is a Λ-labeling of Σ_A. The condition (Λ-1) is obviously satisfied. To see why λ satisfies the condition (Λ-2), fix a saturated chain $C : 1 = \alpha_0 \to \alpha_1 \to \cdots \to \alpha_{q-1} \to \alpha_q$ of Σ_A with $\lambda(C) = (p_1, p_2, \ldots, p_q)$, and let $\alpha_q \to \beta$ and $\alpha_q \to \gamma$ with $\beta \neq \gamma$ such that β covers an element $\alpha' \in \Sigma_A$ with $\alpha_{q-1} \to \alpha'$ satisfying $\lambda((C \setminus \{\alpha_q\}) \cup \{\alpha'\}; \alpha_{q-1} \to \alpha') < \lambda(C; \alpha_{q-1} \to \alpha_q)$ and that γ covers no element $\alpha' \in \Sigma_A$ with $\alpha_{q-1} \to \alpha'$ satisfying $\lambda((C \setminus \{\alpha_q\}) \cup \{\alpha'\}; \alpha_{q-1} \to \alpha') < \lambda(C; \alpha_{q-1} \to \alpha_q)$. Let $(s_{i_1}, \ldots, s_{i_n})$ denote the $(q-1)$-th derived sequence associated with $C \setminus \{\alpha_q\}$ and $\alpha_q = \alpha_{q-1}s_{i_b}$. Let $\alpha' = \alpha_{q-1}s_{i_a}$ with $a < b$ and suppose that β covers α'. Then $\beta/\alpha_q \in (s_{i_a}) : s_{i_b} \subset (s_{i_1}, \ldots, s_{i_{b-1}}) : s_{i_b}$, while $\gamma/\alpha_q \notin (s_{i_1}, \ldots, s_{i_{b-1}}) : s_{i_b}$. Hence, if $(s_{i_1}, \ldots, s_{i_{b-1}}) : s_{i_b} = (s_{j_1}, \ldots, s_{j_c})$ with $j_1 < \cdots < j_c$, then

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$\lambda(C \cup \{\beta\}; \alpha_q \to \beta) \le c < \lambda(C \cup \{\gamma\}; \alpha_q \to \gamma)$, as desired.

We now prove that the Λ-labeling λ is a Λ-shelling of Σ_A. Let $u \in \Sigma_A$ and $M(u)$ the set of maximal chains of $[1, u]$.

Let δ be the cardinality of any maximal chain of $[1, u]$. Let $C, C' \in M(u)$ with $\lambda(C') <_{lex} \lambda(C)$ and $\sharp(C' \cap C) < \delta - 1$. We must show that there exists $C'' \in M(u)$ with $\lambda(C'') <_{lex} \lambda(C)$ such that $C' \cap C \subset C'' \cap C$ with $C' \cap C \ne C'' \cap C$. Let $C : 1 = u_0 \to u_1 \to \cdots \to u_\delta = u$ and $C' : 1 = v_0 \to v_1 \to \cdots \to v_\delta = u$. Let q denote the smallest integer for which $u_{q+1} \ne v_{q+1}$ and let p denote the smallest integer with $q < p$ for which $u_p = v_p$. If $p = q + 2$, then

$$C'' : 1 \to u_1 \to \cdots \to u_q \to v_{q+1} \to u_{q+2} \to \cdots \to u_\delta$$

is a desired maximal chain of $[1, u]$. Next, suppose that $p > q + 2$. Let $\mathbf{s} = (s_{i_1}, \ldots, s_{i_n})$ denote the q-th derived sequence associated with the saturated chain $1 \to u_1 \to \cdots \to u_q$ of length q. Let $v_{q+1} = v_q s_{i_j}$ and $u_{q+1} = u_q s_{i_k}$. Then $j < k$ since $\lambda(C') <_{lex} \lambda(C)$. Hence, $w = u_p/u_{q+1}$ belongs to $(s_{i_1}, \ldots, s_{i_{k-1}}) : s_{i_k}$. Let $\mathbf{s}' = (s_{j_1}, \ldots, s_{j_n})$ denote the $(q+1)$-th derived sequence associated with the saturated chain $1 \to u_1 \to \cdots \to u_q \to u_{q+1}$ of length $q + 1$ and suppose that the colon ideal $(s_{i_1}, \ldots, s_{i_{k-1}}) : s_{i_k}$ is generated by $\{s_{j_1}, \ldots, s_{j_t}\}$. Then $w = w' s_{j_r}$ for some $1 \le r \le t$ and for some $w' \in \Sigma_A$. Let $s_{j_\ell} = u_{q+2}/u_{q+1}$.

We distinguish the two cases; one is the case with $s_{j_\ell} \in (s_{i_1}, \ldots, s_{i_{k-1}}) : s_{i_k}$ and the other is the case with $s_{j_\ell} \notin (s_{i_1}, \ldots, s_{i_{k-1}}) : s_{i_k}$. If s_{j_ℓ} belongs to the colon ideal $(s_{i_1}, \ldots, s_{i_{k-1}}) : s_{i_k}$, then $s_{j_\ell} s_{i_k} = s_{i_m} s_{n'}$ for some m and n' with $1 \le m < k$. Let $w_{q+1} = u_q s_{i_m}$. Thus $u_{q+2} = w_{q+1} s_{n'}$. Then

$$C'' : 1 \to u_1 \to \cdots \to u_q \to w_{q+1} \to u_{q+2} \to \cdots \to u_\delta$$

is a desired maximal chain of $[1, u]$. If $s_{j_\ell} \notin (s_{i_1}, \ldots, s_{i_{k-1}}) : s_{i_k}$, then $r \le t < \ell$. Let $w_{q+2} = u_{q+1} s_{j_r}$. Then $w_{q+2} < u_p$ in Σ_A since $u_p = u_{q+1} w = u_{q+1} w' s_{j_r} = w' w_{q+2}$. We choose any maximal chain $w_{q+2} \to w_{q+3} \to \cdots \to u_p$ of $[w_{q+2}, u_p]$. Then

$$\begin{aligned} C'' : 1 \to u_1 \to \cdots \to u_q \to u_{q+1} \to w_{q+2} \to w_{q+3} \to \cdots \\ \cdots \to u_p \to u_{p+1} \to \cdots \to u_\delta \end{aligned}$$

is a required maximal chain of $[1, u]$.

("if") Let $\alpha_1, \ldots, \alpha_n$ denote the monomial generators of A and λ a Λ-shelling of Σ_A. First, supposing that $\lambda(1 \to \alpha_1) < \cdots < \lambda(1 \to \alpha_n)$, we will prove that the sequence $\alpha = (\alpha_1, \ldots, \alpha_n)$ has linear quotients. Write I_i for the colon ideal $(\alpha_1, \ldots, \alpha_{i-1}) : \alpha_i$ for each $1 < i \leq n$. Let w be a monomial of A belonging to I_i. Then $w\alpha_i = w'\alpha_j$ for some $1 \leq j < i$ and for some monomial w' of A. Let $u = w\alpha_i$. Then $\alpha_i < u$ and $\alpha_j < u$ in Σ_A. Let $C : 1 \to \alpha_i \to \beta \to \cdots \to u$ be the maximal chain of $[1, u]$ such that $(***)$ for all maximal chain C' $(\neq C)$ of $[1, u]$ with $\alpha_i \in C'$, we have $\lambda(C) <_{lex} \lambda(C')$. Since $\lambda(1 \to \alpha_j) < \lambda(1 \to \alpha_i)$, it follows that the maximal chain C cannot be the first maximal chain of $[1, u]$ with respect to $<_{lex}$. Since λ is a Λ-shelling, we can find a maximal chain C'' of $[1, u]$ with $\lambda(C'') <_{lex} \lambda(C)$ and with $\sharp(C'' \cap C) = \delta - 1$, where δ is the cardinality of any maximal chain of $[1, u]$. Let $\alpha_k \in C''$. Then, by $(***)$ and by $\lambda(C'') <_{lex} \lambda(C)$, we have $k \neq i$. Hence $k < i$. Moreover, since $\sharp(C'' \cap C) = \delta - 1$, we have $1 \to \alpha_k \to \beta$. Let $\beta = \alpha_i\alpha_\ell$ with $1 \leq \ell \leq n$. Since $\beta \in (\alpha_k)$ with $k < i$, we have $\alpha_\ell \in I_i$. Moreover, $w \in (\alpha_\ell)$. Hence I_i is generated by a subset of $\{\alpha_1, \ldots, \alpha_n\}$, as desired.

Second, for each $1 < i \leq n$, write $I_i = (\alpha_{j_1}, \ldots, \alpha_{j_{p(i)}})$ with

$$\lambda(C^i_{j_1}; \alpha_i \to \alpha_i\alpha_{j_1}) < \cdots < \lambda(C^i_{j_{p(i)}}; \alpha_i \to \alpha_i\alpha_{j_{p(i)}}),$$

where C^i_ℓ is the saturated chain $1 \to \alpha_i \to \alpha_i\alpha_\ell$ of Σ_A. Let $\{\alpha_{j'_1}, \ldots, \alpha_{j'_{n-p(i)}}\}$ denote the set of all α_j's with $\alpha_j \notin I_i$ and suppose that

$$\lambda(C^i_{j'_1}; \alpha_i \to \alpha_i\alpha_{j'_1}) < \cdots < \lambda(C^i_{j'_{n-p(i)}}; \alpha_i \to \alpha_i\alpha_{j'_{n-p(i)}}).$$

By virtue of the technique appearing in the first paragraph, in order to show that the ordered sequence

$$(\alpha_{j_1}, \ldots, \alpha_{j_{p(i)}}, \alpha_{j'_1}, \ldots, \alpha_{j'_{n-p(i)}})$$

of $\{\alpha_1, \ldots, \alpha_n\}$ has linear quotients, noting that λ is also a Λ-shelling of $[\alpha_i, \infty)$ in the obvious way, it is enough to show that $\lambda(C^i_{j_q}; \alpha_i \to \alpha_i\alpha_{j_q}) < \lambda(C^i_k; \alpha_i \to \alpha_i\alpha_k)$ for all $1 \leq q \leq p(i)$ and for all $k \in \{j'_1, \ldots, j'_{n-p(i)}\}$. Since α_{j_q} belongs to I_i, there is $j < i$ with $1 \to \alpha_j \to \alpha_i\alpha_{j_q}$. Moreover, since $\alpha_k \notin I_i$, there is no $j < i$ with $1 \to \alpha_j \to \alpha_i\alpha_k$. Hence, by ($\Lambda$-2) we have $\lambda(C^i_{j_q}; \alpha_i \to \alpha_i\alpha_{j_q}) < \lambda(C^i_k; \alpha_i \to \alpha_i\alpha_k)$, as required.

Now, repeated application of the discussion in the second paragraph completes the proof of the "if" part of the theorem. □

Let A be a homogeneous semigroup ring and P the minimal system of monomial generators of A. Then A is called a *monomial ASL* if A is an ASL

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with respect to a partial order on P over K. See [10]. Since a monomial ASL is an extendable sequentially Koszul semigroup ring (Theorem 2.4), we immediately obtain

Corollary 3.2. *The divisor poset of a monomial ASL is Λ-shellable.*

Let $A = A_0 \oplus A_1 \oplus \cdots$ be a homogeneous K-algebra. Recall that the d-th *Veronese subring* of A is the subalgebra $A^{(d)} = A_0 \oplus A_d \oplus A_{2d} \oplus \cdots$ of A. If A is a homogeneous semigroup ring with the divisor poset Σ_A, then the divisor poset of its d-th Veronese subring $A^{(d)}$ is the rank-selected subposet of Σ_A of order d. Hence it follows from Theorems 1.1 and 3.1 that

Corollary 3.3. *All Veronese subrings of an extendable sequentially Koszul semigroup ring are extendable sequentially Koszul.*

It is known [5, Proposition 1.4] that a homogeneous semigroup ring is strongly Koszul if and only if its divisor poset is wonderful (locally semimodular). Since a strongly Koszul semigroup ring is extendable sequentially Koszul, we have

Corollary 3.4. *A wonderful divisor poset is Λ-shellable.*

We conclude this paper with a few examples and questions.

Example 3.5. (a) The homogeneous semigroup ring [9, Example 2.2] is extendable sequentially Koszul, but not strongly Koszul. This semigroup ring is generated by the squarefree cubic monomials

$$x_1x_2x_3,\ x_1x_3x_4,\ x_1x_4x_5,\ x_1x_2x_5,\ x_2x_3x_6,\ x_4x_5x_6,\ x_3x_4x_7,\ x_2x_5x_7.$$

Hence the divisor poset of this semigroup ring is Λ-shellable, but not wonderful. Moreover, this ring possesses *no* quadratic Gröbner basis. Thus we can obtain an example of a homogeneous semigroup ring having *no* quadratic Gröbner basis whose divisor poset is Λ-shellable.

(b) Since the second squarefree Veronese subring $R_{n,2}$ is extendable sequentially Koszul for all n, its divisor poset $\Sigma_{R_{n,2}}$ is Λ-shellable. It is shown [8, Theorem 2.3], however, that $R_{n,2}$ is quasi-poset if and only if $n \leq 4$. Hence the shellability of $\Sigma_{R_{n,2}}$ with $n \geq 5$ does *not* follow from [11, Corollary 3.6].

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QUESTION 3.6. *Is the divisor poset of a sequentially Koszul semigroup ring shellable?*

QUESTION 3.7. *Is there a Koszul semigroup ring whose divisor poset is nonshellable?*

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