A THEOREM ON PARTIALLY ORDERED SETS, WITH APPLICATIONS TO FIXED POINT THEOREMS

SMBAT ABIAN AND ARTHUR B. BROWN

In this paper the authors prove Theorem 1 on maps of partially ordered sets into themselves, and derive some fixed point theorems as corollaries.

Here, for any partially ordered set P, and any mapping $f\colon P \to P$ and any point $a \in P$, a well ordered subset $W(a) \subset P$ is constructed. Except when $W(a)$ has a last element ξ greater than or not comparable to $f(\xi)$, $W(a)$, although constructed differently, is identical with the set A of Bourbaki **(3)** determined by a, f, and P_1: $\{x|x \in P, x \leqslant f(x)\}$.

Theorem 1 and the fixed point Theorems 2 and 4, as well as Corollaries 2 and 4, are believed to be new.

Corollaries 1 and 3 are respectively the well-known theorems given in **(1**, p. 54, Theorem 8, and Example 4**)**.

The fixed point Theorem 3 is that of **(1**, p. 44, Example 4**)**; and has as a corollary the theorem given in **(2)** and **(3)**.

The proofs are based entirely on the definitions of partially and well ordered sets and, except in the cases of Theorem 4 and Corollary 4, make no use of any form of the axiom of choice.

In what follows, "$a < b$" implies that a and b are distinct. Furthermore, we shall always deal with elements and subsets of a given partially ordered set P, and "lub T" will denote exclusively "the least upper bound of T in P"; that is, an upper bound z of T such that if s is any other upper bound of T, then $z < s$. The symbol "$\subset$" shall mean "is a subset (not necessarily proper) of."

Definition. Let P be a partially ordered set and f a mapping of P into P. For any $a \in P$, an *a-chain* C_r is a subset of P satisfying the following conditions:

(1) C_r is well ordered, with a as its first element and r as its last element;

(2) If $z \in C_r$ and $z \neq r$, then $f(z) \in C_r$, $z < f(z)$, and there exists no $y \in C_r$ for which $z < y < f(z)$;

(3) If T is a non-empty subset of C_r, then the least upper bound (in P) of T exists and is in C_r.

It will follow from Lemma 4 below that, for given P, f, and a, C_r is uniquely determined by r.

We designate by $W(a)$ the set of all $r \in P$ for which there exists an a-chain C_r having r as its last element. We note that (2) implies that $W(a) = \{a\}$ except when $a < f(a)$.

Received September 1, 1959.

https://doi.org/10.4153/CJM-1961-007-5 Published online by Cambridge University Press

Under the hypotheses of the Definition, we shall first prove the following lemmas.

LEMMA 1. *If* $r \in W(a)$ *and* C_r *is an a-chain with last element* r, *then* $C_r \subset W(a)$.

Proof. If $t \in C_r$, the set of all elements of C_r which are $\leqslant t$ is easily seen to be an a-chain, and hence $t \in W(a)$. Therefore the lemma is true.

LEMMA 2. *If* $r \in W(a)$ *and* $r \leqslant f(r)$, *then* $f(r) \in W(a)$.

Proof. The set $C_r \cup \{f(r)\}$ is obviously an a-chain, and hence $f(r) \in W(a)$.

LEMMA 3. *If* $r, s \in W(a)$ *and* C_r *is an a-chain with last element* r, *then either* $s \in C_r$ *or* $r < s$.

Proof. Let $T = C_r \cap C_s$. By (1), $T \neq \emptyset$ and hence, by (3), $z = \text{lub } T$ exists and $z \in T$. If $s \notin C_r$ then $z \neq s$. If also $z \neq r$ then, by (2), $z < f(z) \in T$, contrary to the fact that $z = \text{lub } T$. Hence $s \notin C_r$ implies that $z = r$, so that $r \in C_s$; and since $r \neq s$, we see by (1) that $r < s$. Since, by (1), $s \in C_r$ and $r < s$ cannot both hold, we infer the truth of Lemma 3.

LEMMA 4. *If* $r \in W(a)$, *there is just one* C_r *with last element* r, *namely the set of all elements of* $W(a)$ *which are* $\leqslant r$.

Proof. This follows from Lemmas 1 and 3.

THEOREM 1. *Let* P *be a non-empty partially ordered set*, f *a map of* P *into* P, *and* a *an arbitrary element of* P. *Then*

(4) $W(a)$ *is well ordered with* a *its first element.*

Moreover, if $\xi = \text{lub } W(a)$ *exists, then*

(5) $W(a)$ *is an a-chain with* ξ *its last element,*

and

$$\xi \nless f(\xi). \tag{6}$$

Proof. Let H be any non-empty subset of $W(a)$, and $r \in H$. Since $r \in H \cap C_r$, we see by (1) that $H \cap C_r$ has a first element, which, in view of Lemma 4, is the first element of H. Hence $W(a)$ is well ordered. By Lemma 1, $a \in W(a)$, and by (1), if $r \in W(a)$ then $a \leqslant r$. Thus we conclude that a is the first element of $W(a)$. Hence (4) is valid.

Next, assume $\xi = \text{lub } W(a)$ exists and let $W^* = W(a) \cup \{\xi\}$. We shall show that W^* is an a-chain. Since $W(a)$ is well ordered, W^* is well ordered too and thus (1) is satisfied for W^*, with a its first and ξ its last element. Now, let $z \in W^*$ and $z \neq \xi$. Then $z \in W(a)$ and $\{x | x \in W(a), z < x\} \neq \emptyset$. Sinec $W(a)$ is well ordered, z has an immediate successor r in $W(a)$, hence in W^*. By Lemma 4, z and r are the last two elements of C_r. Hence, by (2) applied to z as an element of C_r, we see that $f(z) = r$, so that (2) is satisfied for W^*. To prove (3) for W^*, let T be any non-empty subset of W^*. Obviously ξ is an upper bound of T. If there is no element of $W(a)$ which exceeds every element of T then, in view of the well orderedness of $W(a)$, any upper bound of T is also an upper bound of $W(a)$ and hence is $\geqslant \xi$, which implies that $\xi = \text{lub } T$, and thus lub $T \in W^*$. If there is an element $r \in W(a)$ which exceeds every element of T, then $T \subset W(a)$ and, by Lemma 4, $T \subset C_r$. Hence, by (3), lub T exists and is in C_r, and therefore, by Lemma 1, lub $T \in W(a)$, so that again lub $T \in W^*$. Consequently (3) is satisfied for W^*. Therefore W^* is an a-chain with ξ its last element, which implies that $\xi \in W(a)$ and $W(a) = W^*$. Thus (5) is valid.

Now, suppose $\xi < f(\xi)$. By Lemma 2, $f(\xi) \in W(a)$, so that (5) is contradicted. Therefore $\xi \not< f(\xi)$. Thus (6) is valid, and Theorem 1 is proved.

THEOREM 2. *Let P be a partially ordered set in which*

(7) lub *of every non-empty well ordered subset $W \subset P$ exists.*

Let f be a map of P into P such that f is isotone, that is,

(8) *for every two elements $x, y \in P$ with $x \leqslant y$, we have $f(x) \leqslant f(y)$;*

and

(9) *there exists an element $a \in P$ with $a \leqslant f(a)$.*

Then there exists at least one $\xi \in P$ such that $\xi = f(\xi)$. In fact, $\xi =$ lub $W(a)$ *is such an element.*

Proof. If $a = f(a)$, the conclusion is obvious. Now suppose $a < f(a)$.

Consider the set $W(a)$, where a is the element referred to in (9). By (4) and (7), $\xi = \text{lub } W(a)$ exists, and hence by (5), $W(a) = C_\xi$. By (9) and Lemma 2 we see that $f(a) \in W(a)$, and therefore $a < \xi$. Since $W(a)$ is an a-chain and $W(a) - \{\xi\}$ is non-empty, we infer from (3) that $\theta = \text{lub}[W(a) - \{\xi\}]$ is in $W(a) = C_\xi$. According as $\theta = \xi$ or $\theta < \xi$, we have

(10) $$\xi = \text{lub } [W(a) - \{\xi\}]$$

or

(11) ξ is the immediate successor of θ in $W(a)$.

If (10) holds, take any element $z \in [W(a) - \{\xi\}]$. Then $z < \xi$, and by (8), $f(z) \leqslant f(\xi)$. By (2), $z < f(z)$. Consequently $z < f(\xi)$ and therefore $f(\xi)$ is an upper bound for $[W(a) - \{\xi\}]$, and thus, by (10), $\xi \leqslant f(\xi)$.

If (11) holds, by (2), $f(\theta) = \xi$. Also, since $\theta < \xi$, by (8), $f(\theta) \leqslant f(\xi)$, so that again $\xi \leqslant f(\xi)$.

Since $\xi \leqslant f(\xi)$, we see from (6) that $\xi = f(\xi)$. Thus Theorem 2 is proved.

Remark. An alternative proof of Theorem 2 can be given by considering the set $\{x | x \in P, x \leqslant f(x)\}$ and using Theorem 1.

COROLLARY 1. *Let f be any isotope map of a non-empty complete lattice L into itself. Then $\xi = f(\xi)$ for some $\xi \in L$.*

Proof. In view of Theorem 2, we need only verify (9). Choose $a =$ the greatest

https://doi.org/10.4153/CJM-1961-007-5 Published online by Cambridge University Press

lower bound of L. Then clearly (9) is valid, and Corollary 1 follows from Theorem 2.

COROLLARY 2. *Let P be a partially ordered set in which*

(12) *every non-empty well ordered subset $W \subset P$ which is bounded above has a* lub.

Let f be an isotone map of P into P and let there exist two elements $a, b \in P$ such that

(13) $$a \leqslant f(a) \leqslant f(b) \leqslant b.$$

Then there exists $\xi \in P$ such that $\xi = f(\xi)$ and $a \leqslant \xi \leqslant b$. In fact, $\xi = \text{lub}\, W(a)$ is such an element.

Proof. Let $Q = \{x | x \in P, a \leqslant x \leqslant b\}$. Since f is isotone, we see by (13) that if $x \in Q$, then $a \leqslant f(a) \leqslant f(x) \leqslant f(b) \leqslant b$. Hence f maps Q into Q. Moreover, since Q is bounded above by b, we see from (12) that (7) is valid for Q. Therefore the hypotheses of Theorem 2 are satisfied by Q, f, and a. Thus from Theorem 2 we infer the validity of Corollary 2.

COROLLARY 3. *If f is an isotone map of a conditionally complete lattice into itself and if $a \leqslant f(a) \leqslant f(b) \leqslant b$, then $\xi = f(\xi)$ for some ξ with $a \leqslant \xi \leqslant b$.*

THEOREM 3. *Let P be a non-empty partially ordered set in which*

(14) lub *of every non-empty well ordered subset $W \subset P$ exists.*

Let f be a map of P into P such that

(15) $$\textit{for every } x \in P, \qquad x \leqslant f(x).$$

Then there exists at least one $\xi \in P$ such that $\xi = f(\xi)$. In fact, for every $a \in P$, $\xi =$ lub $W(a)$ is such an element.

Proof. Consider an a-chain $W(a) \subset P$. By (4) and (14), $\xi =$ lub $W(a)$ exists. By (15) and (6), $\xi = f(\xi)$. Thus Theorem 3 is proved.

In the following a generalization of Corollary 2 is proved with the help of the axiom of choice.

THEOREM 4. *Let P be a partially ordered set in which*

(16) lub *of every non-empty well ordered subset which is bounded above exists.*

Let g be a map of P into P such that, for every two elements $x, y \in P$,

(17) $$\textit{if } g(x) < g(y), \textit{ then } x < y;$$

and, for $x, y, s \in P$,

(18) $$\textit{if } g(x) \leqslant s \leqslant g(y), \textit{ then } g^{-1}(s) \neq \emptyset.$$

Furthermore, let f be an isotone map of P into P, and let there exist $a, b \in P$, with $a < b$, satisfying

$$g(a) \leqslant f(a) \qquad \textit{and} \qquad f(b) \leqslant g(b).$$

Then the reexists at least one $\xi \in P$ such that $a \leqslant \xi \leqslant b$ and $f(\xi) = g(\xi)$.

Proof. If $f(a) = g(a)$ or $f(b) = g(b)$, the conclusion is obvious. Hence we may assume that

(19) $$g(a) < f(a) \quad \text{and} \quad f(b) < g(b).$$

Consider the set $\{S_i\}$ of all non-empty subsets $S_i \subset P$ such that there exists $s_i \in P$ with $g^{-1}(s_i) = S_i$. Clearly, $\{S_i\} \neq \emptyset$. By the axiom of choice, there exists a function φ mapping $\{S_i\}$ into P, such that $\varphi(S_i) \in S_i$. Hence

(20) $$g\varphi g^{-1}(s_i) = s_i.$$

We observe also that, in view of (17),

(21) if $s_i < s_j$, then every element of $g^{-1}(s_i) <$ every element of $g^{-1}(s_j)$.

We shall show now that the function

(22) $$h = \varphi g^{-1} f$$

maps the set $Q = \{x | x \in P,\ a \leqslant x \leqslant b\}$ into itself. If $x \in Q$, then, since f is isotone, by (19) we have

(23) $$g(a) < f(a) \leqslant f(x) \leqslant f(b) < g(b),$$

and hence by (18) we see that $g^{-1}[f(x)] \neq \emptyset$. By (21) and (23) we find, $a < \varphi g^{-1}[f(x)] < b$. Hence, by (22), $h(x) \in Q$. Taking $x = a$, we infer also that

(24) $$a < h(a).$$

Furthermore, since f is isotone, if $x \leqslant y$ then $f(x) \leqslant f(y)$, and from (21) we infer that $\varphi g^{-1}[f(x)] \leqslant \varphi g^{-1}[f(y)]$, so that by (22) h is isotone on Q.

From (24) we see that a and h satisfy (9) on Q. Also, since Q is bounded above by b, we see from (16) that Q satisfies (7).

Hence Q and h satisfy the hypotheses of Theorem 2, and consequently there exists $\xi \in Q$ such that $h(\xi) = \xi$. Applying g to each side we have, by (22)

$$g\varphi g^{-1}[f(\xi)] = g(\xi),$$

and thus, by (20),

$$f(\xi) = g(\xi).$$

This completes the proof.

COROLLARY 4. *If in Theorem* 4 *instead of condition* (17) *we assume that* g *is isotone, then the conclusion of Theorem* 4 *remains valid provided* P *is a simply ordered set.*

REFERENCES

1. G. Birkhoff, *Lattice theory* (A.M.S. Coll. Publ., *25*, rev. ed.; New York, 1948).
2. N. Bourbaki, *Théorie des ensembles* (Paris, 1956), chapter III, p. 48, Example 6b.
3. ——— *Sur le Théorème de Zorn*, Archiv der Mathematik, *2* (1949–50), 434–437.

University of Pennsylvania
and
Queens College