THE SIMPLE GROUPS RELATED TO M_{24}, II

Dedicated to the memory of Hanna Neumann

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The objective of this paper is to prove the following generalization of the main result in [2]:

THEOREM. *Let G be a finite simple group which possesses an involution t such that the centralizer of t in G is isomorphic to the centralizer of an involution in H. Then G is isomorphic to $L_5(2)$, M_{24}, or H.*

REMARK. By H we denote a simple group of order $2^{10}.3^3.5^2.7^3.17$ discussed in [2]. The group H has precisely two classes of involutions. If $C(t)$ is isomorphic to the centralizer of a 2-central involution of H, the result follows from [2; Theorem. p. 253]. Thus, in what follows, we shall assume that $C(t)$ is isomorphic to the centralizer of an involution of H which is not contained in the center of a S_2-subgroup of H. Using the characterization of $L_5(2)$, M_{24}, and H by their Sylow 2-subgroups due to. Schoenwaelder [3], Deckers [1] has shown that our theorem holds in case that a S_2-subgroup of G has order at most 2^{10}. Thus, we shall assume that 2^{11} divides the order of G.

Familiarity with section 1 and with the lemmas 3.2–3.6 of [2] will be assumed. Table II of [2; p. 279] lists the conjugacy classes of $C(t)$. We use the notation introduced in [2]. In particular, a S_2-subgroup S of $C(t)$ will be described by generators and relations given in section 1 of [2]. We put

$$S = \langle z_1, z_3, z_4, \pi, \tau, \mu, \lambda, \mu'\tau', z_2\tau' \rangle$$

and $t = z_3\pi$. The four-group $V = \langle z_3\pi, z_1\mu\tau \rangle$ is normal in $C(t)$ and $C(t)/V$ contains a unique subgroup H^*/V of index 2 isomorphic to $L_3(4)$. As in [2; Lemma 2.2, p. 257] put $R_1 = \langle z_1, z_3, z_4, \pi, \mu\tau, \mu'\tau' \rangle$ and $R_2 = \langle z_1, z_3, \pi, \mu, \tau, \lambda \rangle$. Then R_1R_2 is a S_2-subgroup of H^*. Since $z_2\tau'$ does not centralize V, we have $C(V) = H^*$. Note that $C(V)$ does not split over V.

It seems worthwhile remarking that $C_H(z_3\pi)$ is very similar to the centralizer of an involution in the Suzuki sporadic group.

1. Preliminary lemmas

LEMMA 1.1. *Under the action of* $N(R_i) \cap C(z_3\pi)$, $i = 1,2$, *we have the following classes of involutions in* R_i: *There is one conjugate of* $z_3\pi$, *there are two conjugates of* $z_1\mu\tau$, *there are* 15 *conjugates of* z_1, *there are* 15 *conjugates of* $z_1z_3\pi$, *and there are* 30 *conjugates of* π.

PROOF. The assertion follows immediately from [2; lemma 3.2, p. 271, and table II, p. 279].

LEMMA 1.2. *For* $i = 1,2$, *the group* $N(R_i) \cap C(z_3\pi)$ *is a splitting extension of* R_i *by* Σ_5. *We have* $Z(S) = \langle z_1, z_3\pi\rangle$ *and* $\langle z_1\rangle$ *char* S. *The involutions* $z_3\pi$ *and* $z_1z_3\pi$ *are conjugate in every subgroup of order* 2^{10} *of* G *which contains* S. *There are only the following possibilities for the fusion of involutions of* R_i *under the action of* $N(R_i)$, $i = 1$ *or* 2:

		$C_{z_3\pi}$	$\lvert N(R_i)\rvert$
(a)	$z_3\pi \sim z_1z_3\pi$	16	$2^{13}\cdot 3\cdot 5$
(b)	$z_3\pi \sim z_1z_3\pi \sim z_1\mu\tau$	18	$2^{10}\cdot 3^3\cdot 5$
(c)	$z_3\pi \sim z_1z_3\pi \sim \pi$	46	$2^{10}\cdot 3\cdot 5\cdot 23$
(d)	$z_3\pi \sim z_1z_3\pi \sim z_1\mu\tau \sim \pi$	48	$2^{13}\cdot 3^2\cdot 5$
(e)	$z_3\pi \sim z_1\mu\tau$	3	$2^9\cdot 3^2\cdot 5$
(f)	$z_3\pi \sim \pi$	31	$2^9\cdot 3\cdot 5\cdot 31$
(g)	$z_3\pi \sim z_1\mu\tau \sim \pi$	33	$2^9\cdot 3^2\cdot 5\cdot 11$
(h)	$z_3\pi$	1	$2^9\cdot 3\cdot 5$

PROOF. The assertions follow easily from lemma 1.1, the fact that $S'' = \langle z_1\rangle$, and [2; lemma 3.2].

LEMMA 1.3. *The cases* (c), (f), *and* (g) *are not possible.*

PROOF. Since 11 and 23 do not divide the order of $L_6(2)$, we see that (c) and (g) cannot happen. If we are in case (f), then an element of order 31 acts trivially on the $N(R_i)$-class of z_1 because $z_1 \sim z_3\pi$ is impossible by (1.2).

LEMMA 1.4. *If a* S_2-*subgroup of* $N(R_i)$ *for* $i = 1$ *or* $i = 2$ *has order* 2^9 *we get a contradiction.*

PROOF. Assume that $\lvert N(R_i)\rvert = 2^9\cdot 3\cdot 5$ or $2^9\cdot 3^2\cdot 5$ for one $i \in \{1,2\}$. Denote by S^* a 2-subgroup of G which contains S as a subgroup of index 2. There is an element $f \in S^*$ such that $S^* = S\langle f\rangle$ and $R_1{}^f = R_2$. Also we have $(z_3\pi)^f = z_1z_3\pi$. If $\lvert N(R_i)\rvert = 2^9\cdot 3\cdot 5$, then $[f, z_3\pi] = 1$ which is impossible. Let $\lvert N(R_i)\rvert = 2^9\cdot 3^2\cdot 5$ We may assume $i = 1$. Then f maps the $N(R_1)$-class of $z_3\pi$ onto the $N(R_2)$-class of $z_1z_3\pi$. However, in $N(R_1)$ the element $z_3\pi$ has precisely three conjugates

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whereas in $N(R_2)$ the element $z_1 z_3 \pi$ has at least 15 conjugates. We have arrived at a contradiction.

LEMMA 1.5. *A S_2-subgroup of $N(R_i)$, $i = 1,2$, has order at least 2^{10}. We are in case* (a), (b), *or* (d).

PROOF. This follows from the above lemmas.

LEMMA 1.6. *Let T be a subgroup of G of order 2^{10} which contains S. Then, R_1 and R_2 are the only elementary abelian subgroups of T which have order* 64.

PROOF. We have $Z(T) = \langle z_1 \rangle$. Denote by R_3 an elementary abelian subgroup of T with $R_1 \neq R_3 \neq R_2$ and $|R_3| = 64$. Thus $SR_3 = T$ and $|S \cap R_3| = 2^5$. As $S \cap R_3 \supseteq R_1 R_2 \cap R_3$ and $2^{10} \geqq |R_1 R_2 R_3| \geqq 2^9$, we must have $2^5 \geqq |R_1 R_2 \cap R_3| \geqq 2_4$. An involution of $R_1 R_2$ lies in R_1 or in R_2. Let r_1, r_2 be involutions contained in $R_1 R_2 \cap R_3$ such that $r_1 \in R_1$ and $r_2 \in R_2 \setminus R_1$. Clearly, $r_1 \in C_{R_1}(r_2) = R_1 \cap R_2 \subset R_2$. Thus, $\langle r_1, r_2 \rangle \subseteq R_2$. It follows $R_1 R_2 \cap R_3 \subseteq R_i$ $i = 1$ or 2, and so, $R_1 R_2 \cap R_3 = R_3 \cap R_i$, $i = 1$ or 2. Assume first that $|R_1 R_2 \cap R_3| = 2^4$. Then, $|R_1 R_2 R_3| = 2^{10}$. Without loss of generality we may assume that $R_1 R_2 \cap R_3 = R_3 \cap R_1$. Clearly, $|R_2 \cap R_3| \geqq 2^2$ and $R_2 \cap R_3 \subseteq R_1 R_2 \cap R_3 = R_3 \cap R_1$. Thus, $R_2 \cap R_3$ would be in $Z(T)$ which is impossible. Assume finally that $|R_1 R_2 \cap R_3| = 2^5$. Then, $|R_1 R_2 R_3| = 2^9$. Without loss of generality we may assume that $R_1 R_2 \cap R_3 = R_1 \cap R_3$. Clearly, $|R_2 \cap R_3| \geqq 2^3$ and $R_2 \cap R_3 \subseteq R_1 R_2 \cap R_3 = R_1 \cap R_3$. Thus, $R_2 \cap R_3$ lies in $Z(R_1 R_2 R_3)$. But $|T : R_1 R_2 R_3| = 2$. From the Jordan-canonical form for an element of $T \setminus R_1 R_2 R_3$ on $\Omega_1(Z(R_1 R_2 R_3))$, we get a contradiction to $|Z(T)| = 2$. The lemma is proved.

LEMMA 1.7. *If ω is an element of order* 5 *of $C(z_3 \pi)$, then a S_2-subgroup of $C(\omega)$ has order 2^2 and a S_2-subgroup of $N(\langle \omega \rangle)$ has order 2^4.*

PROOF. We know that $C(z_3 \pi)$ contains a subgroup isomorphic to Σ_5. Clearly, V lies in $C(\omega)$. Let V^* be a subgroup of order 8 of $C(\omega)$ such that $V \subset V^*$. If $z_3 \pi \nsim z_1 \mu \tau$ in G, we get $z_3 \pi \in Z(V^*)$. If $z_3 \pi \sim z_1 \mu \tau$, then $Z(V^*)$ contains an involution conjugate to $z_3 \pi$ in G. It follows that V must be a S_2-subgroup of $C(\omega)$.

2. The case (*b*)

In this section we suppose that a S_2-subgroup of $N(R_i)$ has order 2^{10} for one $i \in \{1,2\}$.

LEMMA 2.1. *For $i = 1$ and $i = 2$ we have $|N(R_i)| = 2^{10} \cdot 3^3 \cdot 5$.*

PROOF. Without loss of generality let $|N(R_1)| = 2^{10} \cdot 3^3 \cdot 5$. Denote by T a S_2-subgroup of $N(R_1)$ with $S \subset T$. From (1.6) we get that R_1 and R_2 are the only elementary abelian subgroups of order 64 of T. Denote by T^* a subgroup of order 2^{11} of G with $T \subset T^*$. There is an element $f \in T^* \setminus T$ such that $R_1^f = R_2$. The lemma is proved.

LEMMA 2.2. *The order of* $N(R_1R_2)$ *is* $2^{11} \cdot 3^2$.

PROOF. From (1.6) we get that 2^{11} divides the order of $N(R_1R_2)$. Since all involutions of V are conjugate, we get $|N(V)| = 2^9 \cdot 3^3 \cdot 5 \cdot 7$. We know that R_1R_2 is a S_2-subgroup of $C(V)$ and that $|N(R_1R_2) \cap C(V)| = 2^8 \cdot 3$. The Frattini-argument yields $N(V) = (N(R_1R_2) \cap N(V))C(V)$. It follows that $N(R_1R_2)$ contains an elementary abelian group of order 9. Also one has $|N(R_1R_2) \cap C(z_3\pi)| = 2^9 \cdot 3$. It follows $|N(R_1R_2)| = 2^{11} \cdot 3^2$, since $z_3\pi$ cannot have more than 12 conjugates under the action of $N(R_1R_2)$.

LEMMA 2.3. *The involution* z_1 *has precisely three conjugates under the action of* $N(R_1R_2)$.

PROOF. From (1.2) we see that z_1 lies in the center of a S_2-subgroup of $N(R_1R_2)$. Assume by way of contradiction that 3^2 divides $|C(z_1) \cap N(R_1R_2)|$. We know that $N(R_1R_2) = N(R_1 \cap R_2)$ and that $C(R_1 \cap R_2) = R_1R_2$. Our assumption implies that an element of order 3 centralizes $R_1 \cap R_2$ which is not possible. Since $z_3\pi$ has 12 conjugates in $R_1 \cap R_2$ under $N(R_1R_2)$, we see that z_1 has precisely three conjugates under the action of $N(R_1R_2)$.

LEMMA 2.4. *In* $N(R_1R_2)$ *we have* $z_1 \sim \pi$.

PROOF. Assume by way of contradiction that there is an element g in $N(R_1R_2)$ such that $(z_3\pi)^g = \pi$. Then, $R_1^g = R_2$. This is impossible, since in $N(R_1)$ the element $z_3\pi$ has only 18 conjugates, whereas in $N(R_2)$ the element π has at least 30 conjugates. Hence, $z_3\pi \nsim \pi$ holds in $N(R_1R_2)$. Thus, $z_1 \sim \pi$ in $N(R_1R_2)$.

LEMMA 2.5. *The case* (b) *is not possible.*

PROOF. The only elements of $R_1 \cap R_2$ conjugate to z_1 are $z_1, \pi, z_1\pi$. All other involutions of $R_1 \cap R_2$ are conjugate to $z_3\pi$ under $N(R_1R_2)$. We know that $V\tau$ is conjugate to Vz_1 in $N(V)$. All elements of Vz_1 except z_1 are conjugate to $z_3\pi$ in G. The coset $R_1\tau$ has precisely 16 involutions which fuse under S in the following way:

$$\begin{aligned}
&\tau \sim z_1\tau \sim \pi\tau \sim z_1\pi\tau,\\
&z_3\tau \sim z_1z_3\tau \sim z_3\pi\tau \sim z_1z_3\pi\tau,\\
&\mu \sim z_1\mu \sim z_3\pi\mu \sim z_1z_3\pi\mu,\\
&\pi\mu \sim z_1\pi\mu \sim z_1z_3\mu \sim z_3\mu.
\end{aligned}$$

Hence, $R_1\tau$ contains at least eight involuticns conjugate to $z_3\pi$ in G. Thus R_2 contains at least 20 conjugates of $z_3\pi$. Since under the action of $N(R_2)$ the element $z_3\pi$ has only 18 conjugates, we get $z_3\pi \sim z_1$ as $z_1 \sim \pi$ which is impossible. The lemma is proved.

3. The cases (*a*) and (*d*)

In this section we finish the proof of our theorem. We know that $2^{13}\cdot 3\cdot 5$ divides $|N(R_i)|$ and that $|N(R_i)|$ divides $2^{13}\cdot 3^2\cdot 5$ for $i = 1$ and $i = 2$. Remember that $N(R_i)$ contains a subgroup Σ isomorphic to Σ_5.

We study the structure of $N(R_i)$. Put $\boldsymbol{R} = N(R_i)/R_i$. We may choose Σ to be a subgroup of $C(z_3\pi)$. Then, $[\Sigma', V] = \langle 1 \rangle$. The only simple groups which can be involved in $\boldsymbol{R}$ are A_5 and A_6. It is a consequence of (1.7) that $O_2(\boldsymbol{R}) \neq \langle 1 \rangle$. Denote by L the inverse image of $O_2(\boldsymbol{R})$ in $N(R_i)$. Then, $|L| \geqq 2^7$. Because of $C(R_i) = R_i$, we get $Z(L) \subset R_i$. Obviously, $L \cap C(z_3\pi) = R_i$ because $\Sigma \cap L = \langle 1 \rangle$. Assume that $Z(L)$ lies in V. Then, $|Z(L)| = 2$, and $Z(L)$ is centralized by Σ. This would imply $Z(L) = \langle z_3\pi \rangle$, a contradiction. Hence, $Z(L) \not\subseteq V$. It follows that Σ' is faithfully represented on $Z(L)$. Thus, $|Z(L)| \geqq 2^4$. Assume that $V \cap Z(L) \neq \langle 1 \rangle$. This intersection does not contain $\langle z_3\pi \rangle$. Hence, $Z(L) \cap V$ is equal to $\langle z_1\mu\tau \rangle$ or to $\langle z_1 z_3 \pi\mu\tau \rangle$. But $z_2\tau'$ normalizes $Z(L) \cap V$. This gives a contradiction. We have shown that $R_i = V \times Z(L)$. Since Σ' normalizes $Z(L)$ we get that $V \times Z(L)\Sigma'$ splits over V. This is not possible. The theorem is proved.

References

[1] M. Deckers, 'Simple groups having centralizers of involutions isomorphic to those in He', *Archiv Math.* (To appear).

[2] D. Held. 'The simple groups related to M_{24}', *J. Algebra* 13 (1969), 253–296.

[3] U. Schoenwaelder, 'Finite groups with a Sylow 2-subgroup of type M_{24}', *J. Algebra* (To appear).

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