4

The vacuum as a dielectric medium; renormalisation

4.1 Introduction

In this chapter we will consider some major problems in quantum field theory. They are related to the understanding of *polarisation effects in the vacuum state*. Although this state in the mean is empty it nevertheless embraces the continuous production and annihilation of virtual particle-antiparticle pairs due to quantum fluctuations. All the real charges and currents then behave as if they were moving in a dielectric medium. In connection with QED this effect is small (although readily observable). For QCD, on the other hand, it plays a major role.

The first kind of problem is mathematical, related to ill-defined series expansions in perturbation theory and also to undefined integrals. The second is general in physics: it is necessary to isolate the effective dependence on the theoretical parameters in all the calculated expressions for the observables (note that this dependence is in general complicated when one deals with non-linear equations). This is the *renormalisation procedure*, which always must be performed in order to relate the parameters in a theoretical expression to the observables in an experiment.

It is true that physicists are, compared to most other scientists, privileged because the components of many systems in physics can be isolated. In this situation the properties of each component can be determined. Afterwards the whole system can be brought back into interaction, with well-defined values of the parameters which govern the behaviour of each subsystem. For an interacting quantum field it is, however, not straight-forward to isolate the 'real' quanta from the surrounding fields and the quantum fluctuations. This was found for an electron in connection with the method of virtual quanta in Chapter 2: the electron energy can only be isolated from the surrounding field energy by means of an impact parameter cutoff. Similarly the properties of a field quantum in an interacting-

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field theory cannot be described in terms of the corresponding free-field behaviour without some limiting procedures and the introduction of cutoff parameters.

It is a surprising and gratifying result that we are able to solve both the above-mentioned problems at the same time. It turns out that all the 'bad' mathematical expressions occur, for a wide class of field theories, just where we would anyhow have had to redefine these expressions in order that the coupling constants and the masses should have the values observed for the free initial- and and final-state quanta.

We will consider two different examples with some similarities. The first one corresponds to the scalar field theory we exhibited before in Chapter 3, with two ψ-operators coupled to a single ϕ-field. We will choose the quanta of the ϕ-field to be massless, $M_\phi = 0$, in order to connect with the QED and QCD field theories. These will provide our second example, with massless vector particles, photons and gluons, coupled to spin $1/2$ fermions corresponding to the scalar ψ-fields in the first example.

We will find that, apart from *mass renormalisation*, the scalar field theory is a finite field theory, called *super-renormalisable*. This feature is related to the dimensions of the coupling constant. For a super-renormalisable theory the coupling constant dimension is positive in terms of energy dimensions. Then the theory contains at most a finite number of undefined diagrammatic contributions in perturbation theory and this can be understood in the following way. Undefined, divergent, integrals in perturbation theory stem from the fact that there are too many energy-momentum integration variables as compared with the energy denominators (from the propagators). Then the integrals are not sufficiently damped for large values of the energy-momentum variables (and are therefore called *ultraviolet divergent*). If we consider this phenomenon in space-time then the divergences in the energy-momentum integrals correspond to singular behaviour of the space-time integrals for small values of the relative coordinates of the field operators. The singularities stem from the *distribution properties* of the field operators, which we have already encountered in Chapter 3. In general one is not allowed to multiply field operators at the same space-time point (which we would like to do when we consider local interactions between fields).

Let us consider a physical quantity $\mathscr{I}$ which by a suitable rescaling is dimensionless. When it is defined in perturbation theory at the nth order in the coupling constant g one obtains $g^n I_n$ with I_n some integral. The integral I_n must then have the (energy) dimension $\dim I_n = -n \times \dim g$. Therefore in a super-renormalisable theory the (energy) dimension will become more and more negative with n. This means that the number of energy denominators must be increasing faster than the number of

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integration variables, which means that we obtain integrals which are more and more damped for larger energies.

QED and QCD have dimensionless coupling constants and in these cases the argument above does not work. They are nevertheless *renormalisable* in the following sense. One finds that in each order of perturbation theory there will be logarithmically divergent integrals (which in practice stem from non-allowed changes of integration order and undefined limits). It is then necessary to introduce in each order of perturbation theory a method to make the results finite. For renormalisable theories it turns out that *all the undefined quantities can be incorporated as multiplicative constants in the field operators and in the coupling constants* after mass renormalisation has been performed. This means that the 'new' *renormalised field theory* contains just as many parameters as the original one. When these parameters have been fixed by the observed values then *all the remaining observable quantities are finite and predicted by the theory*.

To be more precise we may imagine that we have a fixed external electric charge (size g_0) and that we make use of it in order to measure the properties of a quantum field coupled to the charge via QED. As a thought experiment we will consider the scattering of a field quantum with momentum transfer q^2 from this external source.

Now let us take into account the influence of the quantum fluctuations in the state, i.e. what we have earlier referred to as the dielectric properties of the surrounding vacuum state. All these properties can be calculated in perturbation theory but (unless one is particularly careful about the distribution properties of the fields) the expressions will correspond to ill-defined integrals and series. The calculations can nevertheless be performed with different degrees of sophistication. We will then obtain results which can be expressed in terms of the original (unrenormalised) parameters of the theory together with some suitable cutoff parameters to make the mathematical expressions well defined.

In this way we will obtain an expression for the scattering amplitude (cf. Eq. (3.98)) which should be of the generic shape

$$\frac{g_0 g(q^2)}{q^2} \times \text{k.f.} \tag{4.1}$$

where k.f. corresponds to the necessary kinematical factors and $g(q^2)$ corresponds to the coupling constant at the 'test frequency', q^2, at which we perform the calculation (the theoretically evaluated quantity $g(q^2)$ also depends upon the the cutoff parameters, of course). We may also calculate other quantities, such as the value for which there will be a pole in the field propagator. This obviously corresponds to the squared mass of the

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corresponding quanta (as seen by a probe with the frequency q^2) and from the size of the pole term we may calculate the number of quanta which are available at the scale q^2 in the field (the 'field strength').

These calculations provide us with a value for the effective coupling constant, $g(q^2)$, as well as values for the mass(es) and the normalisation(s) of the field(s) at the 'frequency' q^2, and they are all expressed in terms of the unrenormalised parameters and the cutoff parameters. We may then choose these numbers to coincide with our expectations (giving *coupling constant*, *mass* and *wave function renormalisation*, respectively). But note that this free choice can in general only be made for a single value of the frequency! For other frequencies there will be changes but *in a renormalisable theory all such changes are computable and finite* although all quantities will seemingly depend upon the value of q^2 for which the original definition is made. There is, however, no reason to prefer one value of q^2 to another and we may then freely move between different 'normalisation points'. But the values of our parameters at these different points are all related, i.e. for any given value q^2 and our choices of the parameters at that value we may compute the result for any other frequency value. And for any particular value q_p^2 we will obtain the same observable results, independent of the normalisation point!

This is the content of the renormalisation group theory. After we have sketched the general behaviour of any renormalisable field theory we will derive the *Callan-Symanzik equations*, [108], which relate the behaviour of the matrix elements and the effective coupling constant $g(q^2)$ at different values of the momentum transfer by means of differential equations. We will use these equations again in Chapter 19 to derive the QCD predictions for the scale breaking in the parton structure functions, which governs the behaviour of the inelastic lepto-production cross sections.

We start by introducing the Källén-Lehmann representation as a convenient tool to perform the renormalisation procedure. This will also provide an opportunity to show the occurrence of some of the phase space factors we will meet further on. We also show how to calculate the polarisation correlations which occur when one couples spin 1/2 (Dirac) particles to spin 1 particles (photons or gluons) as is done in QED and QCD. The particular polarisation properties of the QCD field theory are treated in some detail and we will then also consider the relationship between the weight function and the full polarisation function in a Källén-Lehmann representation, i.e. we will introduce the notion of 'cut diagrams'. We will finally show how to calculate the color factors which occur in QCD.

4.2 The Källén-Lehmann representation, the n-particle phase space

We start out with the following general expression for a propagator

$$\begin{aligned}\Delta_{FA}(x) &= \langle 0| \mathscr{T}\{A(0)A(x)\} |0\rangle \\ &= \Theta(x) \langle 0| A(x)A(0) |0\rangle + \Theta(-x) \langle 0| A(0)A(x) |0\rangle \end{aligned} \tag{4.2}$$

where A is a local (for simplicity also self-adjoint, i.e. real) operator of any kind expressed in terms of the in-fields (we omit all i-indices from now on). We may introduce a complete set of states $\sum_n |n\rangle \langle n| = 1$ in between the operators. Further we note that (due to translation invariance)

$$\begin{aligned}\langle 0| A(x) |n\rangle &= \exp(-ik_n x) \langle 0| A(0) |n\rangle \\ \langle n| A(x) |0\rangle &= \exp(ik_n x) \langle n| A(0) |0\rangle \end{aligned} \tag{4.3}$$

with k_n the total energy-momentum of the state n. We then rearrange the expression for Δ_{FA} into

$$\begin{aligned}\Delta_{FA}(x) &= \sum_n [\Theta(x)\exp(-ik_n x) + \Theta(-x)\exp(ik_n x)] |A_{0n}|^2 \\ &= \int \frac{dq}{(2\pi)^3} [\Theta(x)\exp(-iqx) + \Theta(-x)\exp(iqx)]\, da\delta^+(q^2 - a) G_A(a) \\ G_A(q^2) &= (2\pi)^3 \sum_n \delta(q - k_n)|A_{0n}|^2 \end{aligned} \tag{4.4}$$

where we have used the shortened version $A_{0n} = \langle 0| A(0) |n\rangle$.

The fact that G_A is a Lorentz invariant will be exhibited below. Then the resulting expression for Δ_{FA} is

$$\Delta_{FA} = \int da \Delta_F(x, a) G_A(a) = \frac{i}{(2\pi)^4} \int dq \frac{\exp(iqx)}{q^2 - a + i\epsilon} da G_A(a) \tag{4.5}$$

which is the Källén-Lehmann representation for the general propagator. The structure is a sum of ordinary Feynman propagators with contributions from the squared masses of all the possible intermediate states which can be reached by A.

We note that the weight function G_A, if we use the distribution described by Eq. (3.87), is essentially the real part of (the Fourier transform, i.e. the energy-momentum space version, of) Δ_{FA}. This general feature is in Chapter 2 referred to as the Kramers-Kronig relations: the imaginary part of the dielectricity is determined by the real part. From Eq. (4.5) we find the content of this statement, i.e. the total energy-momentum space propagator is determined by its real part. We will elaborate this result further on in this chapter.

In order to investigate the weight function G_A we start by considering the case $A(x) = :\psi^2(x):$. Then there is only a single intermediate state, a two-particle ψ-quantum state, and we obtain for this situation

$$\begin{aligned} G_A &= (2\pi)^3 \sum_n \delta(q - k_n)| < 0| :\psi^2: |n\rangle |^2 \\ &= \int \frac{V^2 d^3k_1 d^3k_2}{(2\pi)^3} \delta(q - k_1 - k_2) \frac{1}{4V^2\omega_1\omega_2} \\ &\to \frac{1}{(2\pi)^{3n_f - 3}} \int \prod_{j_f=1}^{n_f} dk_{j_f} \delta^+(k_{j_f}^2 - M_{j_f}^2) \delta\left(q - \sum_{j_f=1}^{n_f} k_{j_f}\right) \end{aligned} \quad (4.6)$$

where in the last line we have gone over to the result for A, $:\prod_{j_f=1}^{n_f} \psi_{j_f}:$, in order to show the general structure of any G_A-expression containing normal-ordered local-field operators. *The main point is the occurrence of the manifestly Lorentz-invariant n_f-particle phase space* .

For the scalar field theory case the probability of producing real states with the mass square a is given simply by this phase space factor. We will later find a difference when we have spin 1/2 particles coupled to a vector field; then there is also a spin-correlation term.

We will now calculate the phase space integrals, I_{n_f}, for the cases when $n_f = 2, 3$ because we will need them later. We start with I_2:

$$I_2(q^2, a_1, a_2) = \int \frac{dk_1 dk_2}{(2\pi)^3} \delta^+(k_1^2 - a_1) \delta^+(k_2^2 - a_2) \delta(q - k_1 - k_2) \quad (4.7)$$

Evidently q must be a timelike vector with $\sqrt{q^2} \geq \sqrt{a_1} + \sqrt{a_2}$. In order to simplify our formulas, we will make use of the Lorentz invariance to choose the particular system where q is at rest (the cms of particles 1 and 2). Then $q = (W, \mathbf{0})$. Performing the k_2-integral by means of the energy-momentum-conserving δ-distribution we obtain in this frame

$$\begin{aligned} I_2 &= \frac{1}{(2\pi)^3} \int dk_1 \delta^+(k_1^2 - a_1) \delta^+((q - k_1)^2 - a_2) \\ &= \int \frac{k^2 dk d\Omega}{2\omega(2\pi)^3} \delta^+(W^2 - 2\omega W + a_1 - a_2) \end{aligned} \quad (4.8)$$

with the notation $k_1 = (\omega = \sqrt{k^2 + a_1}, \ \mathbf{k})$. We have chosen a spherical coordinate system with $d^3k = k^2 dk d\Omega$. We may then transform to the integration variable ω and obtain

$$I_2 = \frac{1}{(2\pi)^2} \int k d\omega \delta(2\omega W - (W^2 + a_1 - a_2)) = \frac{\sqrt{\lambda(W^2, a_1, a_2)}}{(4\pi)^2 W^2} \quad (4.9)$$

where λ is again the symmetrical function defined in Eq. (3.105). In particular the expression can be written as

$$I_2 = \frac{2p_{cms}}{(4\pi)^2 W} \tag{4.10}$$

Thus the two-particle phase space integral vanishes linearly when the relative velocity vanishes and approaches the constant $1/(4\pi)^2$ for large W^2-values. We note that *the phase space for two particles is dimensionless* (with our conventions, i.e. when $c = \hbar = 1$).

If we look back we notice that for $n = 2$ there are $n \times 4$ integration variables with dimension mass. But there is a four-dimensional δ-distribution, with dimensions $4 \times (-1)$, and $n = 2$ (mass-shell) δ-distributions with dimensions -2. This means that the n-particle phase space has the energy dimension $\dim I_n = 4n - 4 - 2n = 2n - 4$.

We note also that the number of degrees of freedom is $4n - 4 - n$ because the mass-shell δ-distributions fix only one of the four energy-momentum variables describing each particle. There is, however, also the question of orienting the event. It takes three Euler angles (cf. Goldstein) to fix the coordinate system. If there is no outside direction to relate to, these angles will always be integrated out. Thus for the internal dynamics of the n-particle state there are effectively $3n - 7$ degrees of freedom.

For the three-particle phase space we find an energy dimension 2 and also that there are two internal degrees of freedom. This is a sign that it is a density in two energy variables. We note that if we again go to the cms, i.e. choose the vector $q = (W, \mathbf{0})$, then the energy δ-distribution requires the three cms energies to satisfy

$$\sum_{j=1}^{3} \omega_j = W \tag{4.11}$$

We may then choose two of these to be independent variables, e.g. the pair ω_1, ω_2. We will only calculate in detail the result when all the three particles are massless; we then obtain

$$\begin{aligned}
\frac{d^2 I_3}{d\omega_1 d\omega_2} &= \frac{1}{(2\pi)^6} \int d^3k_1 d^3k_2 dk_3 \prod_{j=1}^{3} \delta(k_j^2) \delta\left(q - \sum_{j=1}^{3} k_j\right) \\
&= \frac{1}{(2\pi)^6} \int d^3k_1 d^3k_2 \prod_{j=1}^{2} \delta(k_j^2) \delta\left((W - \omega_1 - \omega_2)^2 - (\mathbf{k}_1 + \mathbf{k}_2)^2\right) \\
&= \frac{\pi}{(2\pi)^5} \omega_1 \omega_2 \int \sin\theta d\theta \delta\left(W^2 - 2W(\omega_1 + \omega_2) + 2\omega_1\omega_2(1 - \cos\theta)\right)
\end{aligned} \tag{4.12}$$

In the second line we have introduced $k_3 = (W - \omega_1 - \omega_2, \ -(\mathbf{k}_1 + \mathbf{k}_2))$ and then performed the integrals over everything besides the relative angle θ

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between the vectors $\mathbf{k}_1$ and $\mathbf{k}_2$. The final step leads to the result

$$\frac{d^2 I_3}{d\omega_1 d\omega_2} = \frac{\pi^2}{(2\pi)^6}\Theta(W-\omega_1-\omega_2) \times\Theta\left(2(\omega_1+\omega_2)-W)(W-2\omega_1)(W-2\omega_2)\right) \tag{4.13}$$

One way to make the whole thing symmetric is to introduce the new dimensionless quantities $x_j = 2\omega_j/W$ for $j = 1, 2, 3$ and to rewrite the distribution as

$$d^3 I_3 = \frac{W^2}{(4\pi)^4}\delta\left(\sum_{j=1}^{3} x_j - 2\right)\prod_{j=1}^{3} dx_j\Theta(x_j)\Theta(1-x_j) \tag{4.14}$$

The expressions for the higher-order phase space factors become more and more complicated to handle. Van Hove [81] devised the idea of 'longitudinal phase space', which means that one projects the total n-particle phase space onto a single direction. He was in that way rather successful in obtaining low-energy dynamical information from the experimental distributions. But even in this simplified case one cannot make do with fewer than n coordinates for n particles so this method fails to give information as soon as we go away from the resonance region.

4.3 A scalar-field-theory propagator in the Källén-Lehmann representation

In this section we will make use of the Källén-Lehmann representation together with the structure of the perturbative expansion as given in Dyson's equation to study some very general properties of the propagator.

We will as an example consider the time-ordered product

$$T_{prop} = \mathscr{T}\{\phi_f(x_1)\phi_f(x_2)\} = \mathscr{T}\{S^*\phi_i(x_1)\phi_i(x_2)S\} \tag{4.15}$$

for the simple $g:\phi\psi^2$:-theory. To second order in the coupling constant the (in-)vacuum expectation value of the operator T_{prop} contains two terms:

$$\langle 0i|T_{prop}|0i\rangle = \Delta_F(x_2-x_1, M_\phi) + 4g^2\int dx_3 dx_4 \Delta_F(x_2-x_3, M_\phi) \times\Delta_F^2(x_3-x_4, M_\psi)\Delta_F(x_4-x_1, M_\phi) \tag{4.16}$$

The result is presented in Feynman graph language in Fig. 4.1.

It is not too difficult to continue towards higher-order approximations (although there are some problems with respect to counting the number of contributions to each particular diagram in accordance with combinatorics). In Fig. 4.2 we show the relevant contributions in the next order; it is then possible to deduce the general structure.

Fig. 4.1. The first two orders in the expansion of the ϕ-propagator described by Feynman diagrams in the simple $g\phi\!:\!\psi^2\!:$-theory. Solid (broken) lines correspond to ψ- (ϕ-)propagators.

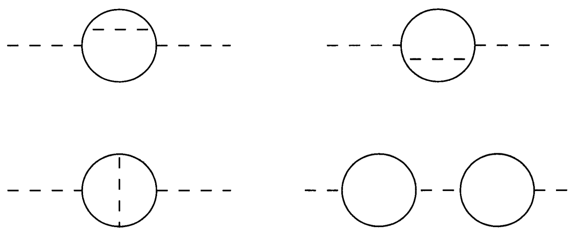

Fig. 4.2. The next-order contributions to the propagator in the $g\phi\!:\!\psi^2\!:$-theory.

There is a 'master' part, ρ, which is called the *polarisation function.* It is the sum of all the contributions from diagrams (with one ϕ-line in and one out) with the following connection structure:

- they are everywhere two-line (at least) connected, i.e. all parts are connected to the rest by at least two lines (this is called *one-particle irreducible*). It means that you cannot disconnect one part from the others by cutting a single line (whether it is broken, corresponding to a ϕ-propagator, or solid, a ψ-propagator).

The contributions in Fig. 4.2 are one-particle irreducible for the first three cases shown but the fourth contribution can be divided easily by cutting the line in between the 'blobs'.

We will assume that it is possible to sum up the contributions to ρ. Unfortunately it can be proved that in a scalar field theory the contributions are, at the $2n$th approximation level, positive and the number of contributions increases more than $n!$ [82]. Therefore the power series in the coupling constant g^2 cannot converge in the usual sense.

This behaviour can be described in very sophisticated mathematical ways but the major physical reason is that the interaction term is not well-behaved, in this case the interaction term $\propto \phi\!:\!\psi^2\!:$ is not *positive definite*. Therefore it is possible to find state configurations with a positive energy in the original free-field case (we may e.g. chose large negative ϕ-field contributions). For the total energy operator $\mathscr{H}$ such configurations

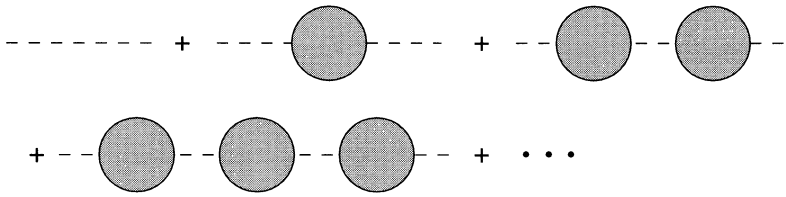

Fig. 4.3. The result of summing all one-line irreducible diagrams into the polarisation function ρ (denoted by a shaded oval) and then adding all these one-particle reducible contributions.

will provide very large negative contributions. Then the Hilbert space of the free-field configurations becomes different from the Hilbert space of the interacting fields, i.e. for some states of the free Hilbert space the interaction term is not well defined.

(You can find a similar behaviour if you introduce e.g. a seemingly small but singular perturbation $\propto \epsilon|x|^{-1-\delta}$ into the one-dimensional Schrödinger equation with a binding potential at the origin; for any $\epsilon, \delta > 0$ there is at least one state, the ground state ψ_0, which is not allowed in the Hilbert space of states of the total hamiltonian because the perturbation term is not defined on the state ψ_0.)

In Fig. 4.2 we note in the fourth contribution the appearance of a repeated part from Fig. 4.1. After a little thought we may conclude that to all orders in the expansion the result can be described as a single (free) ϕ-propagator connecting ρ's in accordance with Fig. 4.3. This means that if we introduce the Fourier transform $\tilde{\rho}$ of ρ by

$$\rho(x) = \frac{i}{(2\pi)^4} \int dq \tilde{\rho}(q) \exp(iqx) \tag{4.17}$$

then we obtain the total result after Fourier transformation (using $\tilde{\Delta}$ for the full Feynman ϕ-propagator in energy-momentum space and $\tilde{\Delta}_0^{-1}(q) = q^2 + i\epsilon$ for the corresponding free ϕ-propagator):

$$\tilde{\Delta} = \tilde{\Delta}_0(q) - \tilde{\Delta}_0(q)\tilde{\rho}(q)\tilde{\Delta}_0(q) + \tilde{\Delta}_0(q)\tilde{\rho}(q)\tilde{\Delta}_0(q)\tilde{\rho}(q)\tilde{\Delta}_0(q) + \cdots$$

This can be expressed as an algebraic equation:

$$\tilde{\Delta} = \tilde{\Delta}_0(q) - \tilde{\Delta}_0(q)\tilde{\rho}(q)\tilde{\Delta}(q) \tag{4.18}$$

with solution

$$\tilde{\Delta} = \frac{\tilde{\Delta}_0(q)}{1 + \tilde{\Delta}_0(q)\tilde{\rho}} = \frac{1}{q^2 + \tilde{\rho}(q) + i\epsilon} \tag{4.19}$$

We have then in effect summed a geometrical series without worrying

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about convergence problems. This has at least a formal meaning in connection with a perturbative expansion. Equation (4.18) was first derived by Dyson and corresponds to his propagator equation. We conclude that *in order to learn about the general propagator it is enough to know the polarisation function* $\tilde{\rho}$ *in energy-momentum space.*

Actually we have in the simple $\phi{:}\psi^2{:}$-theory already calculated the lowest-order contribution to ρ, i.e. the contribution $\rho^{(1)}$ corresponding to the second term in Fig. 4.1. We note that this has exactly the structure of the vacuum expectation value of a time-ordered product (cf. Eq. (4.16)):

$$4\Delta_F^2(x_4 - x_3, M_{\psi^2}) = \left\langle 0 \middle| \mathcal{T}\left\{ :\psi^2(x_3)::\psi^2(x_4): \right\} \middle| 0 \right\rangle \tag{4.20}$$

We may then use the Källén-Lehmann representation for such an expression and pick up the result directly from Eqs. (4.5), (4.6), (4.9):

$$\tilde{\rho}^{(1)}(q^2) = \frac{4g^2}{(4\pi)^2} \int_{4M^2} \sqrt{1 - \frac{4M^2}{a}} \left[\frac{da}{a - q^2 - i\epsilon} \right] \tag{4.21}$$

with $M \equiv M_\psi$. The weight function in the integral is just the size of the intermediate two-particle phase space. The integral does not vanish for $q^2 = 0$; as a matter of fact it does not even converge! This is due to a too-cavalier treatment of limits in the calculations. But even if the integral were finite we would have to (re)define it so that $\tilde{\rho}$ vanishes for $q^2 = 0$. *This is called mass renormalisation and corresponds to the requirement that our physical* ϕ*-field also should have massless quanta.*

It can be done easily in this case:

$$\begin{aligned}
\tilde{\rho}^{(1)}(q^2) &\rightarrow \tilde{\rho}^{(1)}(q^2) - \tilde{\rho}^{(1)}(0) \\
&= \frac{4g^2}{(4\pi)^2} \int_{4M^2} \sqrt{1 - \frac{4M^2}{a}} \left[\frac{da}{a - q^2 - i\epsilon} - \frac{da}{a} \right] \\
&= q^2 \frac{4g^2}{(4\pi)^2} \int_{4M^2} \sqrt{1 - \frac{4M^2}{a}} \left[\frac{da}{a(a - q^2 - i\epsilon)} \right] \equiv q^2 \imath_\phi^{(1)}(q^2)
\end{aligned} \tag{4.22}$$

Then we obtain for the total propagator the expression

$$\Delta_F(x) = \frac{i}{(2\pi)^4} \int dq \frac{\exp(iqx)}{q^2 + i\epsilon} \left[\frac{1}{1 + \imath_\phi(q^2)} \right] \tag{4.23}$$

where the correction term $[1 + \imath_\phi(q^2)]^{-1} \simeq 1 - \imath_\phi(q^2)$ is to lowest order equal to a convergent integral:

$$1 - \imath_\phi^{(1)}(q^2) = 1 - \frac{4g^2}{(4\pi)^2} \int_{4M^2} \sqrt{1 - \frac{4M^2}{a}} \left[\frac{da}{a(a - q^2 - i\epsilon)} \right] \tag{4.24}$$

A very similar calculation can be done for the full ψ-propagator and in this case we obtain as the lowest-order correction term the Källén-Lehmann

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contribution, corresponding to Eq. (4.20):

$$\langle 0|\mathscr{T}\{:\phi\psi(3)::\phi\psi(4):\}|0\rangle = \Delta_F(x_4 - x_3, M_\psi^2)\Delta_0(x_4 - x_3) \quad (4.25)$$

which again leads to the necessity of defining the mass pole for the ψ-propagator. We end up with an expression for the propagator similar to the one we obtained for the ϕ-propagator in Eq. (4.23) and with a denominator in the integral containing an (inverse) correction term $1 + \imath_\psi$.

The quantities in Eqs. (4.25) and (4.20) are usually referred to as 'self-energy contributions', indicating that the $\psi(\phi)$-particle may fluctuate into a $\phi\psi(\psi\psi)$-state and back again, i.e. interact with 'its own field'. Just as for the electron in the method of virtual quanta, cf. Chapter 2, it is then necessary to distinguish between the $\psi(\phi)$-quantum itself and the surrounding quantum field, i.e. it is necessary to define the mass of the quantum.

It turns out that in this field theory there are now no other undefined Feynman diagram integrals. After mass renormalisation it contains in each order of perturbation theory only well-defined expressions. As mentioned above, the number of contributions increases very fast with perturbation order and therefore the theory as a whole is not definable by means of our present formulation of perturbation theory.

There is, however, one particular feature which is valid both for $\imath_\phi$ and $\imath_\psi$: *they are both positive-definite functions* for spacelike values of $q^2 < 0$. *This can be traced back to the properties of the weight functions.* It has the evident consequence that there is a dielectricity function, $\tilde{\epsilon} \simeq 1 - \imath_\phi^{(1)}(q^2)$ to the lowest order, *which must be always smaller than* 1 (to all orders if it can be defined at all).

This is the most general feature we can prove for any renormalisable or super-renormalisable field theory in which the Källén-Lehmann representation is valid in the form Eq. (4.5). The main point is that the *weight function G in the integral is positive-definite because we are in reality calculating the phase space size of the real intermediate states.*

Actually the weight function generally has the meaning of a *production rate*, i.e. the probability of emitting a ψ-quantum pair from an external (unit) ϕ-source, ϕ_e, carrying energy-momentum P with $P^2 \geq 4M_\psi^2$. To see this we note that the matrix element $\mathscr{M}$ and the transition rate ω will be

$$\begin{aligned} \mathscr{M} &= \frac{g}{2V\sqrt{E_2E_3}} \int dx\phi_e(x) \exp\left[i(k_2 + k_3)x\right] \\ \omega &= \frac{g^2}{(2\pi)^3}|\tilde{\phi}_e(P)|^2 dP I_2(P^2, M_\psi^2, M_\psi^2) \end{aligned} \quad (4.26)$$

where $\tilde{\phi}_e$ is the Fourier transform of the external source $\phi_e(x)$ and I_2 is the two-particle phase space in Eq. (4.8). With normalisation such that $\int |\tilde{\phi}_e(P)|^2\delta^+(P^2 - a)dP/(2\pi)^4 = 1$ we obtain directly from the distribution-valued limit in Eqs. (3.85), (3.87) that ω agrees with the (negative) real part of the first-order polarisation contribution in Eq. (4.21).

This is just the Kramers-Kronig result for this case: the absorption cross section for the ϕ-field, i.e. the rate of producing ψ-pairs, determines the real part of the dielectricity function while the imaginary part stems from an integral over that quantity, cf. Eq. (4.21).

4.4 The photon propagator in QED and the gluon propagator in QCD

1 *Introduction*

Before we consider the renormalisation process further we will discuss the results for the propagators in QED and QCD corresponding to those in the previous section. We will start with the properties of the polarisation function and methods for calculating the spin-averaged current matrix elements in QED and QCD.

We will use the results from this calculation repeatedly in the book. It is possible to understand the simple structure without ever entering into the complexities of the Dirac spinors if we use

1 helicity conservation,

2 Lorentz covariance,

3 common sense and simple algebra.

Of these only the first item has not been used before. It is a general property, valid for all massless particles with spin, that the spin must always be directed either along the direction of motion of the particle (*positive helicity*) or in the opposite direction (*negative helicity*). This feature was noticed by Wigner, [112], in his fundamental classification of the Lorentz group. Actually we already know from Chapter 2 that a real (massless) photon, which is a quantum of an electromagnetic radiation field ($\mathscr{E}, \mathscr{B}$) with its motion along the Poynting vector $\mathbf{P} = \mathscr{E} \times \mathscr{B}$, has its polarisation plane in a direction transverse to $\mathbf{P}$ (conventionally along $\mathscr{E}$). Its spin component is then either $+1$ or -1 along the Poynting vector direction (remember how the spherical harmonics Y_1^m look for $m = 0, \pm 1$).

The same goes for a massless spin $1/2$ particle and it is also a good approximation when the particle's rest mass can be neglected compared to its energy ($m \ll e$). For a particle with energy of order its restmass it is always possible to go to its restframe and prepare the spin in any suitable direction and then (although some care is needed in the Lorentz transformations of spins, cf. Chapter 14), it will have a definite direction in any other Lorentz frame. In particular a massive spin 1 particle will have three possible values of its spin, $\pm 1, 0$, along any direction.

There is a precise statement that the electromagnetic current matrix element between an incoming electron and an outgoing electron vanishes (if we neglect the electron's mass) unless they have the same helicity. This is evidently also true for the massless q- and $\bar{q}$-particles in QCD. The implication is that *QED and QCD interactions conserve the helicity of massless charged particles* or in other words the current only couples to the transverse degrees of freedom of the vector potential.

2 *The vector nature of the field theories QED and QCD*

The two major differences between QED (QCD) and the simple scalar version we discussed in section 4.3 are that *QED and QCD are vector theories*, which means that all the operators carry Lorentz vector or tensor indices and that *they have different dimensional properties.*

The fact that the currents are conserved also means restrictions on the different operator matrix elements. In particular the polarisation distribution will in this connection be a tensor, $\rho^{\mu\nu} \equiv \rho^{\mu\nu}(x)$, where

$$\rho^{\mu\nu}(x) = \langle 0 | \mathscr{T} \{ j^{\mu}(x) j^{\nu}(0) \} | 0 \rangle \tag{4.27}$$

which in space-time and energy-momentum space must fulfil

$$\partial_{\mu} \rho^{\mu\nu} = \partial_{\nu} \rho^{\mu\nu} = 0 \quad \Rightarrow \quad q_{\mu} \tilde{\rho}^{\mu\nu} = q_{\nu} \tilde{\rho}^{\mu\nu} = 0 \tag{4.28}$$

because it is constructed from conserved currents.

There is only one Lorentz-covariant tensor fulfilling Eq. (4.28) that can be built from a *single* vector q; its Fourier transform has the shape

$$\tilde{\rho}_{\mu\nu} = (q^2 g_{\mu\nu} - q_{\mu} q_{\nu}) \tilde{\rho}(q^2), \quad \tilde{\rho}(q^2) = \alpha \tilde{\rho}'(q^2) \tag{4.29}$$

In this way we have defined the *polarisation function* $\tilde{\rho}$ and in the second equation indicated that it is proportional to the fine structure constant, i.e. the squared electric coupling constant $\alpha = e^2/4\pi$. As well as having tensor indices $\tilde{\rho}_{\mu\nu}$ must be expressible in a Källén-Lehmann representation because it fulfils all the requirements needed to derive Eq. (4.4) (note in particular that the current is a real operator). Therefore it should be possible to write for the polarisation function

$$\tilde{\rho}(q^2) = \int \frac{da \sigma(a)}{a - q^2 - i\epsilon} \tag{4.30}$$

where the *polarisation weight function* $\sigma(a)$ stems from the sum over intermediate states with squared mass a. (We note that it is in this case also necessary to be able to sum over the spin of the quanta in these states and we will devise methods for that in the next subsection.)

Further the free photon propagator is

$$\langle 0|\mathscr{T}\{A_\mu(0)A_\nu(x)\}|0\rangle_0 = \frac{i}{(2\pi)^4}\int dq \exp(iqx)\tilde{D}_0(q^2)(g_{\mu\nu} + \text{g.t.}) \quad (4.31)$$

$$\tilde{D}_0(q^2) = \frac{1}{q^2 + i\epsilon}$$

The notation g.t. stands for gauge terms and we have used the conventional notation D_0 for the photon propagator in QED. We have already, in Chapter 2, pointed out that owing to gauge invariance it is possible to make the change $A_\mu \to A_\mu + \partial_\mu \Lambda$ without changing the physical results in any calculation. This is due to the fact that the interaction term can be expressed as follows:

$$\int d^4x g j^\mu(x) A_\mu(x) \to \int d^4x g j^\mu(x) A_\mu(x) - \int d^4x g \Lambda(x)\partial_\mu j^\mu(x) \quad (4.32)$$

On the right-hand side we have performed a partial integration and we find that the added gauge term vanishes owing to current conservation. Evidently gauge invariance and current conservation are intimately connected! Depending upon the gauge choice there are different tensor-indexed contributions to the gauge term g.t. in Eq. (4.31) but *when the field and its propagator are coupled to a conserved current we can ignore these terms.*

The second difference between the simple scalar version and the full QED is the dimensions of the currents. For a scalar field we have already noted that the field operator formally has (positive) energy dimension 1. Therefore the term $:\psi^2:$, which in the last subsection corresponds to the current, has energy dimension 2. In order to obtain the right dimensions for the interaction term it is necessary that the coupling constant, g, multiplying $\phi:\psi^2:$ in the interaction term, also has energy dimension 1. The theory is then super-renormalisable, according to the introduction to this chapter.

For QED and QCD (fermion) currents, which are constructed from Dirac operators, we have instead an energy dimension 3. This means that the coupling constant in Eq. (4.32) is dimensionless and also that *the polarisation tensor has energy dimension* 2 in this case. It corresponds to the matrix element in Eq. (4.27). Comparing to Eq. (4.17) we note that the (positive) energy dimension 6 of the coordinate space $\rho^{\mu\nu}$ is after Fourier transform changed to 2, for $\tilde{\rho}^{\mu\nu}$.

This means that the quantity $\tilde{\rho}$ in Eq. (4.29) is dimensionless and it is also obviously a Lorentz invariant and has a Källén-Lehmann representation. We will now provide a more detailed expression for this quantity.

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3 The current matrix elements

In order to obtain the correspondence to Eq. (4.21) for the quantity $\tilde{\rho}$ in Eq. (4.29) we need a method to sum over the spins in the intermediate states. We start with the contribution to the polarisation tensor from the lowest-mass state. We need the matrix element between the vacuum state and any state containing an electron-positron pair, $\langle k_1, k_2 | j_\mu | 0 \rangle$. Then we may define the sum over the spin states of the tensor γ (we will only write out tensor indices when it is necessary to avoid confusion):

$$\gamma_{\nu\mu} = \sum_{spin} \langle 0 | j_\nu | k_1, k_2 \rangle \langle k_1, k_2 | j_\mu | 0 \rangle \tag{4.33}$$

It is useful to introduce the *reduced matrix element*, denoted by $\{\}$:

$$\langle k_1, k_2 | j_\mu | 0 \rangle \equiv \frac{1}{2V\sqrt{k_{10}k_{20}}} \{k_1, k_2 | j_\mu | 0\} \tag{4.34}$$

i.e. we take out the 'ordinary' volume and energy factors from the matrix element. This means that the energy dimension of the reduced matrix element is 1. We obtain the corresponding tensor γ^r (which is Lorentz-invariant due to our conventions in the definition of the weight function in Eq. (4.4) and has energy dimension 2) in terms of these reduced matrix elements:

$$\gamma = \frac{1}{4V^2 k_{10}k_{20}} \gamma^r \tag{4.35}$$

We note that, in order to keep the current conservation condition, γ and therefore also γ^r must fulfil

$$q^\mu \gamma^r_{\nu\mu} = q^\nu \gamma^r_{\nu\mu} = 0 \tag{4.36}$$

with $q = k_1 + k_2$. Further, due to the fact that electromagnetic interactions are parity conserving it must be constructed directly from the vectors k_1, k_2 or from the $g_{\mu\nu}$. This means that γ^r must be constructed from the two tensors T_j, $j = 1, 2$ because these are the only independent combinations that fulfil Eq. (4.36):

$$T_{1\mu\nu} = g_{\mu\nu}q^2 - q_\mu q_\nu, \quad T_{2\mu\nu} = (k_1 - k_2)_\mu (k_1 - k_2)_\nu \tag{4.37}$$

In order to have the right energy dimension, γ^r must then be a linear combination of the T's with coefficients which are dimensionless:

$$\gamma^r = uT_1 + wT_2 \tag{4.38}$$

If the coefficients u, w are to be Lorentz-invariant they can only depend upon the available Lorentz invariants k_1^2, k_2^2, k_1k_2 and if they are to be dimensionless then the dependence must be upon the ratios of these three quantities. For massless particles they must then be plain numbers and,

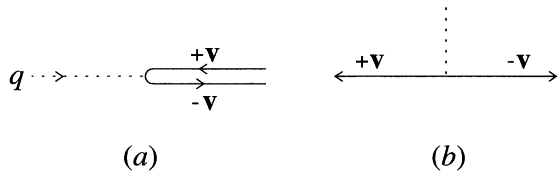

Fig. 4.4. The Breit frame and the cms description of an electron coming in and bouncing back due to a momentum transfer q and an electron-positron pair going apart, respectively.

unless the theory is very singular when the mass approaches 0 (which it is not in this connection), then u, w *must be plain numbers in the general case, too.*

Before we continue we also consider the matrix element $\langle k_1 | j_\mu | k_2 \rangle$, i.e. the current matrix element between the electron energy-momentum states k_1 and k_2. This will be of interest in connection with lepton scattering, cf. Chapter 5. In that case, in order to calculate the cross section we will need the spin-summed matrix element combination

$$\hat{\gamma}_{\nu\mu} = \sum_{spin} \langle k_2 | j_\nu | k_1 \rangle \langle k_1 | j_\mu | k_2 \rangle \tag{4.39}$$

We may again introduce the corresponding reduced matrix element, defined in an obvious way, and the corresponding Lorentz-covariant tensor $\hat{\gamma}^r$. Current conservation again must hold *but this time we must change the definition of* q *to* $q = k_1 - k_2$.

The fact that with the reduced matrix elements and tensors we obtain the same result for γ^r and $\hat{\gamma}^r$ with the exchange $k_2 \to -k_2$ is obvious for the scalar field theory we discussed in the earlier section. It is called *crossing symmetry*. It takes a little effort to prove that it also works for the vector theories QED and QCD but it is nevertheless true and *it is one of the few very general properties which is valid in any field theory.*

To see that u and w must be equal we take recourse to helicity conservation for a vanishing lepton mass. This means that the current matrix elements only couple to the transverse degrees of freedom of the electromagnetic (four)-potential A. The transverse directions are well defined when the electron and positron go out in opposite directions e.g. along the 3-axis, as they do in the cms, for the tensor γ^r. For the tensor $\hat{\gamma}^r$ the same is true in the so-called Breit frame (see Fig. 4.4). This is sometimes called the 'brick-wall frame' for easily understood reasons, i.e. the electron comes in and after the interaction bounces out again with the same energy backwards along the 3-axis.

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Thus helicity conservation means that if $k_j^2 = 0$ then the 00- and 33-components of the tensors γ^r (in the cms) and $\hat{\gamma}^r$ (in the Breit frame) should vanish. We leave it to the reader to convince him-/herself that this happens if and only if $w = u$.

The fact that the sum $T_1 + T_2 \equiv T_\gamma$ only has tensor components in the directions transverse to the momentum transfer (lepton scattering in the Breit frame) or the production axis (e^+e^- annihilation in the cms) means that all its time components vanish and its space part is proportional to the tensor $t(\gamma)$ (using $\mathbf{k}$ as a vector along one of these directions):

$$t(\gamma)_{jl} = \delta_{jl} - \frac{k_j k_l}{\mathbf{k}^2} \tag{4.40}$$

This space tensor occurs when we use transverse wave solutions to describe a photon (gluon in QCD) with energy-momentum $k = (k_0, \mathbf{k})$, i.e. $\mathbf{A} = \epsilon \exp(ikx)$, and would like to sum over the polarisation directions of the square of the wave function:

$$\sum_{polarisation} \epsilon_k \epsilon_l \equiv t(\gamma)_{kl} \tag{4.41}$$

(note that 'transverse' means that $\mathbf{k} \cdot \boldsymbol{\epsilon} = 0$ and the normalisation comes from the fact that there are two transverse directions). The tensor T_γ is a continuation of $t(\gamma)$ to values of k outside the mass-shell $k^2 = 0$ for a real photon (gluon). The result is a consequence of the relationship between current conservation and gauge invariance, cf. Eq. (4.32).

We may now calculate the polarisation weight function σ, occurring in Eq. (4.30), to lowest order:

$$\begin{aligned}(q^2 g_{\mu\nu} - q_\mu q_\nu)\sigma^{(1)}(q^2) &= (2\pi)^3 \sum \delta(k_1 + k_2 - q)\gamma_{\mu\nu} \\ &= \frac{e^2}{(2\pi)^3} \int dk_1 dk_2 \delta(k_1^2 - M^2)\delta(k_2^2 - M^2) \\ &\quad \times \delta(k_1 + k_2 - q)\gamma^r_{\mu\nu}\end{aligned} \tag{4.42}$$

The simplest way to obtain an expression for $\sigma^{(1)}(q^2)$ is to take the trace of the tensors on both sides of the equation. We note that $\operatorname{tr}(q^2 g_\mu^\nu - q_\mu q^\nu) = 3q^2$ and that $\operatorname{tr}\gamma^r = u(2q^2 + 4M^2)$ (prove that with $u = w$!) and therefore we obtain immediately

$$\begin{aligned}\sigma^{(1)}(q^2) &= \frac{2e^2 u}{3(2\pi)^3}\left(1 + \frac{2M^2}{q^2}\right) \\ &\quad \times \int dk_1 dk_2 \delta(k_1^2 - M^2)\delta(k_2^2 - M^2)\delta(k_1 + k_2 - q)\end{aligned} \tag{4.43}$$

We recognize in the integrand the expression for the polarisation function of the simpler case in Eq. (4.21) (*the two-particle phase space*). It is multi-

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plied by a factor $1+2M^2/q^2$ from the tensor structure (*the spin-correlation factor*) and a different factor $2e^2u/[3(2\pi)^3]$ in front.

The (squared) scalar coupling constant (which due to combinatorics is multiplied by the factor 2 in Eq. (4.21)) is exchanged for $2ue^2/3$. The factor 2/3 stem from the fact that massless fermions only couple to two (the transverse ones) of the three vector degrees of freedom (cf. the discussion of the tensor $t(\gamma)$ in Eq. (4.40)). Therefore the unknown quantity u should equal unity, which is confirmed in more elaborate calculations with the full Dirac formalism.

It is worthwhile to note that the spin-correlation factor, within the large parentheses, contains a term proportional to M^2/q^2 which corresponds to a correction for massive particles. Such terms occur frequently but evidently vanish in the limit of large squared momentum transfer (or cms energy) q^2. They are known as 'higher-twist corrections'.

In this way we obtain the result for the first-order perturbative correction to $\tilde{\rho}$:

$$\tilde{\rho}^{(1)}_{QED} = \frac{\alpha}{3\pi}\int_{4M^2}\sqrt{1-\frac{4M^2}{a}}\left(1+\frac{2M^2}{a}\right)\frac{da}{a-q^2-i\epsilon} \tag{4.44}$$

Before we end this subsection we note that the tensor T_2 defined above can be written solely in terms of the initial electron energy-momentum k $(=k_1)$ and the momentum transfer $q = k_2 - k_1$. Thus

$$k_1 + k_2 = 2k + q = 2(k - (kq/q^2)q) \equiv 2\hat{k} \tag{4.45}$$

which is true for elastic scattering because of the identity

$$k_2^2 = (q+k)^2 = k^2 + 2qk + q^2 \Rightarrow q^2 = -2kq \tag{4.46}$$

when the lepton is on the mass shell before and after the interaction. Note, however, that the vector $\hat{k}$ fulfils $\hat{k}q = 0$ independently of the mass-shell condition. We will meet this vector later in connection with inelastic scattering situations.

4 *Dyson's equation for QED*

Dyson's equation, Eq. (4.18), is for the full photon propagator $\tilde{D}$

$$\tilde{D}_{\mu\nu} \equiv \tilde{D}g_{\mu\nu} + \text{g.t.} = \tilde{D}_0(q^2)g_{\mu\nu} - \tilde{D}_0(q^2)\tilde{\rho}_{\mu\lambda}\tilde{D}^{\lambda}_{\nu} + \text{g.t.} \tag{4.47}$$

From this expression we obtain, using the results of the earlier subsections, the solution

$$\alpha_u\tilde{D} = \frac{\alpha_u}{(q^2+i\epsilon)[1+\alpha_u\tilde{\rho}'(q^2)]} \tag{4.48}$$

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The gauge terms, which do not contribute if the propagator is coupled to a conserved current, may be neglected. We have explicitly exhibited the dependence on the (unrenormalised) coupling α_u according to Eq. (4.44).

There are two features of this result worth pondering:

- Owing to the tensor character and the (energy) dimensions of the polarisation tensor we have in Eq. (4.48) obtained the mass-renormalised photon propagator without the subtraction necessary in Eq. (4.23). *The photon must always be massless and this can be traced back to gauge invariance and current conservation.*

- The function $\tilde{\rho}'$ is defined by a non-convergent integral. This is noticeable for the lowest-order term in Eq. (4.44). A few further terms are known in the perturbation-theoretical expansion of $\tilde{\rho}'$. They exhibit the same sign and scaling behaviour as the one written out in Eq. (4.44). The *sign of the correction term can again be traced back to the positive-definiteness of the corresponding weight function G in Eq. (4.5)*, i.e. to the fact that we obtain positive contributions from the real intermediate states in the weight function.

Before we perform the necessary renormalisations for QED we will consider the differences for the equations derived above in QCD. In this case the current coupling to the gluon propagator contains contributions both from the quark-antiquark currents and from the field self-interaction, the *three-gluon vertex coupling* (there is also a 'local' four-gluon vertex necessary to keep to the symmetries of the theory but it does not change the conclusions). This field self-interaction is different because it corresponds to a coupling between three *vector* particles. We will find that this contribution means a large difference between the polarisation function in QCD and that in QED, where there is no such interaction possible between the chargeless photons (although they also are vector particles).

The fermion contribution is the same as we have met before. Thus the $q\bar{q}$ intermediate state will give a contribution per flavor (evidently each flavor provides an independent contribution) equal to the result in Eq. (4.44) with the exchange $\alpha_{QED} \rightarrow \alpha_s/2$. The factor 2 is due to an unfortunate convention in the normalisation of the QCD coupling constant and we will meet it further on also.

For the gluonic contributions to the weight function we find the surprising result that the total contribution is no longer positive, [68]. This is very disturbing because we have repeatedly pointed out that the definite sign in the Källén-Lehmann description of the polarisation function stems from the fact that we sum over positive contributions from the intermediate physical states. Depending upon the gauge choice there are different ways

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to obtain the result but the gauge-independent result is a function with the same properties as in Eq. (4.44) although with the opposite sign.

4.5 Two reasons why in QCD the polarisation tensor behaves differently; the introduction of cut diagrams

In this subsection we will provide two ways of getting an intuitive understanding for the negative contributions to the polarisation weight function for the gluons (in subsection 2 of the next section we will present a third way to see the difference between QED and QCD bremsstrahlung emission). At the same time it will provide us with the possibility of introducing higher-order corrections, such as the *vertex corrections* (usually termed 'virtual corrections'), in a natural way. In order to clarify the relationship between a Feynman diagram and the weight function of its Källén-Lehmann representation we will define the notion of *cut diagrams.*

The first argument for the behaviour of σ_{QCD} in Eq. (4.29) is that the negative contributions stem from a lack of phase space for the real emitted gluons in the intermediate states. The second reason we provide is that there is a difference between the states containing transversely polarised gluons and those containing Coulomb interaction gluons.

According to the first argument, when we calculate to a certain order of perturbation theory and two gluons are emitted too close in phase space (i.e. too close in angle or rapidity) then they will be reabsorbed into a single gluon again, at the next order. This is at the specified order noticeable as an available phase space for real gluon emission and as a *larger* phase space for the absorption, i.e. for the virtual corrections to this emission process. *This will result in a negative contribution* to the polarisation weight function σ in Eq. (4.30). (The implication is that the theory should be formulated in terms of 'effective gluons', which are not reabsorbed; we will do that in sections 18.5 and 18.6, where we introduce an approximation method called discrete QCD.)

For the second argument we note that the Coulomb gluons are not real degrees of freedom to be quantised in the QCD field (there is always a Coulomb field around any gauge theory charge). If, nevertheless, the interactions with the Coulomb fields are incorporated into the Feynman diagrammatical description then the occurrence of Coulomb gluons in a state provides negative contributions to the state sum (they have a negative metric in the Hilbert space of the states, cf. the Gupta-Bleuler formalism in e.g. [30]). Therefore the weight function σ in the Källén-Lehmann representation does not need to provide positive contributions from the states containing Coulomb gluons (needless to say the two descriptions of the phenomena are equivalent!).

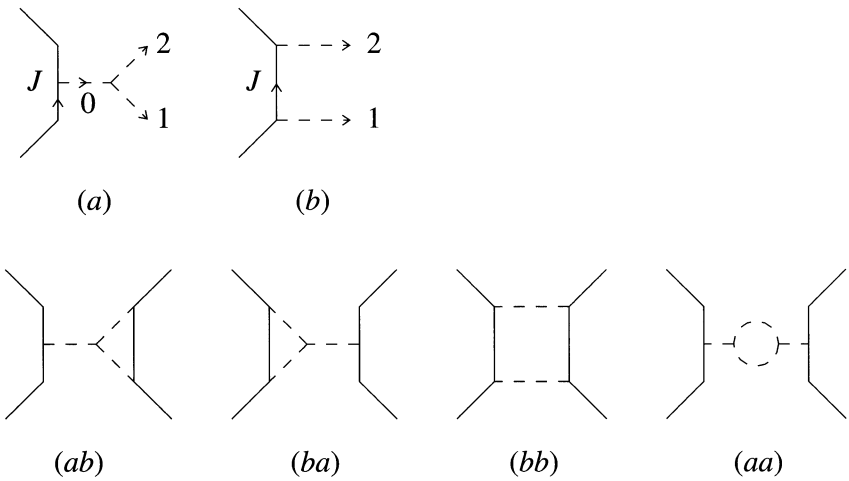

Fig. 4.5. The diagrammatic description of the matrix elements between the current and the two gluons: the contributions $\mathscr{J}_k, k = a, b$, and the three contributions to the polarisation function obtained by squaring the matrix element $\mathscr{J}$ (note that there is a second symmetric vertex contribution in C_{ab}).

In order to relate to the QED calculations above we consider a current emitting two gluons g_1, g_2. (Gluon bremsstrahlung emission is treated in great detail in Chapters 16, 17 and 18.) We consider the process in a transverse gauge, i.e. with the $g_j, j = 1, 2$ polarised e.g. transverse to the current direction. This process can occur according to perturbative QCD diagrams in two different ways. Either there is a 'first' emission of a gluon: $J \to Jg_0$, with J the current. Afterwards the gluon decays via the three-gluon vertex as $g_0 \to g_1 g_2$, see Fig. 4.5(a). This is similar to the decay into a (fermionic) e^+e^--pair of a massive (i.e. off-shell) photon stemming from the emission of a QED current. *But this time we are dealing with two vector particles in the final state and this makes a major difference.*

There is, to the same order in perturbation theory, a second way to emit the gluons, i.e. sequentially as $J \to Jg_1 \to Jg_1 \oplus g_2$, see Fig. 4.5($b$). The total matrix element for the bremsstrahlung emission is therefore a sum of two contributions, $\mathscr{J} \equiv \mathscr{J}_a + \mathscr{J}_b$ (in easily understood notation). As the contribution to the polarisation weight function contains the absolute square $|\mathscr{J}|^2$ there will be in principle three contributions, two from real gluon emission, $C_{aa} = |\mathscr{J}_a|^2$, $C_{bb} = |\mathscr{J}_b|^2$, and a correction from the interference term $C_{ab} = 2\,\mathrm{Re}(\mathscr{J}_a^\star \mathscr{J}_b)$. A closer examination tells us, however, that (see Figs. 4.5(aa), (bb), (ab) and (ba)) only C_{aa} and C_{ab} correspond to corrections to the polarisation function of second order in the coupling constant.

To clarify this statement we note the simple relationship between the weight function and the polarisation function in the Källén-Lehmann representation. The weight function will contain the square of the matrix elements (obtained in a certain perturbative order) between the initial state and a state containing some particular on-the-mass-shell configuration, e.g. the particles $p_1, \ldots, p_n$. For the case discussed above the initial state is a current in the vacuum and the intermediate state contains also a two-particle state, which may be emitted from the current.

If the matrix element contains several terms, each leading to this state, then we must consider the overlap of all the terms. To obtain the sum over the intermediate state it is necessary to consider the product of one term, say $\mathscr{J}_a$, and the complex conjugate of another term, say $\mathscr{J}_b^\star$, etc. All these overlap integral terms can be considered as diagrammatic contributions as exhibited in Fig. 4.5. But we note that C_{bb} in this way corresponds to two-gluon exchange for the current, i.e. it is not part of the corrections to single-gluon emission.

The difference between the weight function and the polarisation function is that the weight function is obtained by putting the intermediate state on the mass shell, i.e. each line corresponds to $\delta^+(p^2 - m^2)$, while the polarisation function corresponds to using the corresponding Feynman propagator $(p^2 - m^2 - i\epsilon)^{-1}$. Actually we are again invoking the distribution-valued relationship obtained in Eqs. (3.85), (3.87). The operation of introducing δ-distribution(s) instead of propagator(s) is called *cutting the diagrams* and we will meet this notion later on in the book.

We will now consider the contributions in more detail, using the transverse gauge. We assume that the two gluons g_1 and g_2 are emitted with compensating transverse momenta $\pm\mathbf{k}_\perp$ with respect to the polarisation direction. Further we assume that their combined squared mass a (corresponding to the 'virtuality' of g_0 and to the a-variable in Eq. (4.44)) is very large, $a \gg \mathbf{k}_\perp^2$. Then the available rapidity region for the emission in the contribution C_{aa} is $\Delta y = \log(a/\mathbf{k}_\perp^2) - 11/6$. The result (including the peculiar number $11/6$) is further clarified in section 18.5.

There are two comments on the result. The first is that this is evidently a large rapidity region, growing logarithmically with $a/\mathbf{k}_\perp^2$, and secondly it is a result typical of vector emission. If we consider the emission of massless fermions, i.e. the contribution $g_0 \to q\bar{q}$, then *there is no such logarithmic contribution to the available rapidity region.*

The difference is that if we emit two spin $1/2$ particles from a vector then helicity conservation (cf. section 4.4 above) implies that they would like to be close together in phase space (to make $1/2+1/2=1$ with respect to the helicity states). Then the contribution to the weight function is constant

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for large values of a, as seen in Eq. (4.44) (the result $\alpha/3\pi$ means that the effective rapidity difference will be 2/3 as we will see in detail in section 18.5). But for the vector emissions the final-state vector gluons must go in different directions to conserve the helicity. Therefore *vectors will tend to spread apart in rapidity space*. A more precise mathematical statement is that the (relative) rapidity (y) dependence for a given $k_\perp$ is proportional to dy for the vector (gg) emissions and to $dy\exp(-y)$ for the $q\bar{q}$ emission.

The vector emission contribution will therefore provide a factor proportional to the available rapidity region, i.e. it grows logarithmically with the integration variable a in the Källén-Lehmann representation. It is not difficult to see that for states containing more gluons there will be logarithmic factors with a power growing with the number of gluons in the intermediate state.

It is nevertheless a fact that QCD is renormalisable (although t'Hooft, who was first to provide the proof, had to work very hard!). The reason is that the logarithmic rapidity-difference term from C_{aa} is cancelled by the C_{ab} corrections, the 'vertex corrections'. If we calculate the interference term C_{ab} in the transverse gauge we find that, just as for the gluon emission in C_{aa}, it depends upon the rapidity difference $\delta y = \log(a/\mathbf{k}_\perp^2)$. *It will provide a contribution δy with the opposite sign to the contribution Δy of the emission term C_{aa}*. Therefore to this order in the coupling constant (and it can be shown to all orders, too, which actually is necessary for the renormalisability property) there is no δy-dependence in the weight function of the polarisation tensor in QCD.

There is, however, the term $-11/6$ left over from combining the vector emission and vertex correction terms and this really has the meaning, according to section 18.5, that there is a depletion of gluon emission close to an already emitted gluon. Therefore the gluon contribution to the polarisation weight function in QCD will for large a-values go to a constant, just as do the fermionic contributions $(N_c\alpha/2\pi)(-11/6)$, with $N_c = 3$ the number of colors, cf. section 18.6, subsection 1.

Another way to understand this result is to note that every charged particle is surrounded by a Coulomb field and this also goes for the gluonic (octet) charges. As soon as we produce a 'physical transverse' gluon then it is necessary to handle the interaction between this gluon and its Coulomb field. Therefore gluons in QCD do not behave like the photons described by the method of virtual quanta (MVQ) (cf. section 2.5). The gluons are not independent of the fields, i.e. they will reinteract on the way out. Actually such Coulomb vector particle interactions do not provide positive-definite contributions to the Källén-Lehmann weight function because the wave functions are not positive-definite in the state space. We may intuitively say that in order to be able to have room for the vector Coulomb fields the two vectors must have an effective rapidity

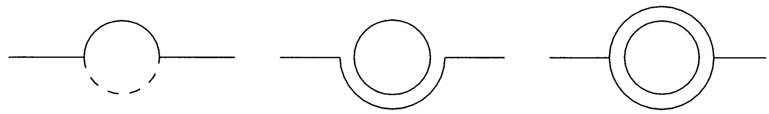

Fig. 4.6. (*a*) The diagrammatic description of a self-energy contribution, i.e. in QCD the (color-3) q-field propagator (full line) turns into a qg-state firstly emitting and afterwards absorbing the color-8 g-quantum (broken line); (*b*) the color flow in diagram (*a*); (*c*) the corresponding color flow in a $g \to gg$ intermediate state.

difference $-11/6$. In the last subsection of this chapter we provide one further intuitive picture of the result, this time related to one of its major implications, asymptotic freedom.

1 *The color factors of QCD*

In the last subsection we considered the emission of gluon states from a QCD current but we did not specify the current in any detail. Suppose, however, that the current is a quark current so that we consider the emission of color-8 gluons from a color-3 current. Then there is a subtle but necessary color factor correction in the sum over colors in the squared matrix element. To see this we consider Fig. 4.6(*a*), which is a self-energy correction corresponding to Eq. (4.25), i.e. a quark q (propagator) fluctuates into a qg-state and back again.

We may compare that to the situation when a gluon decays into two gluons and afterwards rearranges into a single gluon as in Fig. 4.5(*aa*). In both cases we find that there is principally a new color produced, i.e. we may draw the color lines as in Figs. 4.6(*b*) and (*c*) with a closed color ring in the middle.

For the q-state we note that we start out and end in a coherent color-3 state, containing r, g and b. For the sake of argument we may project e.g. onto the initial state color r. In the intermediate state this color-3 can then turn into a g or b by the emission of a $r\bar{g}$ or a $r\bar{b}$, which are both true color-8 states, i.e. they correspond to the gluon in the intermediate state.

But if the gluon emission corresponds to $r\bar{r}$ then there is a color-coherence suppression factor because only two out of the three possible states are really color octets. The third color combination is a color singlet, i.e. not a gluon. Therefore only $2 + 2/3$ of the possible 3 choices are really gluons. We obtain a factor $3 - 1/3 \to N_c - 1/N_c$ multiplying

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the kinematical matrix element instead of the expected factor N_c from the closed color ring, with N_c the number of colors.

For the gluon propagator we are, however, reassured that due to the coupling we always obtain a true gluon and therefore the relative color weight between the two states is $1 - 1/N_c^2 = 8/9$. This is also the relative coupling between the emission of gluons from q- or $\bar{q}$-currents and the emission from a g-current.

2 *The operations in multiplicative renormalisation*

We will in this subsection exhibit the way one can rearrange the propagator equations by means of a *multiplicative renormalisation scheme*. One basic assumption for what we are going to do is that we already have performed mass renormalisation for the photon (gluon) propagator. We have seen that within QED this is trivially possible by making explicit use of gauge invariance and current conservation for the photon propagator in Eq. (4.48), and there is a correspondence in QCD.

We will start with the result in Eq. (4.48) and note that we may rearrange it in the following way:

$$
\begin{aligned}
\alpha_u \tilde{D} &= \frac{1}{(q^2 + i\epsilon)} \frac{1}{\{[1/\alpha_u + \tilde{\rho}'(-\mu^2)] + [\tilde{\rho}'(q^2) - \tilde{\rho}'(-\mu^2)]\}} \Rightarrow \\
\alpha_\mu \hat{D}_\mu &= \frac{1}{(q^2 + i\epsilon)} \frac{1}{\{1/\alpha_\mu + \hat{\rho}'(q^2, \mu^2)\}} \Rightarrow \\
\hat{D}_\mu &= \frac{1}{(q^2 + i\epsilon)} \frac{1}{\{1 + \alpha_\mu \hat{\rho}'(q^2, \mu^2)\}}
\end{aligned} \tag{4.49}
$$

with

$$
\begin{aligned}
\frac{1}{\alpha_\mu} &= \frac{1}{\alpha_u} + \tilde{\rho}'(-\mu^2) \\
\hat{D}_\mu &= Z_{3,\mu^2}^{-1} \tilde{D} \\
Z_{3,\mu^2} \alpha_u &= \alpha_\mu \\
\hat{\rho}'(q^2, \mu^2) &= (q^2 + \mu^2) \int_{4M^2} \frac{da\sigma(a)}{(a + \mu^2)(a - q^2 - i\epsilon)} \\
\sigma^{(1)}(a) &= \frac{\alpha}{3\pi} \sqrt{1 - \frac{4M^2}{a}} \left(1 + \frac{2m^2}{a}\right)
\end{aligned} \tag{4.50}
$$

In Eq. (4.49) we have item by item rearranged the unrenormalised quantities $\alpha_u, \tilde{D}, \tilde{\rho}'(q^2)$ so that only the renormalised correspondences (defined at the effective frequency $q^2 = -\mu^2$) $\alpha_\mu, \tilde{D}_\mu, \tilde{\rho}'(q^2, \mu^2)$ occur. In the last line of Eq. (4.50) we have written out the lowest-order approximation to the weight function σ occurring in the Källén-Lehmann representation for the polarisation function (cf. Eq. (4.30)).

What we have achieved by these operations can be formulated in the following way:

R1 We have introduced a *coupling constant* α_μ, *renormalised at the scale* μ^2, by multiplying the unrenormalised coupling constant α_u by the quantity Z_{3,μ^2}, which is formally defined by

$$Z_{3,\mu^2} = 1 - \alpha_\mu \tilde{\rho}'(-\mu^2) \tag{4.51}$$

R2 We have introduced a *propagator* $\hat{D}_\mu$ *renormalised at the scale* μ^2, by multiplying the (unrenormalised) photon propagator $\tilde{D}$ by the inverse quantity Z_{3,μ^2}^{-1}. This is equivalent to exchanging the unrenormalised photon-field operator for a new scaled operator, $A_\nu^{(\mu^2)}$, the *renormalised photon operator*:

$$A_\nu^{(u)} \to A_\nu^{(u)} = A_\nu^{(\mu^2)} Z_{3,\mu^2}^{1/2} \tag{4.52}$$

Note that the 'size' of a field operator is not observable. The only requirement is that the propagator, i.e. the expectation value of the square of the field operator in the vacuum state, should correspond to the contribution from a single massless quantum at the renormalisation scale $q^2 = -\mu^2$.

R3 We have defined all the new renormalised quantities by a subtraction at the arbitrary (negative) value $q^2 = -\mu^2$. We could, of course, also have done it at $\mu^2 = 0$ or any other value $q^2 < 4M^2$ such that our integrals converge. If we make the exchange $\mu^2 \to \mu_1^2$ we can again do all the changes in the same way and obtain a new set, $\alpha_{\mu_1}, A_\nu^{(\mu_1^2)}, Z_{3,\mu_1^2}$, which is related to the old one by the same equations. The fact that we may do repeated changes of scale $\mu_1^2 \to \mu_2^2 \to \cdots \to \mu_n^2$ and still end up with the same μ_n^2-dependent renormalised quantities means that there is a group character to the procedure, that of the renormalisation group.

R4 In particular, if we chose to define α at the point $\mu^2 = 0$ we would find for the fine structure constant the well-known value $\alpha \equiv \alpha(0) \sim 1/137$, which is observed for static interactions. It is of interest to note that at LEP with a (cms-)energy value of the annihilating e^+e^--pair ~ 90 GeV one obtains an effective coupling $\alpha(90) \simeq 1/128$. This is in accordance with this finite renormalisation group prediction of a change in α of $0 \to 90$ GeV.

In this way we have exhibited in some detail the procedure for redefining the photon field, the photon propagator and the coupling constant in QED. Both within QED and QCD there are other undefined quantities and also other integrals which need a redefinition in the same way as the photon operator in Eq. (4.52). In particular the fermion operators will need a renormalisation such that $\psi^{(u)} \to Z_{2,\mu^2}^{-1/2}\psi^{(\mu^2)}$, with the conventional wave function renormalisation constant Z_{2,μ^2}. The quantity $Z_{2,\mu^2}^{-1/2}$ is defined at the same effective frequency $q^2 = -\mu^2$ as that of the photon, Z_{3,μ^2}, and in this way the renormalised operator $\psi^{(\mu^2)}$ will describe a single quantum at this frequency. There is finally the vertex renormalisation constant, conventionally called Z_{1,μ^2}, which corresponds to a renormalisation $\Gamma^{(u)} \to Z_{1,\mu^2}\Gamma^{\mu^2}$ of every Feynman graph vertex, so that $\Gamma^{\mu^2} = 1$ for the particular momentum transfer $q^2 = -\mu^2$ at the vertex.

One essential result in QED, which also has a correspondence in QCD, is called the Ward identity: $Z_{1,\mu^2} = Z_{2,\mu^2}$. This relation stems from the current conservation and gauge invariance properties of the theory. The consequence is that for every vertex in QED for which two fermion operators and one photon operator are connected, one obtains the rescaling factor $Z_3^{-1/2}(Z_2^{-1/2})^2$. At the same time the vertex itself provides the rescaling Z_1 and the coupling constant the rescaling $Z_3^{1/2}$ according to the third line in Eq. (4.50). In this way we obtain the result that at every vertex there is a renormalised contribution $Z_3^{-1/2}(Z_2^{-1/2})^2 Z_1 Z_3^{1/2} \equiv 1$. *The result is that all the Feynman diagrams in the theory will have the same form of expression as before but now in terms of renormalised propagators and charges.* And now everything is finite (although μ^2-dependent)!

Thus the result is that if we consider a scattering situation or a multiparticle production diagram in QED containing n_γ in- or out-going photons and n_ψ in- or out-going Dirac particles, which in the unrenormalised form looks like

$$F^{(u)}_{n_\gamma,n_\psi}(k_1,\ldots,k_{n_\gamma};p_1,\ldots,p_{n_\psi};\alpha_u) \tag{4.53}$$

it will after renormalisation look like (note that the 'external' renormalisations are not cancelled!)

$$Z_1^{-n_\psi/2}Z_3^{-n_\gamma/2}F_{n_\gamma,n_\psi}(k_1,\ldots,k_{n_\gamma};p_1,\ldots,p_{n_\psi};\alpha) \tag{4.54}$$

While the quantity in Eq. (4.53) is independent of the renormalisation point, the one in Eq. (4.54) will contain a μ-dependence both in the scaled out Z-factors, in the renormalised coupling constant and in all the renormalised propagators. We will not in this book go into further details of the renormalisation process because we do not need it here. The

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formalism can be found in any field theory text-book and does not provide much more physical insight than the results we have already encountered.

4.6 The Callan-Symanzik equations for the renormalisation group

1 *The equations and what they imply for QCD*

There is evidently nothing sacred about the particular value μ we have chosen in connection with the renormalisation procedure described in the last section. The quantity μ can be varied at will (within the region of no singularities). Therefore we can formulate the dependence upon μ easily. The *unrenormalised function F in Eq. (4.53) is independent of* μ:

$$\mu \frac{dF^{(u)}}{d\mu} = 0 \tag{4.55}$$

If we rewrite this in terms of the renormalised function we obtain immediately a partial differential equation:

$$\left(\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial \alpha} - \frac{n_\gamma \gamma_\gamma}{2} - \frac{n_\psi \gamma_\psi}{2} \right) F = 0 \tag{4.56}$$

with

$$\beta = \mu \frac{\partial \alpha}{\partial \mu}, \quad \gamma_\gamma = \mu \frac{\partial \log Z_3}{\partial \mu}, \quad \gamma_\psi = \mu \frac{\partial \log Z_1}{\partial \mu} \tag{4.57}$$

In the partial differentiations of Eq. (4.57) the unrenormalised coupling constant α_u and any cutoff parameters used in order to make the integrals finite should be kept fixed. Equation (4.56) is the *Callan-Symanzik equation*, [108] and it connects different possible renormalisation points (note that as it may contain several related functions it may be of a matrix character). The β-function in Eq. (4.57) for QED is thus, in the lowest order of perturbation theory, given by differentiating the following expression (cf. the first line of Eq. (4.50)),

$$\alpha = \frac{\alpha_u}{1 + \alpha_u \tilde{\rho}'(-\mu^2)} \tag{4.58}$$

with

$$\begin{aligned} \tilde{\rho}'(-\mu^2) &= \frac{1}{3\pi} \int_{4M^2}^{\Lambda^2} \sqrt{1 - \frac{4M^2}{a}} \left(1 - \frac{2M^2}{a}\right) \frac{da}{a + \mu^2} \\ &\simeq \frac{1}{3\pi} \log \left(\frac{\Lambda^2}{\mu^2} \right) \end{aligned} \tag{4.59}$$

where we have assumed that $4M^2 \ll \mu^2 \ll \Lambda^2$ so that we may neglect all the dependence upon the finite-mass parameters and only keep the logarithmic singularity of the integral.

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It is then easy to see that

$$\beta_{QED}^{(1)} = \frac{2\alpha^2}{3\pi} \tag{4.60}$$

In general it is evident that in the limit in which we only keep the (logarithmically) divergent terms then *all the functions β and γ, defined in Eq. (4.55), are solely functions of the renormalised coupling constant.*

The β-function in QCD can also be calculated and one obtains to the same order as in Eq. (4.60) the result, [68],

$$\beta_{QCD}^{(1)} = -\alpha^2 \left(\frac{11}{4\pi} - \frac{n_f}{6\pi} \right) \tag{4.61}$$

We note the different signs in front of the squared coupling constants for QED and for QCD (at least as long as there are less than 16 flavors!).

We will end this subsection by solving the Callan-Symanzik equations for the two cases of QED and QCD. We will use the following notation for the β- and γ-functions:

$$\begin{aligned} \beta_{QED} &= b_e \alpha^2, \quad \beta_{QCD} = -b_c \alpha^2 \\ \gamma_e(\alpha) &= \quad d_e \alpha, \quad \gamma_c(\alpha) = d_c \alpha \end{aligned} \tag{4.62}$$

where b_e, b_c are positive numbers. The choice for γ, that it is linear in α, is the case we are going to use in Chapter 19 when we encounter the following Callan-Symanzik equation:

$$\left(\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial \alpha} - \gamma \right) F(\log(Q^2/\mu^2), \alpha) = 0 \tag{4.63}$$

We here assume that the distribution F depends (logarithmically) upon a single scaled Lorentz invariant variable Q^2/μ^2 and upon the coupling constant α and we neglect all other dimensional scales in the problem, such as e.g. mass thresholds etc. (cf. [102] where possible observables stemming from the contributions from the mass thresholds are given).

The variable q^2 we used before for the propagators is related to the variable $Q^2 = -q^2$, i.e. we assume that the (Lorentz-invariant) function $F = F(q)$ is taken for large spacelike energy-momentum vectors.

The Callan-Symanzik equations are linear partial differential equations of a kind which occurs very often both in physics and in other disciplines. They are usually called *gain-loss equations*. They correspond to situations when a distribution changes in 'time', which here corresponds to

$$t = \log \mu, \tag{4.64}$$

by a gain term, in this case γ times the value of the distribution, and by a loss term, in this case β times the derivative of the distribution with respect to some variable, here the coupling constant α.

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As an example, taken from Coleman's Erice Lectures, [45], assume that the distribution F corresponds to the density of a population of bacteria moving with a fluid along a pipe. The fluid has velocity $\beta(\alpha)$ with α a position coordinate along the pipe. As they move along there is a changing illumination γ, which determines their rate of reproduction.

We will later consider another example corresponding to the increase in gluon multiplicity and phase space size due to the change in the resolution scale of a parton cascade, cf. Chapter 18. There is a very simple way, called the method of rays, by means of which we can solve this kind of equation.

We start by noting that if we define the *effective coupling constant*, α_{eff}, by means of the equation

$$\frac{d\alpha_{eff}}{dt} = \beta(\alpha_{eff}) \tag{4.65}$$

in terms of the variable t in Eq. (4.64) then Eq. (4.63) becomes

$$\left(\frac{d}{dt} - \gamma(\alpha_{eff})\right) F(L_Q - 2t, \alpha_{eff}(t)) = 0 \tag{4.66}$$

We have then used the notation $L_Q = \log Q^2$ and rearranged the dependence upon α into a dependence upon the effective coupling constant. The earlier partial differential equation is in this way changed into an ordinary differential equation with a t-dependent $\alpha \equiv \alpha_{eff}$.

This means that the quantity $\mathscr{F}$ is a constant

$$\mathscr{F} = F(L_Q - 2t, \alpha_{eff}) \exp\left[-\int^t dt' \gamma(\alpha_{eff}(t'))\right] \tag{4.67}$$

along all 'rays'; those correspond to the solutions for Eq. (4.65).

For QCD and QED we may construct these rays from Eq. (4.62):

$$\alpha_{eff,QED} = \frac{1}{c_e - b_e t}, \quad \alpha_{eff,QCD} = \frac{1}{c_c + b_c t} \tag{4.68}$$

where c_e, c_c are constants. The main property is that independently of the value of the constant c_c if we choose the scale $\mu = \exp t$ sufficiently large *then for QCD the effective coupling constant will vanish but for QED the effective coupling will instead increase with* μ (cf. the result in connection with the LEP experiments in remark R4 above).

This means that the exponential factor in Eq. (4.67) is for QCD given by (introducing the expression for $\gamma(\alpha_{eff}(t))$ from Eq. (4.62))

$$\exp\left[-\int^t dt' \gamma(\alpha_{eff}(t'))\right] = (c_c + b_c t)^{-d_c/b_c} \equiv \left[\alpha_{eff\,QCD}(t)\right]^{d_c/b_c} \tag{4.69}$$

and for QED with obvious changes there is a corresponding result.

Now, let us assume that we would like to know the function F for some scale corresponding to μ_1, where the coupling constant is α_1. From Eq.

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(4.67) we may then immediately write for the QCD case

$$F(\log(Q^2/\mu_1^2, \alpha_1)(\alpha_1)^{d_c/b_c}$$
$$= F(\log(Q^2/\mu^2), \alpha_{eff,QCD}(t)) \left[\alpha_{eff,QCD}(t)\right]^{d_c/b_c} \tag{4.70}$$

In particular there is nothing to stop us from choosing the scale $t = L_Q/2$. From this we conclude that

$$\lim_{Q^2 \equiv \mu^2 \to \infty} \left\{ F(\log(Q^2/\mu_1^2, \alpha_1) \right\}$$
$$= \lim_{Q^2 \to \infty} \left\{ F(0, \alpha_{eff QCD}(L_Q/2)) \left[\frac{\alpha_{eff,QCD}(L_Q/2)}{\alpha_1} \right]^{d_c/b_c} \right\} \tag{4.71}$$

As the effective coupling constant for QCD vanishes in this limit we may write in the second line $F(0, \alpha_{eff,QCD}(L_Q/2)) \simeq F(0,0)$. Thus we have found a simple and powerful way to calculate the limiting behaviour of F as just a power in the coupling constant times a number $F(0,0)$ corresponding to the behaviour of the function F for a free-field theory, for which the coupling constant is 0!

2 *The running coupling constant of QCD*

The above procedure does not work at all for QED, nor as a matter of fact for any other kind of theory known to date besides nonabelian gauge theories. The positive-definiteness of the weight function in the Källén-Lehmann representation of the polarisation function results for other theories in a positive value of the β-function, which means that the effective coupling increases with the scale.

The β-function may evidently turn over to negative values again for larger-order terms in the perturbation series (although this would mean that the theory contains states which effectively provide a negative phase space contribution according to the Källén-Lehmann representation!). Such a behaviour would lead to an *attractive fixed point for the coupling* at the value α^* for which $\beta(\alpha^*) = 0$. This means that when the energy increases the effective coupling constant will be attached to this value. We will, however, not pursue this discussion any further because there is for the cases of interest in this book no known example of such behaviour.

The very fact that the β-function goes from 0 for $\alpha = 0$ to negative values for a nonabelian gauge theory like QCD (and it is known to have the same behaviour also for the next order in perturbation theory) means that there is an attractive fixed point for a vanishing coupling constant. And a vanishing coupling constant in principle means a free-field theory.

In reality, though, we find that the theory is not completely free. There are evidently some logarithmic power corrections and we will see in Chapter 19 that this means scale-breaking corrections to the parton model.

The QCD effective coupling, usually referred to as the *running coupling of QCD*, can be written (with the number of colors $N_c = 3$)

$$\frac{\alpha_s(Q^2)}{4\pi} = \frac{3}{(11N_c - 2n_f)\log(Q^2/\Lambda^2_{QCD})} \tag{4.72}$$

by a suitable redefinition of the constant c_c in Eq. (4.69) and the introduction of the value for b_c given above.

We will end with a simple picture of why the coupling constants in QED and QCD behave so differently. We consider an ordinary electric charge in the vacuum and note that this will imply that the vacuum will be polarised in the way described above. In particular there will be some screening of the bare charge, because all the time it will be surrounded by a (virtual) cloud of charged particle-antiparticle pairs. These pairs will arrange themselves in a dipole-like manner so that viewed from afar we will see a diminished charge.

Now suppose that we send a set of probes towards this (pointlike) charge, corresponding to shorter and shorter wavelengths, i.e. we will observe the results from larger and larger values of the momentum transfer Q^2. The probes will evidently come closer and closer to the original bare charge and therefore 'see' more and more of it without the charge screening. Thus the effective charge will become larger with increasing Q^2. The main point in this argument is that the virtual pairs can in effect move and spread freely around the original charge. But note that the field quanta, i.e. the photons, are uncharged so that the charge is pointlike inside the virtual cloud of dipole pairs.

Let us now consider the corresponding situation in QCD. In this case, the field itself also contains charge, because the gluons are color-8's. This means that any original color charge will be smeared out over the region where the field is. A long-wavelength probe will then not be affected, i.e. it will see the whole, bare, charge. On the other hand, of course, as always in quantum mechanics short-wavelength probes will either 'see' the whole charge or nothing. But there will be a decreasing probability of finding the charge the smaller the region that is probed. In this way *the effective QCD charge actually corresponds to a charge multiplied by a 'form factor'*. We will show in Chapter 18 that the size and the behaviour of the β-function in QCD do in fact correspond to an interval in rapidity space within which we can expect modifications of the field.