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STRONG APPROXIMATION THEOREM FOR ABSOLUTELY IRREDUCIBLE VARIETIES OVER THE COMPOSITUM OF ALL SYMMETRIC EXTENSIONS OF A GLOBAL FIELD

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Abstract. Let K be a global field, $\mathcal{V}$ a proper subset of the set of all primes of K, $\mathcal{S}$ a finite subset of $\mathcal{V}$, and $\tilde{K}$ (resp. $K_{\rm sep}$) a fixed algebraic (resp. separable algebraic) closure of K with $K_{\rm sep} \subseteq \tilde{K}$. Let $\mathrm{Gal}(K) = \mathrm{Gal}(K_{\rm sep}/K)$ be the absolute Galois group of K. For each $\mathfrak{p} \in \mathcal{V}$, we choose a Henselian (respectively, a real or algebraic) closure $K_\mathfrak{p}$ of K at $\mathfrak{p}$ in $\tilde{K}$ if $\mathfrak{p}$ is non-archimedean (respectively, archimedean). Then, $K_{{\rm tot},\mathcal{S}} = \bigcap_{\mathfrak{p}\in\mathcal{S}} \bigcap_{\tau\in\mathrm{Gal}(K)} K_\mathfrak{p}^\tau$ is the maximal Galois extension of K in $K_{\rm sep}$ in which each $\mathfrak{p} \in \mathcal{S}$ totally splits. For each $\mathfrak{p} \in \mathcal{V}$, we choose a $\mathfrak{p}$-adic absolute value $|\ |_\mathfrak{p}$ of $K_\mathfrak{p}$ and extend it in the unique possible way to $\tilde{K}$. Finally, we denote the compositum of all symmetric extensions of K by $K_{\rm symm}$. We consider an affine absolutely integral variety V in $\mathbb{A}^n_K$. Suppose that for each $\mathfrak{p} \in \mathcal{S}$ there exists a simple $K_\mathfrak{p}$-rational point $\mathbf{z}_\mathfrak{p}$ of V and for each $\mathfrak{p} \in \mathcal{V} \smallsetminus \mathcal{S}$ there exists $\mathbf{z}_\mathfrak{p} \in V(\tilde{K})$ such that in both cases $|\mathbf{z}_\mathfrak{p}|_\mathfrak{p} \leq 1$ if $\mathfrak{p}$ is non-archimedean and $|\mathbf{z}_\mathfrak{p}|_\mathfrak{p} < 1$ if $\mathfrak{p}$ is archimedean. Then, there exists $\mathbf{z} \in V(K_{{\rm tot},\mathcal{S}} \cap K_{\rm symm})$ such that for all $\mathfrak{p} \in \mathcal{V}$ and for all $\tau \in \mathrm{Gal}(K)$, we have $|\mathbf{z}^\tau|_\mathfrak{p} \leq 1$ if $\mathfrak{p}$ is archimedean and $|\mathbf{z}^\tau|_\mathfrak{p} < 1$ if $\mathfrak{p}$ is non-archimedean. For $\mathcal{S} = \emptyset$, we get as a corollary that the ring of integers of $K_{\rm symm}$ is Hilbertian and Bezout.

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Introduction. The strong approximation theorem for a global field K gives an $x \in K$ that lies in given $\mathfrak{p}$-adically open discs for finitely many given primes $\mathfrak{p}$ of K such that the absolute $\mathfrak{p}$-adic value of x is at most 1 for all other primes $\mathfrak{p}$ except possibly one [**2**, p. 67]. A possible generalization of that theorem to an arbitrary absolutely integral affine variety V over K fails, because in general, $V(K)$ is a small set. For example, if V is a curve of genus at least 2, then $V(K)$ is finite (by Faltings). This obstruction disappears as soon as we switch to appropriate 'large Galois extensions' of K.

Extensions of K of this type occur in our work [**6**]. In that work, we fix an algebraic closure $\tilde{K}$ of K, set $K_{\rm sep}$ to be the separable closure of K in $\tilde{K}$, and consider a non-negative integer e. We equip $\mathrm{Gal}(K)^e$ with the normalised Haar measure [**3**, Section 18.5] and use the expression 'for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$' to mean 'for all $\boldsymbol{\sigma}$ in $\mathrm{Gal}(K)^e$ outside a set of measure 0'. For each $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_e) \in \mathrm{Gal}(K)^e$ let

$K_{\mathrm{sep}}(\boldsymbol{\sigma}) = \{x \in K_{\mathrm{sep}} \mid x^{\sigma_i} = x \text{ for } i = 1, \dots, e\}$ and let $K_{\mathrm{sep}}[\boldsymbol{\sigma}]$ be the maximal Galois extension of K in $K_{\mathrm{sep}}(\boldsymbol{\sigma})$.

Further, let $\mathbb{P}_K$ be the set of all primes of K, let $\mathbb{P}_{K,\mathrm{fin}}$ be the set of all non-archimedean primes, and let $\mathbb{P}_{K,\mathrm{inf}}$ be the set of all archimedean primes. We fix a proper subset $\mathcal{V}$ of $\mathbb{P}_K$, a finite subset $\mathcal{T}$ of $\mathcal{V}$, and a subset $\mathcal{S}$ of $\mathcal{T}$ such that $\mathcal{V} \smallsetminus \mathcal{T} \subseteq \mathbb{P}_{K,\mathrm{fin}}$. For each $\mathfrak{p}$, we fix a completion $\hat{K}_\mathfrak{p}$ of K at $\mathfrak{p}$ and embed $\tilde{K}$ in an algebraic closure $\widetilde{\hat{K}}_\mathfrak{p}$ of $\hat{K}_\mathfrak{p}$. Then, we extend a normalized absolute value $|\ |_\mathfrak{p}$ of $\hat{K}_\mathfrak{p}$ to $\widetilde{\hat{K}}_\mathfrak{p}$ in the unique possible way. In particular, this defines $|x|_\mathfrak{p}$ for each $x \in \tilde{K}$.

Next, we set $K_\mathfrak{p} = \tilde{K} \cap \hat{K}_\mathfrak{p}$, and note that $K_\mathfrak{p}$ is a Henselian closure of K at $\mathfrak{p}$ if $\mathfrak{p} \in \mathbb{P}_{K,\mathrm{fin}}$ and a real or the algebraic closure of K at $\mathfrak{p}$ if $\mathfrak{p} \in \mathbb{P}_{K,\mathrm{inf}}$. Thus,

$$K_{\mathrm{tot},\mathcal{S}} = \bigcap_{\mathfrak{p}\in\mathcal{S}} \bigcap_{\tau\in\mathrm{Gal}(K)} K_\mathfrak{p}^\tau$$

is the maximal Galois extension of K in which each $\mathfrak{p} \in \mathcal{S}$ totally splits. For each $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$, we set $K_{\mathrm{tot},\mathcal{S}}[\boldsymbol{\sigma}] = K_{\mathrm{sep}}[\boldsymbol{\sigma}] \cap K_{\mathrm{tot},\mathcal{S}}$.

For each extension M of K in $\tilde{K}$ and every $\mathfrak{p} \in \mathbb{P}_{\mathrm{fin}} \cap \mathcal{V}$, we consider the valuation ring $\mathcal{O}_{M,\mathfrak{p}} = \{x \in M \mid |x|_\mathfrak{p} \le 1\}$ of M at $\mathfrak{p}$. For each subset $\mathcal{U}$ of $\mathcal{V}$, we let

$$\mathcal{O}_{M,\mathcal{U}} = \{x \in M \mid |x^\tau|_\mathfrak{p} \le 1 \text{ for all } \mathfrak{p} \in \mathcal{U} \text{ and } \tau \in \mathrm{Gal}(K)\}.$$

Then, the main result of **[6]** is the following theorem.

THEOREM A. *Let $K, \mathcal{S}, \mathcal{T}, \mathcal{V}, e$ be as above. Then, for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ the field $M = K_{\mathrm{tot},\mathcal{S}}[\boldsymbol{\sigma}]$ satisfies the strong approximation theorem:*

Let V be an absolutely integral affine variety over K in $\mathbb{A}_K^n$ for some positive integer n. For each $\mathfrak{p} \in \mathcal{S}$ let $\Omega_\mathfrak{p}$ be a non-empty $\mathfrak{p}$-open subset of $V_{\mathrm{simp}}(K_\mathfrak{p})$. For each $\mathfrak{p} \in \mathcal{T} \smallsetminus \mathcal{S}$ let $\Omega_\mathfrak{p}$ be a non-empty $\mathfrak{p}$-open subset of $V(\tilde{K})$, invariant under the action of $\mathrm{Gal}(K_\mathfrak{p})$. Finally, for each $\mathfrak{p} \in \mathcal{V} \smallsetminus \mathcal{T}$ we assume that $V(\mathcal{O}_{\tilde{K},\mathfrak{p}}) \neq \emptyset$. Then, $V(\mathcal{O}_{M,\mathcal{V}\smallsetminus\mathcal{T}}) \cap \bigcap_{\mathfrak{p}\in\mathcal{T}} \bigcap_{\tau\in\mathrm{Gal}(K)} \Omega_\mathfrak{p}^\tau \neq \emptyset$.

The main result of the present work establishes the strong approximation theorem for much smaller fields. To this end, we call a Galois extension L of K *symmetric* if $\mathrm{Gal}(L/K)$ is isomorphic to the symmetric group S_n for some positive integer n. We denote the compositum of all symmetric extensions of K by K_{symm}.

THEOREM B. *Let $K, \mathcal{S}, \mathcal{T}, \mathcal{V}, e$ be as above. Then, for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ the field $M = K_{\mathrm{symm}} \cap K_{\mathrm{tot},\mathcal{S}}[\boldsymbol{\sigma}]$ satisfies the strong approximation theorem: (as in Theorem A). In particular, $K_{\mathrm{symm}} \cap K_{\mathrm{tot},\mathcal{S}}$ satisfies the strong approximation theorem.*

Additional interesting information about the fields mentioned in Theorem B and their rings of integers is contained in the following result.

THEOREM C. *Let K be a global field and e a non-negative integer. Then, for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ the field $M = K_{\mathrm{symm}} \cap K_{\mathrm{sep}}[\boldsymbol{\sigma}]$ is PAC (Definition 1.2) and Hilbertian, hence $\mathrm{Gal}(M) \cong \hat{F}_\omega$. Moreover, the ring of integers of M is Hilbertian and Bezout (Definition 2.1).*

Note that the statement about the Hilbertianity of M in Theorem C is due to **[1]**. See also the proof of Proposition 2.6. The authors are indebted to the anonymous referee for pointing out that proposition and its proof.

1. Weakly symmetrically K-stably P$\mathcal{S}$C fields over holomorphy domains. Let K be a global field, that is K is either a number field or an algebraic function field of one variable over a finite field. Throughout this work, we use the notation $\mathbb{P}_K$, $\tilde{K}$, K_{sep}, $\mathrm{Gal}(K)$, $K_{\mathfrak{p}}$ and $|\ |_{\mathfrak{p}}$ for $\mathfrak{p} \in \mathbb{P}_K$, introduced in the introduction. For each $\mathfrak{p} \in \mathbb{P}_K$ and every subfield M of $\tilde{K}$, we consider the closed disc

$$\mathcal{O}_{M,\mathfrak{p}} = \{x \in M \mid |x|_{\mathfrak{p}} \leq 1\}$$

of M at $\mathfrak{p}$. If $\mathfrak{p}$ is non-archimedean, then $\mathcal{O}_{M,\mathfrak{p}}$ is a valuation ring of rank 1 of M.

Next, we consider a subset $\mathcal{U}$ of $\mathbb{P}_K$ and a field $K \subseteq M \subseteq \tilde{K}$. A *prime* of M is an equivalence class of absolute values of M, where two absolute values on M are *equivalent* if they define the same topology on M. Let $\mathcal{U}_M$ be the set of all primes of M that lie over $\mathcal{U}$. If $\mathfrak{q} \in \mathcal{U}_M$ lies over $\mathfrak{p} \in \mathcal{U}$, then we denote the unique absolute value of M that extends $|\ |_{\mathfrak{p}}$ to M and represents $\mathfrak{q}$ by $|\ |_{\mathfrak{q}}$. In this case, there exists $\tau \in \mathrm{Gal}(K)$ such that $|x|_{\mathfrak{q}} = |x^{\tau}|_{\mathfrak{p}}$ for each $x \in M$. Conversely, the latter condition defines $\mathfrak{q}$. We set

$$\mathcal{O}_{M,\mathcal{U}} = \bigcap_{\mathfrak{q} \in \mathcal{U}_M} \{x \in M \mid |x|_{\mathfrak{q}} \leq 1\}$$

for the *$\mathcal{U}$-holomorphy domain* of M. If $\mathcal{U}$ consists of non-archimedean primes, then $\mathcal{O}_{M,\mathcal{U}}$ is the integral closure of $\mathcal{O}_{K,\mathcal{U}}$ in M. If $\mathcal{U}$ is arbitrary but M is Galois over K, then

$$\mathcal{O}_{M,\mathcal{U}} = \bigcap_{\mathfrak{p} \in \mathcal{U}} \bigcap_{\tau \in \mathrm{Gal}(K)} \mathcal{O}_{M,\mathfrak{p}}^{\tau} .$$

In the number field case (i.e., $\mathrm{char}(K) = 0$), we denote the (cofinite) set of all non-archimedean primes of K by $\mathbb{P}_{K,\mathrm{fin}}$. In the function field case, where $p = \mathrm{char}(K) > 0$, we fix a separating transcedence element t_K for $K/\mathbb{F}_p$ and let $\mathbb{P}_{K,\mathrm{fin}} = \{\mathfrak{p} \in \mathbb{P}_K \mid |t_K|_{\mathfrak{p}} \leq 1\}$. In both cases, we set

$$\mathcal{O}_K = \mathcal{O}_{K,\mathbb{P}_{K,\mathrm{fin}}} = \{x \in K \mid |x|_{\mathfrak{p}} \leq 1 \text{ for all } \mathfrak{p} \in \mathbb{P}_{K,\mathrm{fin}}\}.$$

If K is a number field, then $\mathcal{O}_K$ is the integral closure of $\mathbb{Z}$ in K. In the function field case, $\mathcal{O}_K$ is the integral closure of $\mathbb{F}_p[t_K]$ in K. In both cases, $\mathcal{O}_K$ is a Dedekind domain. Following the convention in algebraic number theory, we call $\mathcal{O}_K$ the *ring of integers* of K.

Next, we consider a finite (possibly empty) subset $\mathcal{S}$ of $\mathbb{P}_K$. We set

$$K_{\mathrm{tot},\mathcal{S}} = \bigcap_{\mathfrak{p} \in \mathcal{S}} \bigcap_{\tau \in \mathrm{Gal}(K)} K_{\mathfrak{p}}^{\tau}$$

as in the introduction. If $\mathcal{S} = \emptyset$, then $K_{\mathrm{tot},\mathcal{S}} = K_{\mathrm{sep}}$.

We also choose a non-empty proper subset $\mathcal{V}$ of $\mathbb{P}_K$ that contains $\mathcal{S}$.

DEFINITION 1.1. [**6**, Definition 12.1] Let M be an extension of K in $K_{\mathrm{tot},\mathcal{S}}$ and let $\mathcal{O}$ be a subset of M. We say that M is *weakly symmetrically K-stably* P$\mathcal{S}$C *over* $\mathcal{O}$ if for every polynomial $g \in K[T]$ with $g(0) \neq 0$ and for every absolutely irreducible polynomial $h \in K[T, Y]$ monic in Y with $d = \deg_Y(h)$ satisfying

(1a) $h(0, Y)$ has d distinct roots in $K_{\mathrm{tot},\mathcal{S}}$, and
(1b) $\mathrm{Gal}(h(T, Y), K(T)) \cong \mathrm{Gal}(h(T, Y), \tilde{K}(T))$ and is isomorphic to the symmetric group S_d,

there exists $(a, b) \in \mathcal{O} \times M$ such that $h(a, b) = 0$ and $g(a) \neq 0$.

Note that in that case, if $M \subseteq M' \subseteq K_{\mathrm{tot},\mathcal{S}}$, then M' is also weakly symmetrically K-stably P$\mathcal{S}$C over $\mathcal{O}$.

If $\mathcal{S} = \emptyset$, we say that M is *weakly symmetrically K-stably* PAC *over* $\mathcal{O}$.

DEFINITION 1.2. [**6**, Definition 13.1] Let M be an extension of K in $K_{\mathrm{tot},\mathcal{S}}$ and let $\mathcal{O}$ be a subset of M. We say that M is *weakly* P$\mathcal{S}$C *over* $\mathcal{O}$ if for every absolutely irreducible polynomial $h \in M[T, Y]$ monic in Y such that $h(0, Y)$ decomposes into distinct monic linear factors over $K_{\mathrm{tot},\mathcal{S}}$ and every polynomial $g \in M[T]$ with $g(0) \neq 0$ there exists $(a, b) \in \mathcal{O} \times M$ such that $h(a, b) = 0$ and $g(a) \neq 0$. In particular, $\mathcal{O}$ is infinite.

If $\mathcal{S} = \emptyset$, then M is *PAC over* $\mathcal{O}$ [**6**, Definition 13.5], i.e. for every absolutely irreducible polynomial $f \in M[T, X]$, which is separable in X there exist infinitely many points $(a, b) \in \mathcal{O} \times M$ such that $f(a, b) = 0$.

Indeed, let $f \in M[T, X]$ be an absolutely irreducible polynomial, which is separable in X. Let $\Delta \in M[T]$ be the discriminant of f, let $g \in M[T]$ be the leading coefficient of f, and let $d = \deg_X(f)$. Since $\mathcal{O}$ is infinite, we can choose $c \in \mathcal{O}$ with $\Delta(c)g(c) \neq 0$. Let $Y = g(T)X$, let $h'(T, Y) = g(T)^{d-1} f(T, g(T)^{-1} Y)$, and let $h(T, Y) = h'(T + c, Y)$. Then, $h \in M[T, Y]$ is an absolutely irreducible polynomial, monic in Y, such that $h(0, Y)$ decomposes into distinct monic linear factors over K_{sep}. By assumption, there exist infinitely many $(a, b) \in \mathcal{O} \times M$ such that $h(a, b) = 0$ and $g(a) \neq 0$, hence $f(a + c, g(a)^{-1} b) = 0$.

Note that in that case, M is a *PAC* field, i.e., every absolutely integral variety over M has an M-rational point [**7**, Lemma 1.3].

LEMMA 1.3. *Let M_0 be an extension of K in K_{sep}, let $M = M_0 \cap K_{\mathrm{tot},\mathcal{S}}$, and let $\mathcal{O}$ be a subset of $\mathcal{O}_{M,\mathcal{S}}$ such that $\mathcal{O}_{K,\mathcal{V}} \cdot \mathcal{O} \subseteq \mathcal{O}$. Suppose that M_0 is weakly symmetrically K-stably PAC over $\mathcal{O}$. Then, M is weakly symmetrically K-stably* P$\mathcal{S}$C *over* $\mathcal{O}$.

Proof. Let g be a polynomial in $K[T]$ with $g(0) \neq 0$ and let h be an absolutely irreducible polynomial in $K[T, Y]$, monic in Y, with $d = \deg_Y(h)$ satisfying (1). By [**5**, Lemma 1.9], there exists $c \in \mathcal{O}_{K,\mathcal{V}}$, which is sufficiently $\mathcal{S}$-close to 0 such that for each $a \in \mathcal{O}_{K_{\mathrm{tot},\mathcal{S}},\mathcal{S}}$ all the roots of $h(ac, Y)$ are simple and belong to $K_{\mathrm{tot},\mathcal{S}}$. Consider the polynomial $h(cT, Y) \in K[T, Y]$. Then, since M_0 is weakly symmetrically K-stably PAC over $\mathcal{O}$, there exists $a \in \mathcal{O}$ and $b \in M_0$ such that $h(ac, b) = 0$ and $g(a) \neq 0$. Then, $ac \in \mathcal{O}$ and $b \in M_0 \cap K_{\mathrm{tot},\mathcal{S}} = M$, as desired. □

LEMMA 1.4. [**6**, Lemma 13.2] *Let M be an extension of K in $K_{\mathrm{tot},\mathcal{S}}$, which is weakly symmetrically K-stably* P$\mathcal{S}$C *over $\mathcal{O}_{K,\mathcal{V}}$. Then, M is weakly* P$\mathcal{S}$C *over $\mathcal{O}_{M,\mathcal{V}}$.*

2. Composita of symmetric extensions of a global field. A *symmetric extension* of K is a finite Galois extension of K with Galois group isomorphic to S_m for some positive integer m. Let K_{symm} be the compositum of all symmetric extensions of K.

Using the notation of the introduction, we prove that for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$, the field $K_{\mathrm{symm}}[\boldsymbol{\sigma}]$ is PAC and Hilbertian, so $\mathrm{Gal}(K_{\mathrm{symm}}[\boldsymbol{\sigma}]) \cong \hat{F}_\omega$. Moreover, if $\mathcal{V}$ contains only non-archimedean primes, then the ring $\mathcal{O}_{K_{\mathrm{symm}}[\boldsymbol{\sigma}],\mathcal{V}}$ is Hilbertian and Bezout. Finally, the field $M = K_{\mathrm{tot},\mathcal{S}} \cap K_{\mathrm{symm}}[\boldsymbol{\sigma}]$ is weakly P$\mathcal{S}$C over $\mathcal{O}_{M,\mathcal{V}}$. This leads in Section 3 to a strong approximation theorem for M.

DEFINITION 2.1. Let $\mathcal{O}$ be an integral domain with quotient field F. We consider variables $T_1, \ldots, T_r, X$ over F and abbreviate $(T_1, \ldots, T_r)$ to $\mathbf{T}$. Let $f_1, \ldots, f_m$ be irreducible and separable polynomials in $F(\mathbf{T})[X]$ and let g be a non-zero polynomial in $F[\mathbf{T}]$. Following [**3**, Section 12.1], we write $H_F(f_1, \ldots, f_n; g)$ for the set of all $\mathbf{a} \in F^r$ such that $f_1(\mathbf{a}, X), \ldots, f_m(\mathbf{a}, X)$ are defined, irreducible, and separable in $F[X]$ with $g(\mathbf{a}) \neq 0$. Then, we call $H_F(f_1, \ldots, f_m; g)$ a *separable Hilbert subset* of F^r. We say that the ring $\mathcal{O}$ is *Hilbertian* if for every positive integer r every separable Hilbert subset of F^r has a point with coordinates in $\mathcal{O}$. Finally, we say that $\mathcal{O}$ is *Bezout* if every finitely generated ideal of $\mathcal{O}$ is principal.

EXAMPLE 2.2. Taking $\mathfrak{q}_0 \in \mathbb{P}_K \smallsetminus \mathcal{V}$ in [**3**, Theorem 13.3.5(b), p. 241], we find that $H \cap \mathcal{O}_{K,\mathcal{V}}^r \neq \emptyset$ for each $r \geq 1$ and every separable Hilbert subset H of K^r. In particular, if $\mathcal{V}$ contains only non-archimedean primes, then $\mathcal{O}_{K,\mathcal{V}}$ is a Hilbertian domain.

Let d be a positive integer. Denote the set of all absolutely irreducible polynomials $h \in K[T, Y]$, monic in Y with $d = \deg_Y(h)$, that satisfy (1) of Section 1 with $\mathcal{S} = \emptyset$, i.e.,

(1a) $h(0, Y)$ has d distinct roots in K_{sep}, and
(1b) $\mathrm{Gal}(h(T, Y), K(T)) \cong \mathrm{Gal}(h(T, Y), \tilde{K}(T)) \cong S_d$

by $\mathcal{H}_d$. Let $\mathcal{H} = \bigcup_{d=1}^{\infty} \mathcal{H}_d$.

LEMMA 2.3. *Let e be a non-negative integer. Then, for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ every separable algebraic extension M of $K_{\mathrm{symm}}[\boldsymbol{\sigma}]$ is weakly symmetrically K-stably PAC over $\mathcal{O}_{K,\mathcal{V}}$.*

In particular, the field K_{symm} is weakly symmetrically K-stably PAC over $\mathcal{O}_{K,\mathcal{V}}$.

Proof. By Definition 1.1, it suffices to consider the case $e \geq 1$ and to prove that for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ the field $K_{\mathrm{symm}}[\boldsymbol{\sigma}]$ is weakly symmetrically K-stably PAC over $\mathcal{O}_{K,\mathcal{V}}$. Moreover, since the set $\mathcal{H}$ is countable, it suffices to consider a positive integer d, a polynomial $h \in \mathcal{H}_d$, and a non-zero polynomial $g \in K[T]$, and to prove that for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ there exists $(a, b) \in \mathcal{O}_{K,\mathcal{V}} \times K_{\mathrm{symm}}[\boldsymbol{\sigma}]$ such that $h(a, b) = 0$ and $g(a) \neq 0$.

By Borel–Cantelli [**3**, Lemma 18.5.3(b), p. 378], it suffices to construct a sequence of pairs $(a_1, b_1), (a_2, b_2), (a_3, b_3), \ldots$ that satisfies for each $n \geq 1$ the following conditions:

(2a) $a_n \in \mathcal{O}_{K,\mathcal{V}}$ and $h(a_n, X)$ is separable,
(2b) the splitting field K_n of $h(a_n, X)$ over K has Galois group S_d,
(2c) $h(a_n, b_n) = 0$, in particular $b_n \in K_n$, and $g(a_n) \neq 0$,
(2d) $K_1, K_2, \ldots, K_n$ are linearly disjoint over K.

Indeed, inductively suppose that n is a positive integer and $(a_1, b_1), \ldots, (a_{n-1}, b_{n-1})$ satisfy Condition (2) (for $n-1$ rather than for n). Let $L = K_1, K_2, \ldots, K_{n-1}$. By [**3**, Proposition 16.1.5, p. 294] and [**3**, Corollary, 12.2.3, p. 224], K has a separable Hilbert subset H such that for each $a \in H$ the polynomial $h(a, X)$ is separable, $\mathrm{Gal}(h(a, X), K) \cong \mathrm{Gal}(h(a, X), L) \cong S_d$, and $g(a) \neq 0$. Using Example 2.2, we choose an element $a_n \in H \cap \mathcal{O}_{K,\mathcal{V}}$ and a root $b_n \in K_{\mathrm{sep}}$ of $h(a_n, X)$. Then, b_n lies in the splitting field K_n of $h(a_n, X)$, so all of the statements (2a)–(2d) are satisfied. □

By Lemmas 1.3 and 1.4, we get the following corollary.

COROLLARY 2.4. *Let e be a non-negative integer. Then, for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ each extension M of $K_{\mathrm{tot},\mathcal{S}} \cap K_{\mathrm{symm}}[\boldsymbol{\sigma}]$ in $K_{\mathrm{tot},\mathcal{S}}$ is weakly symmetrically K-stably* P$\mathcal{S}$C *over $\mathcal{O}_{K,\mathcal{V}}$. Hence, M is weakly* P$\mathcal{S}$C *over $\mathcal{O}_{M,\mathcal{V}}$.*

In particular, the field $M = K_{\mathrm{tot},\mathcal{S}} \cap K_{\mathrm{symm}}$ *is weakly symmetrically* K*-stably* P$\mathcal{S}$C *over* $\mathcal{O}_{K,\mathcal{V}}$*, so it is also weakly* P$\mathcal{S}$C *over* $\mathcal{O}_{M,\mathcal{V}}$.

When $\mathcal{S} = \emptyset$, we get from Definition 1.2 the following result.

COROLLARY 2.5. *Let* e *be a non-negative integer. Then, for almost all* $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ *each separable algebraic extension* M *of the field* $K_{\mathrm{symm}}[\boldsymbol{\sigma}]$ *is PAC over* $\mathcal{O}_{M,\mathcal{V}}$.
In particular, the field $M = K_{\mathrm{symm}}$ *is PAC over* $\mathcal{O}_{M,\mathcal{V}}$.

PROPOSITION 2.6. *Let* L *be a Hilbertian field and* M *an extension of* L *in* L_{symm}. *Then,* M *is Hilbertian.*

Proof. Following [**1**, Section 2.1], we say that a profinite group G has *abelian-simple length* n if there is a finite series $\mathbf{1} = G^{(n)} \lhd \cdots \lhd G^{(1)} \lhd G^{(0)} = G$ of closed subgroups, where for $i = 0, \dots, n-1$, the group $G^{(i+1)}$ is the intersection of all open normal subgroups N of $G^{(i)}$ such that $G^{(i)}/N$ is abelian or simple.

As mentioned in the proof of [**1**, Theorem 5.5], the abelian-simple length of each symmetric group S_n is at most 3. Hence, by [**1**, Proposition 2.8], the abelian-simple length of $\mathrm{Gal}(L_{\mathrm{symm}}/L)$ is at most 3. Therefore, by [**1**, Theorem 3.2], every field M between L and L_{symm} is Hilbertian. □

COROLLARY 2.7. *Let* e *be a positive integer. Suppose that* $\mathcal{V}$ *contains only non-archimedean primes. Then, for almost all* $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ *the rings* $\mathcal{O}_{K_{\mathrm{sep}}[\boldsymbol{\sigma}],\mathcal{V}}$ *and* $\mathcal{O}_{K_{\mathrm{symm}}[\boldsymbol{\sigma}],\mathcal{V}}$ *are Hilbertian. In addition, the ring* $\mathcal{O}_{K_{\mathrm{symm}},\mathcal{V}}$ *is Hilbertian.*

Proof. By Proposition 2.6, for all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ the field $K_{\mathrm{symm}}[\boldsymbol{\sigma}]$ is Hilbertian. By [**3**, Theorem 27.4.8, p. 669], for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ the field $K_{\mathrm{sep}}[\boldsymbol{\sigma}]$ is Hilbertian. By [**11**, Proposition 2.5 and Corollary 2.6], if a field M is PAC over a subring $\mathcal{O}$ and M is Hilbertian, then the ring $\mathcal{O}$ is Hilbertian. It follows from Corollary 2.5 that for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ the rings $\mathcal{O}_{K_{\mathrm{sep}}[\boldsymbol{\sigma}],\mathcal{V}}$ and $\mathcal{O}_{K_{\mathrm{symm}}[\boldsymbol{\sigma}],\mathcal{V}}$ are Hilbertian.

Finally, by Proposition 2.6, the field K_{symm} is also Hilbertian. By Corollary 2.5, K_{symm} is PAC over $\mathcal{O}_{K_{\mathrm{symm}},\mathcal{V}}$. Hence, by the preceding paragraph, the ring $\mathcal{O}_{K_{\mathrm{symm}},\mathcal{V}}$ is Hilbertian. □

COROLLARY 2.8. *Let* e *be a non-negative integer. Then, for almost all* $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ *the field* $K_{\mathrm{symm}}[\boldsymbol{\sigma}]$ *is PAC, Hilbertian, and* $\mathrm{Gal}(K_{\mathrm{symm}}[\boldsymbol{\sigma}]) \cong \hat{F}_\omega$.

Proof. By Corollary 2.5, Definition 1.2, and Corollary 2.7, for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ the field $M = K_{\mathrm{symm}}[\boldsymbol{\sigma}]$ is PAC and Hilbertian. Hence, by [**9**, Theorem 5.10.3, p. 90], $\mathrm{Gal}(M) \cong \hat{F}_\omega$, as claimed. □

Remark 2.9. (a) It is not true that $K_{\mathrm{symm}}[\boldsymbol{\sigma}]$ is PAC for every $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$. For example, [**3**, Remark 18.6.2, p. 381] gives $\sigma \in \mathrm{Gal}(\mathbb{Q})$ such that $\tilde{\mathbb{Q}}(\sigma)$ is not a PAC field. Hence, by [**3**, Corollary 11.2.5, p. 196] also the subfield $\mathbb{Q}_{\mathrm{symm}}[\sigma]$ of $\tilde{\mathbb{Q}}(\sigma)$ is not PAC.

(b) In a forthcoming note, we make some mild changes in the proof of Theorem 1.1 of [**1**] and in some lemmas on which it depends in order to prove in the setup of Proposition 2.6 that if L is the quotient field of a Hilbertian domain R and S is the integral closure of R in M, then S is also a Hilbertian domain. In particular, in view of the proof of Proposition 2.6, the latter result applies to every extension M of L in L_{symm}. It will follow, in the notation of Corollary 2.7, that each of the rings $\mathcal{O}_{K_{\mathrm{symm}}[\boldsymbol{\sigma}],\mathcal{V}}$ is Hilbertian.

By [**12**, Lemma 4.6], if M is an algebraic extension of K, which is PAC over its ring of integers $\mathcal{O}_M = \mathcal{O}_{M,\mathbb{P}_{K,\mathrm{fin}}}$, then $\mathcal{O}_M$ is a Bezout domain. Thus, Corollary 2.5, applied to $\mathcal{V} = \mathbb{P}_{K,\mathrm{fin}}$ yields the following result.

COROLLARY 2.10. *Let e be a non-negative integer. Then, for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ the ring of integers of each separable extension of $K_{\mathrm{symm}}[\boldsymbol{\sigma}]$ is Bezout.*

In particular, the ring $\mathcal{O}_{K_{\mathrm{symm}}}$ is Bezout.

3. Strong approximation theorem. In the notation of Section 1, we prove that for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ the field $K_{\mathrm{tot},\mathcal{S}} \cap K_{\mathrm{symm}}[\boldsymbol{\sigma}]$ satisfies the strong approximation theorem for absolutely integral affine varieties.

Given a variety V, we write V_{simp} for the Zariski-open subset of V that consists of all *simple* (= non-singular) points of V. We cite two results from [**6**]. The first one is Proposition 12.4 of [**6**].

PROPOSITION 3.1 (Strong approximation theorem). *Let M be a subfield of $K_{\mathrm{tot},\mathcal{S}}$ that contains K and is weakly symmetrically K-stably* P$\mathcal{S}$C *over $\mathcal{O}_{K,\mathcal{V}}$. Then, $(M, K, \mathcal{S}, \mathcal{V})$ satisfies the following condition, abbreviated as $(M, K, \mathcal{S}, \mathcal{V}) \models \mathrm{SAT}$:*

Let $\mathcal{T}$ be a finite subset of $\mathcal{V}$ that contains $\mathcal{S}$ such that $\mathcal{V} \smallsetminus \mathcal{T} \subseteq \mathbb{P}_{K,\mathrm{fin}}$. Let V be an absolutely integral affine variety over K in $\mathbb{A}_K^n$ for some positive integer n. For each $\mathfrak{p} \in \mathcal{T}$ let $L_\mathfrak{p}$ be a finite Galois extension of $K_\mathfrak{p}$ such that $L_\mathfrak{p} = K_\mathfrak{p}$ if $\mathfrak{p} \in \mathcal{S}$ and let $\Omega_\mathfrak{p}$ be a non-empty $\mathfrak{p}$-open subset of $V_{\mathrm{simp}}(L_\mathfrak{p})$, invariant under the action of $\mathrm{Gal}(L_\mathfrak{p}/K_\mathfrak{p})$. Assume that $V(\mathcal{O}_{\tilde{K},\mathfrak{p}}) \neq \emptyset$, for each $\mathfrak{p} \in \mathcal{V} \smallsetminus \mathcal{T}$. Then, there exists $\mathbf{z} \in V(\mathcal{O}_{M,\mathcal{V}\smallsetminus\mathcal{T}})$ such that $\mathbf{z}^\tau \in \Omega_\mathfrak{p}$ for all $\mathfrak{p} \in \mathcal{T}$ and all $\tau \in \mathrm{Gal}(K)$.

The second result is Proposition 13.4 of [**6**], applied (for simplicity) to the case where $\mathcal{S}$ consists only of finite primes of K and $\mathcal{V} = \mathbb{P}_{K,\mathrm{fin}}$.

PROPOSITION 3.2 (Local-global principle). *Let M be a subfield of $K_{\mathrm{tot},\mathcal{S}}$ that contains K and is weakly symmetrically K-stably* P$\mathcal{S}$C *over $\mathcal{O}_{K,\mathcal{V}}$. Then, $(M, \mathcal{S})$ satisfies the following condition, abbreviated as $(M, \mathcal{S}) \models \mathrm{LGP}$:*

Let V be an absolutely integral affine variety over M in $\mathbb{A}_M^n$ for some positive integer n such that $V_{\mathrm{simp}}(\mathcal{O}_{M_\mathfrak{q},\mathfrak{q}}) \neq \emptyset$ for each $\mathfrak{q} \in \mathcal{S}_M$ and $V(\mathcal{O}_{M_\mathfrak{q},\mathfrak{q}}) \neq \emptyset$ for each $\mathfrak{q} \in \mathbb{P}_{M,\mathrm{fin}} \smallsetminus \mathcal{S}_M$. Then, $V(\mathcal{O}_M) \neq \emptyset$.

Recall that an extension M of K in $K_{\mathrm{tot},\mathcal{S}}$ is said to be P$\mathcal{S}$C (=*pseudo-$\mathcal{S}$-closed*) if every absolutely integral variety V over M with a simple $K_\mathfrak{p}^\tau$-rational point for each $\mathfrak{p} \in \mathcal{S}$ and every $\tau \in \mathrm{Gal}(K)$ has an M-rational point [**4**, Definition 1.3]. Also, a field M is *ample* if the existence of an M-rational simple point on V implies that $V(M)$ is Zariski-dense in V [**9**, Lemma 5.3.1, p. 67]. In particular, every P$\mathcal{S}$C field is ample.[1]

The next lemma is observed in [**8**, Corollary 2.7].

LEMMA 3.3. *Let M be an extension of K in $K_{\mathrm{tot},\mathcal{S}}$. Suppose that $(M, K, \mathcal{S}, \mathcal{S}) \models$ SAT. Then, M is a* P$\mathcal{S}$C *field, hence ample.*

Proof. Consider an absolutely integral variety V over M with a simple $K_\mathfrak{p}^\tau$-rational point for each $\mathfrak{p} \in \mathcal{S}$ and every $\tau \in \mathrm{Gal}(K)$. Replacing K by a finite extension K' in

[1]The work [**10**] uses the adjective "large" rather than "ample".

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$K_{\mathrm{tot},\mathcal{S}}$ and $\mathcal{S}$ by $\mathcal{S}_{K'}$, we may assume that V is defined over K and has a simple $K_{\mathfrak{p}}$-rational point for each $\mathfrak{p} \in \mathcal{S}$. Moreover, we may assume that V is affine. Thus, we may apply Proposition 3.1 to the case $\mathcal{V} = \mathcal{T} = \mathcal{S}$ and $\Omega_{\mathfrak{p}} = V_{\mathrm{simp}}(K_{\mathfrak{p}})$ for each $\mathfrak{p} \in \mathcal{S}$. Observe that in this case $\mathcal{O}_{M,\mathcal{V}\smallsetminus\mathcal{T}} = M$. □

Corollary 2.4, Lemma 3.3, Proposition 3.1, and Proposition 3.2 yield the following result.

THEOREM 3.4. Let e be a non-negative integer. Then, for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$, every extension M of $K_{\mathrm{tot},\mathcal{S}} \cap K_{\mathrm{symm}}[\boldsymbol{\sigma}]$ in $K_{\mathrm{tot},\mathcal{S}}$ has the following properties.

(a) $(M, K, \mathcal{S}, \mathcal{V}) \models \mathrm{SAT}$.
(b) M is P$\mathcal{S}$C, hence ample.
(c) If $\mathcal{S}$ consists only of finite primes of K, then $(M, \mathcal{S}) \models \mathrm{LGP}$.

In particular, $M = K_{\mathrm{tot},\mathcal{S}} \cap K_{\mathrm{symm}}$ satisfies (a)–(c).

Proof. By Corollary 2.4, for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ every extension M of the field $K_{\mathrm{tot},\mathcal{S}} \cap K_{\mathrm{symm}}[\boldsymbol{\sigma}]$ in $K_{\mathrm{tot},\mathcal{S}}$ is weakly symmetrically K-stably P$\mathcal{S}$C over $\mathcal{O}_{K,\mathcal{V}}$. Hence, by Proposition 3.1, $(M, K, \mathcal{S}, \mathcal{V}) \models \mathrm{SAT}$, so (a) holds. It follows from Lemma 2.3 that M is P$\mathcal{S}$C, as (b) states. Finally, if in addition, $\mathcal{S}$ consists only of finite primes, then by Proposition 3.2, $(M, \mathcal{S}) \models \mathrm{LGP}$, which establishes (c). □

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