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ANALOGY IN SOLVING POISSON'S EQUATION

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The passage from ordinary to partial differential equations is a difficult one. While many properties of ordinary differential equations admit analogies in the framework of partial differential equations, the techniques used to establish them may bear little resemblance to those which suffice in the simpler setting.

Poisson's equation is a case in point, but here, as we shall show below, a remarkable degree of analogy is possible and can be used to motivate an otherwise long and complicated proof. Our technique relies on a multidimensional generalization of Leibniz's rule for differentiation under the integral which is of some interest in itself. Such a generalization was considered by Flanders [1] in the context of the theory of differential forms, but the present discussion is based on a more elementary approach.

The standard Leibniz rule deals with a function $F(t)$ defined by

$$F(t) = \int_{\alpha(t)}^{\beta(t)} f(t, \xi)\, d\xi \tag{1}$$

where $\alpha(t)$ and $\beta(t)$ are differentiable for all t in an interval I and $f_t(t, \xi)$ exists and is continuous for $t \in I$ and $\alpha(t) \leq \xi \leq \beta(t)$. It states that $F'(t)$ then exists for $t \in I$ and is given by

$$F'(t) = \int_{\alpha(t)}^{\beta(t)} f_t(t, \xi)\, d\xi + f(t, \beta(t))\beta'(t) - f(t, \alpha(t))\alpha'(t). \tag{2}$$

To obtain a multidimensional generalization, we denote points in E^{n+1} by $(t, \boldsymbol{\xi}) = (t, \xi_1, \ldots, \xi_n)$ and consider a sufficiently smooth domain $G \subset E^{n+1}$. Defining $D_{t_0} = \{(t, \boldsymbol{\xi}) \in G \mid t = t_0\}$, let a function $F(t)$ be given by

$$F(t) = \int_{D_t} f(t, \boldsymbol{\xi})\, d\boldsymbol{\xi}. \tag{1'}$$

A formal computation of $F'(t)$ can be achieved by introducing hyperspherical coordinates $(r, \boldsymbol{\theta})$ into the n-dimensional $\boldsymbol{\xi}$-space and assuming that ∂G can be given by a continuous function $r = r(t, \boldsymbol{\theta})$. Then

$$F(t) = \int_{\Omega} \int_0^{r(t, \boldsymbol{\theta})} f(t, r, \boldsymbol{\theta}) r^{n-1}\, dr\, d\boldsymbol{\theta} \tag{1''}$$

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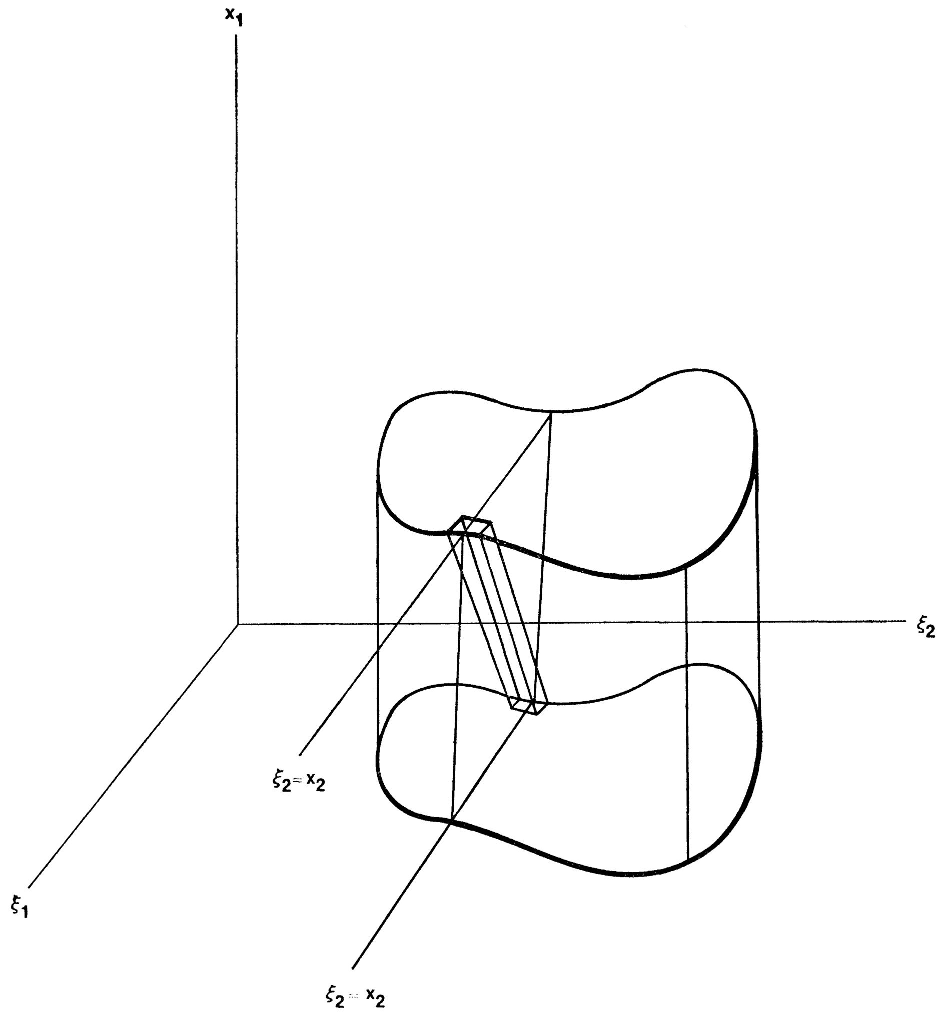

FIGURE 1

where Ω denotes the full range of the angular coordinate $\boldsymbol{\theta}$. The standard rule (2) then yields

$$F'(t) = \int_\Omega \int_0^{r(t,\boldsymbol{\theta})} f_t(t, r, \boldsymbol{\theta}) r^{n-1}\, dr\, d\boldsymbol{\theta} + \int_\Omega f(t, r(t, \boldsymbol{\theta}), \boldsymbol{\theta}) r_t(t, \boldsymbol{\theta}) r^{n-1}(t, \boldsymbol{\theta})\, d\boldsymbol{\theta}$$

or

$$\text{(2')} \qquad F'(t) = \int_{D_t} f_t(t, \boldsymbol{\xi})\, d\boldsymbol{\xi} - \int_{\partial D_t} f(t, \boldsymbol{\xi}) \cot \alpha(t, \boldsymbol{\xi}) d\boldsymbol{\xi}$$

where $\alpha(t, \boldsymbol{\xi})$ is the angle between the positive t-axis and the exterior normal to ∂G at a point $(t, \boldsymbol{\xi}) \in \partial D_t$. (In cases where D_t is not star-shaped relative to $(t, \mathbf{0})$ the description of ∂G in the form $r = r(t, \boldsymbol{\theta})$ will have to be accomplished in several steps.) From the well-known case $n=1$, one can derive hypotheses which assure the validity of (2′)—e.g. $r_t(t, \boldsymbol{\theta})$ exists in $I\times\Omega$ and $f(t, \boldsymbol{\xi})$ and $f_t(t, \boldsymbol{\xi})$ continuous for $(t, \boldsymbol{\xi})\in I\times D_t$.

As an application of (2′) we consider the problem of showing that in E^2 Poisson's equation

$$-\Delta u = f(\mathbf{x}) \tag{3}$$

is satisfied by a solution of the form

$$u(\mathbf{x}) = \int_D g(\mathbf{x}, \boldsymbol{\xi}) f(\boldsymbol{\xi})\, d\boldsymbol{\xi} \tag{4}$$

with

$$g(\mathbf{x}, \boldsymbol{\xi}) = -\frac{1}{2\pi}\log|\mathbf{x}-\boldsymbol{\xi}|.$$

While our computations will be carried out in E^2, they generalize readily to E^n for $n>2$. If $f(\mathbf{x})$ is of Class C^1 in D, then the validity of (4) is easily established by Green's theorem [2; p. 366–368]. However under the weaker hypothesis that $f(\mathbf{x})$ is Hölder continuous in D, the validity of (4) requires a careful computation of the first and second partial derivatives of a function $u(\mathbf{x})$ given by (4) (see [3; p. 246–250] or [4; p. 157–163]), and it is in this connection that (2′) proves useful. Our computations will involve

$$D(\mathbf{x}; \varepsilon) = \{\boldsymbol{\xi}\in D \mid |\xi_i - x_i| > \varepsilon;\ i = 1, 2;\ \mathbf{x}\in D\}$$

and

$$u_\varepsilon(\mathbf{x}) = \iint_{D_\varepsilon} g(\mathbf{x}, \boldsymbol{\xi}) f(\boldsymbol{\xi})\, d\boldsymbol{\xi}.$$

In our considerations, the function $f(\mathbf{x})$ is assumed to be *uniformly* Hölder continuous in D—i.e. there exist positive constants C and γ such that

$$|f(\mathbf{x})-f(\mathbf{y})| \le C\,|\mathbf{x}-\mathbf{y}|^\gamma$$

for all $\mathbf{x}$, $\mathbf{y}$ in D. The uniformity assumption can be removed by an approximation procedure as in [3] but we shall not pursue this refinement here.

The use of (2′) to compute $\partial u_\varepsilon/\partial x_1$ and $\partial^2 u_\varepsilon/\partial x_1^2$ involves a domain G in E^{n+1} which is "a cylinder with a square hole drilled through it, making a 45° angle in the ξ_1-direction".
The angle $\alpha(x_1, \boldsymbol{\xi})$ will be $\pi/2$ on the entire boundary of G except for two opposite sides of the square hole where it will be $\pi/4$ and $3\pi/4$, respectively. Thus (2′) yields

$$\begin{aligned}\frac{\partial u_\varepsilon}{\partial x_1} &= \iint_{D(x,\varepsilon)} g_{x_1}(\mathbf{x}, \boldsymbol{\xi}) f(\boldsymbol{\xi})\, d\boldsymbol{\xi} + \int_{x_2-\varepsilon}^{x_2+\varepsilon} g(x_1, x_2; x_1-\varepsilon, \xi_2) f(x_1-\varepsilon, \xi_2)\, d\xi_2 \\ &\quad + \int_{x_2-\varepsilon}^{x_2+\varepsilon} g(x_1, x_2; x_1+\varepsilon, \xi_2) f(x_1+\varepsilon, \xi_2)\, d\xi_2 \\ &= \frac{1}{2\pi}\iint_{D(x,\varepsilon)} -\frac{x_1-\xi_1}{(x_1-\xi_1)^2+(x_2-\xi_2)^2} f(\xi_1, \xi_2)\, d\xi_1\, d\xi_2 + I_1 + I_2\end{aligned}$$

where I_1+I_2 is absolutely dominated by $(2M\varepsilon/\pi)\,|\log \varepsilon|$, M being an upper bound for $|f(\boldsymbol{\xi})|$ in the square $|x_i-\xi_i|<\varepsilon$. It follows readily from the integrability of f that

$$\lim_{\varepsilon\to 0}\frac{\partial u_\varepsilon}{\partial x_1}=\frac{1}{2\pi}\iint_D -\frac{x_1-\xi_1}{(x_1-\xi_1)^2+(x_2-\xi_2)^2}f(\xi_1,\xi_2)\,d\xi_1\,d\xi_2$$

uniformly in D, and this uniformity, together with the fact that $\lim_{\varepsilon\to 0}u_\varepsilon(\mathbf{x})=u(\mathbf{x})$ uniformly in D, justifies the formula

$$\frac{\partial u}{\partial x_1}=\frac{1}{2\pi}\iint_D -\frac{x_1-\xi_1}{|\mathbf{x}-\boldsymbol{\xi}|^2}f(\xi_1,\xi_2)\,d\boldsymbol{\xi}. \tag{5}$$

Proceeding to the second derivative, we apply (2′) to obtain

$$\begin{aligned}\frac{\partial^2 u_\varepsilon}{\partial x_1^2}=&\frac{1}{2\pi}\iint_{D(x,\varepsilon)} -\frac{(x_2-\xi_2)^2-(x_1-\xi_1)^2}{|\mathbf{x}-\boldsymbol{\xi}|^4}f(\boldsymbol{\xi})\,d\boldsymbol{\xi}\\ &-\frac{1}{2\pi}\int_{x_2-\varepsilon}^{x_2+\varepsilon}\frac{\varepsilon}{\varepsilon^2+(x_2-\xi_2)^2}f(x_1-\varepsilon,\xi_2)\,d\xi_2\\ &+\frac{1}{2\pi}\int_{x_2-\varepsilon}^{x_2+\varepsilon}\frac{-\varepsilon}{\varepsilon^2+(x_2-\xi_2)^2}f(x_1+\varepsilon,\xi_2)\,d\xi_2.\end{aligned}\tag{6}$$

Dealing first with the case where $f(\xi)$ is constant—i.e. $f(\xi)=f(x)$, the last two integrals of (6) each become

$$\begin{aligned}-\frac{f(x)}{2\pi}\int_{x_2-\varepsilon}^{x_2+\varepsilon}\frac{\varepsilon}{(x_2-\xi_2)^2+\varepsilon^2}\,d\xi_2&=-\frac{f(x)}{2\pi}\operatorname{Arctan}\frac{(\xi_2-x_2)}{\varepsilon}\Big|_{x_2-\varepsilon}^{x_2+\varepsilon}\\ &=-\frac{f(x)}{2\pi}[\operatorname{Arctan}1-\operatorname{Arctan}(-1)]\\ &=-\frac{f(x)}{4}.\end{aligned}$$

However the uniform Hölder continuity can be used to write $f(\xi)=f(x)+[f(\xi)-f(x)]$, where

$$|f(\xi)-f(x)|\le C\,|x-\xi|^\gamma$$

for all $\xi\in D$, and this decomposition leads to estimates which imply that

$$\lim_{\varepsilon\to 0}\frac{\partial^2 u_\varepsilon}{\partial x_1^2}=\frac{1}{2\pi}\iint_D -\frac{(x_2-\xi_2)^2-(x_1-\xi_1)^2}{|\mathbf{x}-\boldsymbol{\xi}|^4}f(\boldsymbol{\xi})\,d\boldsymbol{\xi}-\frac{f(\mathbf{x})}{2}$$

uniformly in D. This uniformity implies that in the sense of principal values

$$\frac{\partial^2 u}{\partial x_i^2}=\frac{(-1)^i}{2\pi}\iint_D -\frac{(x_2-\xi_2)^2-(x_1-\xi_1)^2}{|\mathbf{x}-\boldsymbol{\xi}|^4}f(\boldsymbol{\xi})\,d\boldsymbol{\xi}-\frac{f(\mathbf{x})}{2};\qquad i=1,2$$

and the validity of (4).

The advantages of this approach include its close analogy to the much simpler case of $n=1$ and the relative ease with which one can evaluate the last two integrals of (6).

References

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