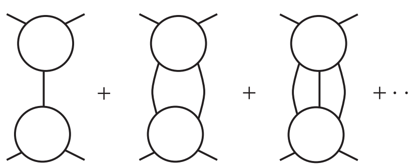
6
t-channel unitarity and growing interaction radius

Until now we have been exploiting analyticity and unitarity in the s-channel. We saw, in particular, how the s-channel unitarity put restrictions on the picture of strong interactions in the impact parameter plane and gave rise to the Pomeranchuk and Froissart theorems. As you remember, analyticity is related to causality and unitarity means that the sum of probabilities of all possible channels of particle *creation* equals one.

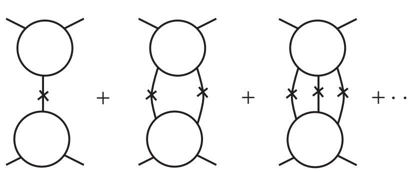
There is, however, one more condition, the one that is not easy to formulate. Namely, the probability that colliding particles *exchange* something,

$A(s,t) \sim$ 

also *cannot be bigger than one*. But in what sense?

The problem one faces trying to formulate such a restriction lies in the fact that it is *real* (on-mass-shell) particles that we can measure and 'count' while the exchange particles are *virtual*. Talking about virtual particles we would have to abandon our general picture in which all what matters are particle masses and on-mass-shell amplitudes ('imaginary parts').

We could make exchange particles real (and thus 'countable') if we chose positive t above corresponding thresholds,



(6.1)

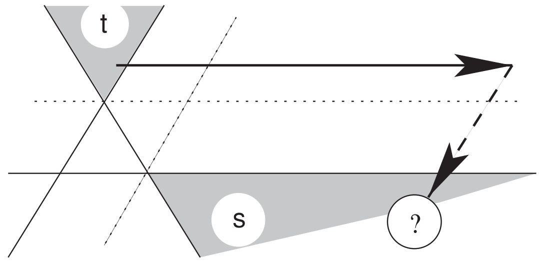
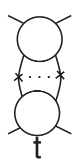


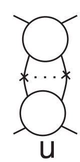
Fig. 6.1 Analytic continuation from t-channel to large imaginary scattering angles, $z_t \propto s \to \infty$ is close to the high energy s-channel scattering region.

This chain reminds unitarity relation written for t-channel scattering. Where does the t-channel unitarity operate? It holds for $t > 4\mu^2$ and negative s (momentum transfer) while we are interested in s positive, and large. This region is unphysical from the point of view of the s-channel so that (6.1) seems to be of little relevance.

On second thought, our amplitudes are analytic functions. We may try to formulate the new restriction we are looking for, by starting from $t > 4\mu^2$ and then *continuing* the t-channel unitarity condition to large s. In so doing we will find ourselves not far from an important physical region of the s-channel which describes high-energy processes with finite momentum transfer $|t| \ll s$, see Fig. 6.1.

By expressing the s-channel amplitude via its imaginary parts (discontinuities) in t- and u-channels,

$$A_3(s,t) \equiv \mathrm{Im}_t\, A = \frac{1}{2}\sum_n$$ 

, $$A_2(s,t) \equiv \mathrm{Im}_u\, A = \frac{1}{2}\sum_n$$ 

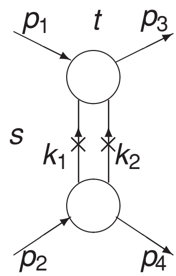
,

we would obtain *specific for the relativistic theory* consequences of the fact that the s-channel interaction is not arbitrary but occurs via exchange of particles in cross-channels. It is interesting to understand, what sort of new restrictions upon $f(\rho, s)$ the t-channel unitarity will impose.

Regretfully, the programme of analytic continuation of t-channel unitarity conditions was not fully completed. We only know how to carry out such continuation in simple cases, the simplest of which is the region $4\mu^2 < t < 16\mu^2$ (for pions) where the two-particle unitarity holds. This was done for the first time by Mandelstam.

https://doi.org/10.1017/9781009290227.007 Published online by Cambridge University Press

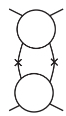
6.1 Analytic continuation of two-particle unitarity



In Lecture 3 we have discussed the s-channel two-particle unitarity condition (3.1). Let us rewrite it for the t-channel scattering, that is treating $t = (p_1 + [-p_3])^2$ as energy, $t > 4\mu^2$, and $s < 0$ (and $u < 0$) as momentum transfer(s).

The unitarity condition takes the form

$$\mathrm{Im}_t\, A(t,s) = \frac{1}{2i}\left[A(t+i\epsilon, s) - A(t-i\epsilon, s)\right]$$

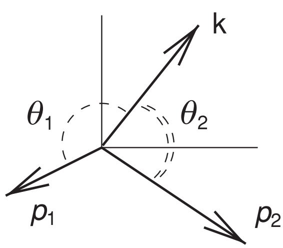
$$\equiv A_3(t,z) \;=\; \frac{1}{2}\cdot$$  $$= \tau \int \frac{d\Omega}{4\pi} A(\mathbf{p}_1, \mathbf{k}) A^*(\mathbf{k}, \mathbf{p}_2), \tag{6.2a}$$

where

$$z = \cos\Theta_{12} = 1 + \frac{2\,s}{t - 4\mu^2} \tag{6.2b}$$

and τ is now the t-channel phase-space volume factor

$$\tau \;=\; \tau(t) = \frac{k_c(t)}{8\pi\sqrt{t}} = \frac{1}{16\pi}\sqrt{\frac{t-4\mu^2}{t}}. \tag{6.2c}$$



The internal amplitudes $A(t, z_1)$ and $A^*(t, z_2)$ depend on the energy t and on the corresponding scattering angles,

$$z_i \equiv \cos\Theta_i = 1 + \frac{2\,s_i}{t - 4\mu^2}, \quad (i = 1, 2)$$

where $\Theta_{1(2)}$ is the angle between the initial (final) cms momentum $\mathbf{p}_{1(2)}$ and the intermediate-state momentum $\mathbf{k}$.

In order to continue (6.2) to large $z \propto s$, we are going to analyse analytic properties of the t-channel imaginary part A_3 in z. As a first step it is convenient to trade the angular integration for symmetric integrals over z_1 and z_2. Choosing the polar axis $\mathbf{z}$ along $\mathbf{p}_1$, we write

$$d\Omega = d(\cos\Theta_1)\cdot d\phi = dz_1 \cdot dz_2 \;\times\; J^{-1}.$$

The trigonometric relation

$$z_2 = z z_1 + \sqrt{(1-z^2)(1-z_1)^2}\cdot\cos\phi$$

https://doi.org/10.1017/9781009290227.007 Published online by Cambridge University Press

gives us the dependence $z_2 = z_2(\phi)$ and we derive

$$\left(\frac{dz_2}{d\phi}\right)^2 = (1-z^2)\big(1-z_1^2\big)\sin^2\phi = (1-z^2)\big(1-z_1^2\big) - (z_2-zz_1)^2.$$

The Jacobian is proportional to $|\sin\phi|$ and equals $J = \frac{1}{2}\sqrt{-K}$, where

$$\begin{aligned} K &\equiv (z_2 - zz_1)^2 - (1-z^2)(1-z_1^2) = \big(z^2 + z_1^2 + z_2^2\big) - 1 - 2zz_1z_2 \\ &= (z - z_1z_2)^2 - \big(1 - z_1^2\big)\big(1 - z_2^2\big). \end{aligned} \tag{6.3}$$

(The symmetry of the Jacobian $z_1 \leftrightarrow z_2$ should have been expected. The fact that it turned out to be symmetric with respect to *all three* cosines is less obvious, though true. We will exploit it in what follows.) We arrive at

$$A_3(t,z) = \frac{\tau}{2\pi}\iint \frac{dz_1\,dz_2}{\sqrt{-K(z,z_1,z_2)}}A(t,z_1)A^*(t,z_2), \tag{6.4}$$

where integration limits are determined by the condition $-K \geq 0$. Examining the integral in new variables we note that the dependence on z is localized in K so that we have a kind of an integral representation with the kernel $(-K)^{-1/2}$.

6.1.1 z_2 integration: pinch

What sort of integral is this? Let us move step by step and study first the integration over z_2 while keeping z_1 fixed. The z_2 integral runs from z_2^- to z_2^+, that is, between two zeros of K (where $\sin\phi = 0$):

$$z_2^{\pm} = zz_1 \pm \sqrt{(1-z^2)(1-z_1^2)}. \tag{6.5}$$

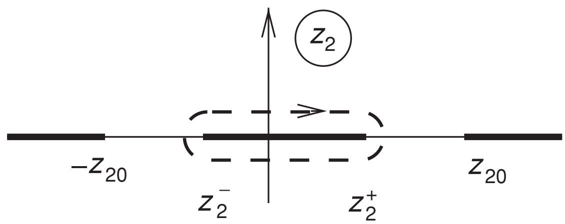
After that we will have to integrate over z_1 in the interval $[-1,1]$. Now that the double integral has been explicitly written, we need to find out what happens to it when we move outside the physical t-channel region $(-1 \leq z \leq 1)$ and keep increasing z.

Quite an exercise in the 'theory of functions of complex variable' is awaiting us. The task of continuing our integral would have been hopeless if we did not possess the knowledge of analytic properties of the amplitude. What should we expect? We know that in unphysical regions of the Mandelstam plane there are 'spectral domains' where A_3 becomes complex. Increasing s, I will inevitably hit these domains. It makes sense therefore to prepare ourselves to this eventuality.

A clever thing to do is to replace the integral over a fixed interval by path integration in the complex z_2-plane along the contour embracing the cut of the function $1/\sqrt{K}$:

$$\int_{z_2^-}^{z_2^+} \frac{dz_2}{\sqrt{-K}} = \frac{i}{2}\int_{\mathcal{C}} \frac{dz_2}{\sqrt{K}}.$$

In addition to the square-root branch cut $[z_2^-, z_2^+]$, our integrand as a function of z_2 has physical singularities of the amplitude $A(t, z_2)$ on the z_2-plane. These are s- and u-thresholds that start at $z_2 = \pm z_{20}$ and run to $\pm\infty$, correspondingly:



Now we increase z and pass through $z = 1$. When $z = 1$, the endpoints of the cut (6.5) *collide* at $z_2^- = z_2^+ = z_1$ and for $z > 1$ they become complex conjugate. At this point nothing dramatic happens to the answer since I will keep deforming calmly the integration contour by following the metamorphosis of the cut. How might our integral develop a singularity? Only if the tip of the cut would collide with one of the threshold singularities $\pm z_{20}$, pinching the contour.

At which value of the external variable z does the *pinch* occur? One needs to solve the equation, for example, $z_2^+ = z_{20}$. It is easy to realize that this equation has the same structure as (6.5) namely,

$$z = z^{\text{pinch}}(z_1) = z_1 z_{20} \pm \sqrt{(1 - z_1^2)(1 - z_{20}^2)}. \tag{6.6}$$

Since $z_{20} > 1$, the position of the pinch point corresponds to a complex z.

6.1.2 z_1 integration: contour trapping

The time has come to look for singularities of the integrand as a function of z_1. Introducing

$$f(t, z, z_1) = \frac{i\tau}{4\pi}\int_{\mathcal{C}} \frac{dz_2}{\sqrt{K(z, z_1, z_2)}} A^*(t, z_2),$$

we have

$$A_3 = \int_{-1}^{1} dz_1\, A(t, z_1) f(t, z, z_1).$$

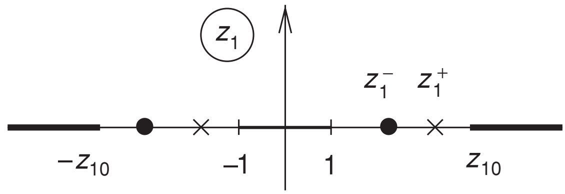


Fig. 6.2 Integration contour $[-1, 1]$ and singularities in the z_1 plane: right (A_1) and left cuts (A_2) of the amplitude $A(t, z_1)$ and pinch points z_1^- (circle) and z_1^+ (cross). The left pair of points solves an alternative pinch condition $z^+ = -z_{20}$.

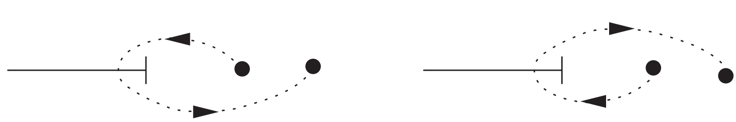
First of all, there are thresholds $z_1^\pm$ of the amplitude $A(t, z_1)$. Secondly, the singularities of the function $f(z_1)$ whose position $z_1 = z_1(z)$ we determine by inverting the pinch condition (6.6):

$$z_1^{\text{pinch}}(z) \equiv z_1^\pm \; = \; zz_{20} \pm \sqrt{(z^2-1)(z_{20}^2-1)}. \tag{6.7}$$

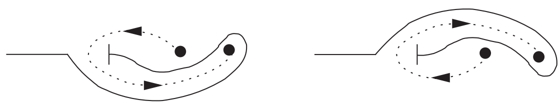
Mark that $z_1^\pm$ are real since $z_{20} > 1$ and $z \geq 1$. The structure of the z_1-plane will look as shown in Fig. 6.2 A symmetric pair of singularities $\tilde{z}_1^\pm$ on the left side of the z_1-plane solves the pinch equation $z_2^+ = -z_{20}$ complementary to (6.6). We will follow those on the right side of the plane, $z_1^\pm$.

With z increasing, the two singular points start off from $z_1^\pm = z_{20}$ at $z = 1$ and separate, z_1^- moving to the left and z_1^+ to the right. Can z_1^- collide with the integration interval $[-1, 1]$? From (6.7) it is clear that $z_1^- = +1$ indeed takes place at $z = z_{20}$. This is, however, the *absolute minimum* of $z_1^-(z)$ for a real z. (It becomes obvious if we parameterize $z = \cosh\eta$, $z_{20} = \cosh\eta_{20}$ resulting in $z_1^- = \cosh(\eta - \eta_{20}) \geq 1$.) This means that with z moving above z_{20}, the position of singularity reflects from $+1$ and increases indefinitely. A peculiar situation: z_1^- barely touches the integration interval and bounces off.

So is there a singularity or not? Let us show that the point $z = z_{02}$ is in fact *not singular*. We face here a curious phenomenon (I wonder if you have met anything of this sort in your maths course.) Imagine that while changing some external parameter, a singularity of the integrand touches the tip of the integration contour. To determine whether the answer for the *integral* will be singular at this value of the parameter we have to compare two ways of passing by this point, from above and from below:

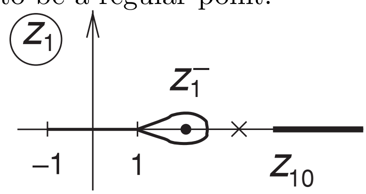


In order to have everything smooth and well defined, we will have to deform the contour correspondingly, and differently in two cases:



The analytic continuation of the integral that was initially defined on $[-1, 1]$ acquired an additional piece running from $+1$ to the new position of the singularity and back, the only difference between the two ways being the *direction* of the loop. If the two paths led to different results then we found a singularity of the integral. There exists, however, a trivial case when the two expressions coincide: when the singularity is a square root, so that the values on the sides of the cut are just opposite in sign. This being our case, we conclude $z = z_{20}$ to be a regular point.

However, something did happen. Namely, in spite of the fact that the function is non-singular, its explicit representation in terms of a contour integral has changed. The phenomenon we encountered is called 'contour trapping'. Now that we have the added loop that follows the movement of the point $z_1^-(z)$, a real possibility to develop a singularity finally emerges. The integral for A_3 becomes singular at the value of z when point $z_1^-(z)$ bumps on the threshold of the amplitude $A(t, z_1)$ at z_{10} and pinches the contour that it trapped and dragged along.



So, would there have been no singularity if not for z_{10}? Sure. In this case we would have $A_3(z) = \text{const}$ (or a polynomial in z at most). This was in fact implicit from the beginning: if the integrand $A(z_1)$ did not depend on the scattering angle Θ_1, the l.h.s. of (6.2a) would have been independent of the angle Θ as well; in other words, it would not have singularities in z.

Finally, solving the 'collision' equation $z_1^- = z_{10}$ for the position of singularity, we obtain

$$z \;=\; z_{10}z_{20} + \sqrt{(z_{10}^2 - 1)(z_{20}^2 - 1)}. \tag{6.8}$$

Substituting explicit expressions for z_{i0} we derive again the familiar equation describing the *Karplus curve* – the boundary of the double spectral function ρ_{st}.

https://doi.org/10.1017/9781009290227.007 Published online by Cambridge University Press

6.1.3 Imaginary part of the imaginary part

Let us take z above the singularity, $z_1^- > z_{10}$, and calculate the imaginary part of A_3 that is the discontinuity in s (in z):

$$\rho_{st} = \mathrm{Im}_s\, A_3(t,z) = \int_{z_{10}}^{z_1^-} dz_1\, \mathrm{Im}_s\, A(t,z_1) \cdot \Delta f(t,z,z_1). \tag{6.9a}$$

What is Δf in this expression? Recall that we had the contour pinched in the z_2-integration as well; Δf stands for the corresponding discontinuity over the cut of the amplitude in the z_2-plane:

$$\Delta f(t,z,z_1) \sim \int_{z_{20}}^{z_2^+} \frac{dz_2}{\sqrt{K}}\, \mathrm{Im}_s\, A(t,z_2). \tag{6.9b}$$

We arrive at the expression of the same structure as (6.4) for the A_3 itself but integrated over a different region,

$$\rho_{st} \sim \iint \frac{dz_1\, dz_2}{\sqrt{K(z,z_1,z_2)}}\, \mathrm{Im}_s\, A(t,z_1)\, \mathrm{Im}_s\, A^*(t,z_2). \tag{6.10a}$$

We don't need to worry about the lower limits of the integrals since the factors $\mathrm{Im}\, A(t,z_i)$ themselves know about z_{10}, z_{20}. As for the upper limits, they are given by the inequality

$$z_1 z_2 + \sqrt{(z_1^2-1)(z_2^2-1)} \,\le\, z \tag{6.10b}$$

which is equivalent to $K > 0$, see (6.3).

Anything else? Until now we have been studying only positive z_1, z_2. Considering analogously left-side singularities on Fig. 6.2 we will restore the u-channel contribution. Using our old notation for imaginary parts of the amplitude in s and u channels, $A_1 \equiv \mathrm{Im}_s\, A$ and $A_2 \equiv \mathrm{Im}_u\, A$, the final formula reads

$$\rho_{st} = \frac{\tau}{\pi} \iint \frac{dz_1\, dz_2}{\sqrt{K(z,z_1,z_2)}} [\, A_1(t,z_1) A_1^*(t,z_2) + A_2(t,z_1) A_2^*(t,z_2)\,], \tag{6.11a}$$

where the integration is performed over the region (6.10b); $z_1 > 1$, $z_2 > 1$.

If I chose to continue analytically the t-channel unitarity condition to $s \to -\infty$ (instead of $+\infty$), I would obtain a similar integral expression for another double spectral function,

$$\rho_{ut} = \frac{\tau}{\pi} \iint \frac{dz_1\, dz_2}{\sqrt{K(z,z_1,z_2)}} [\, A_1(t,z_1) A_2^*(t,z_2) + A_2(t,z_1) A_1^*(t,z_2)\,]. \tag{6.11b}$$

Mandelstam equations (6.11) solve the problem of analytic continuation of the t-channel unitarity condition. Thus we learned how to express 'imaginary parts of the imaginary parts' ρ_{ij} via the imaginary parts A_i of the amplitude themselves!

6.1.4 Mandelstam representation

We have obtained the double discontinuity ρ_{st} in the following order: we were sitting in the t channel at $t > 4\mu^2$, took $A_3 = \text{Im}_t\, A$, then, by continuing A_3 to $|z_t| > 1$, moved to the s-channel and there evaluated $\text{Im}_s\, A_3$. We could have done it in the opposite order, namely start from $A_1 = \text{Im}_s\, A$ in the s-channel, $t < 0$, and then increase t to access $\text{Im}_t\, A_1$ at $t > 4\mu^2$. It is natural to expect that this way we would have got the same expression (6.11a) for ρ_{st},

$$\rho_{st}(s,t) = \text{Im}_s\, A_3(s,t) = \text{Im}_t\, A_1(s,t).$$

Although a formal proof does not exist, this statement would be definitely correct if the amplitude admitted the double integral representation

$$A(s,t) = \frac{1}{\pi^2} \iint \frac{\rho_{st}(s',t')\, ds'\, dt'}{(s'-s)(t'-t)} + [s \to u] + [t \to u], \tag{6.12}$$

where the integration region is restricted by the Karplus curve in the s'–t' plane. Since 1958, when Mandelstam suggested the representation (6.12) for the invariant amplitude Mandelstam (1958), no Feynman graph has been found which would violate it (provided all participating particles are stable, $m_a < m_b + m_c + m_d$).

The spectral density ρ_{st} corresponds to simultaneously evaluating discontinuities over s in t and bears information about unitarity in both channels. This object is therefore well suited to support our expectation that probabilities of particle *creation* and particle *exchange* are not independent. Such inter-dependence is a specific feature of the relativistic theory, in marked difference to non-relativistic quantum mechanics.

6.2 $\rho_0 = \text{const}$, $\sigma_{\text{tot}} = \text{const}$ contradicts *t*-channel unitarity

The Froissart theorem provided us only with *upper bounds* for growth rates of $\rho_0(s)$ and $\sigma_{\text{tot}}(s)$. Now we will show that in a relativistic theory the radius $\rho_0(s)$ *must* virtually always *grow* with s. (To be precise, it is allowed *not to grow* only if the total cross section falls faster than $1/\ln s$ at asymptotically high energies.)

As we have discussed above, the hypothesis $\rho(s) \to \text{const}$ implies that the s- and t-dependence of the scattering amplitude factorize,

$$A(s,t) \overset{s\to\infty}{=} s \cdot F(t), \tag{6.13}$$

https://doi.org/10.1017/9781009290227.007 Published online by Cambridge University Press

where we choose $A \propto s$ to ensure asymptotically constant $\sigma_{\rm tot}$. Once the amplitude has such a form in the physical region of the s channel, then, by virtue of analyticity, it has to have the same structure at positive t as well, it seems. We will suppose that (6.13) holds for finite t (of any sign), but then it has to satisfy the equations (6.11) that we have derived for moderate positive t (in the interval $4\mu^2 < t < 16\mu^2$). Let us see if it really does. From (6.13) we get

$$A_1(s,t) \simeq s \,\mathrm{Im}\, F \equiv s \cdot F_1(t); \quad \rho_{st} \simeq s \cdot \mathrm{Im}\, F_1(t) \quad \text{for } t > 4\mu^2. \tag{6.14a}$$

Analogously for the *antiparticle* scattering amplitude, in the crossing channel, $u \to \infty$,

$$A_2(u,t) \simeq u \cdot F_2(t); \qquad \rho_{ut} \simeq u \cdot \mathrm{Im}\, F_2(t) \quad \text{for } t > 4\mu^2. \tag{6.14b}$$

Thus, we wrote down explicitly all the ingredients of the Mandelstam relations (6.11) for the double spectral densities ρ_{st} and ρ_{ut}. This means that we can verify our model (6.13) provided the dominant contribution to the integral comes from the region of large internal energies. Let us start calculating the integral (6.11a) *supposing* that $z_1, z_2 \gg 1$ and then verify that this is indeed true.

Approximating the Jacobian

$$\begin{aligned} -K &= \left[z - z_1 z_2 + \sqrt{(z_1^2-1)(z_1^2-1)}\right]\left[z - z_1 z_2 - \sqrt{(z_1^2-1)(z_1^2-1)}\right] \\ &\simeq z(z - 2z_1 z_2), \end{aligned}$$

and substituting the asymptotic approximation (6.14) for the block amplitudes A_1 and A_2 we obtain

$$\rho_{st} \simeq \frac{\tau}{\pi} \int \frac{dz_1\, dz_2 \cdot z_1 z_2}{\sqrt{z(z-2z_1 z_2)}} \cdot \left[\frac{t-4\mu^2}{2}\right]^2 \left[F_1 F_1^* + F_2 F_2^*\right].$$

The integrand depends only on the product $z_1 z_2 = x$, therefore

$$\begin{aligned} \rho_{st} &\propto \int_{z_{10}}^{z/z_{20}} \frac{dz_1}{z_1} \int_{z_1 z_{20}}^{z/2} \frac{x\, dx}{\sqrt{z(z-2x)}} \simeq z \int_{z_{10}}^{z/z_{20}} \frac{dz_1}{z_1} \int_0^{1/2} \frac{y\, dy}{\sqrt{1-2y}} \\ &\propto z \ln \frac{z}{z_{10} z_{20}} \propto s \ln s. \end{aligned} \tag{6.15}$$

The inconsistency of our calculation with (6.14a) is apparent:

$$\mathrm{Im}\, F_1(t) = \lim_{s\to\infty} \frac{\rho(s,t)}{s} \quad \overset{?}{\simeq} \quad \frac{c}{\mu^2} \ln s + \frac{1}{\mu^2} \cdot \mathcal{O}(1). \tag{6.16}$$

https://doi.org/10.1017/9781009290227.007 Published online by Cambridge University Press

The unwanted dominant contribution $\mathcal{O}(\ln s)$ came from the specific integration region $z_1 z_2 \sim z$, $z_0 \ll z_1, z_2 \ll z$ (which, by the way, confirms our initial decision to use asymptotic formulae for the internal blocks). To scrutinize other regions won't help since the integrand is positively definite so that there can be no cancellation.

6.2.1 $\rho_0(s)$ for arbitrary $\sigma_{\text{tot}}(s)$

Let us release the $\sigma_{\text{tot}} = \text{const}$ condition and look whether the radius can stay asymptotically constant in the general case. The generalization reads

$$\begin{aligned} A_1(s,t) &\simeq s \cdot h(s) \cdot F_1(t), \\ \rho_{st} &\simeq s \cdot h(s) \cdot \operatorname{Im} F_1(t), \qquad (\rho \equiv 0 \ \text{ for } \ t < 4\mu^2). \end{aligned} \tag{6.17}$$

Then (cf. (6.15))

$$\rho(s,t) \propto \int \frac{dz_1\, dz_2\, z_1 z_2 \cdot h(z_1) h(z_2)}{\sqrt{z(z-2z_1z_2)}} \simeq z \int\limits_{z_{10}}^{z/z_{20}} \frac{dz_1}{z_1} h(z_1) \int\limits_0^{1/2} \frac{y\, dy}{\sqrt{1-2y}} h\left(\frac{zy}{z_1}\right).$$

Since the y-integral converges, we can substitute a constant $c = \langle y \rangle = \mathcal{O}(1)$ for y in the argument of the second h-function to obtain

$$\operatorname{Im} F_1(t) \propto \frac{\rho(s,t)}{zh(z)} \sim \frac{1}{h(z)} \int_{z_{10}}^{z/z_{20}} \frac{dz_1}{z_1} h(z_1) h\left(\frac{z}{z_1} c\right). \tag{6.18}$$

To avoid contradiction, the r.h.s. of (6.18) has to have a finite $z \to \infty$ limit. It is easy to see that this is possible only if

$$h(z) < \frac{\text{const}}{\ln z}, \qquad z \to \infty. \tag{6.19}$$

Only in this case which corresponds to a falling total cross section,

$$\sigma_{\text{tot}}(s) \ < \ \frac{\text{const}}{\ln s}, \qquad s \to \infty,$$

the constant interaction radius would not contradict t-channel unitarity.

We wrote the unitarity condition valid for $4\mu^2 < t < 16\mu^2$, made use of the concrete form of the amplitude at $s \to \infty$ and finite t and came to a contradiction with the hypothesis $\sigma_{\text{tot}} = \text{const}$.

What is the reason for that?

The picture that caused us trouble is that of Fig. 6.3(a). It is related to the production process of two showers of particles in a high energy $\pi\pi$ collision with the exchange of a pion, Fig. 6.3(b).

From the very beginning we supposed that the total $\pi\pi$ interaction cross section is constant at high energy. But it is this $\sigma_{\pi\pi}$ that twice

https://doi.org/10.1017/9781009290227.007 Published online by Cambridge University Press

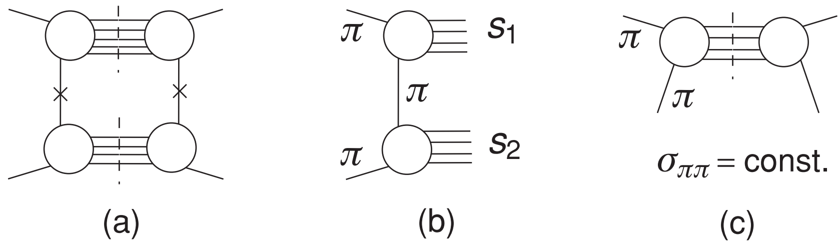


Fig. 6.3 On the 'black disc' ansatz (6.13) versus t-channel unitarity.

enters the graph we have selected. Now, however, we can vary the total energy partitioning between showers, $s\mu^2 \sim s_1 s_2$, adding contributions with different s_1 and s_2. Since for each of the two pion–pion interaction sub-processes Fig. 6.3(c) $\sigma_{\rm tot} \to$ const, we obtained an additional $\ln s$ enhancement due to integration over shower masses.

Does our contradiction mean that $\sigma_{\pi\pi}$ cannot be constant in the high energy limit? No. This only tells us that it is wrong to think that the t dependence of the amplitude is determined exclusively by the nearest singularity due to one-pion exchange: $\rho_0 \simeq (2\mu)^{-1} =$ const. Multi-meson exchanges must be important, interfering with one pion; the higher the energy s, the more the amplitude has to 'remember' about the faraway singularities in t. In other words, we can no longer consider the interaction radius to be energy independent, unless $\sigma_{\rm tot}$ falls with s.

Thus, trying to preserve asymptotic constancy of the total cross section, we have to abandon the factorization ansatz (6.13) and look for a more complicated structure of the amplitude; we have to have ρ_0 changing with energy.

There were times when the constancy of the interaction radius was held in deep respect, people thought that it had a deep physical meaning. Later it transpired that the truth is just the opposite: it is *practically impossible* to have it not growing with energy.

6.2.2 Numerical estimate

How 'serious' is the contradiction with unitarity that we have faced? Look more attentively at our relation:

$$\operatorname{Im} F_1 \ \gtrsim\ \frac{\tau}{4\pi}\cdot\left[\frac{t-4\mu^2}{2}\right]\cdot 2\,|F_1|^2\cdot\ln s.$$

For $t \sim \mu^2$ we can take

$$\operatorname{Im} F_1(t) \sim F_1(t) \sim F_1(0) = \sigma_{\rm tot} \sim \frac{1}{\mu^2}$$

https://doi.org/10.1017/9781009290227.007 Published online by Cambridge University Press

as a rough estimate. Stepping away from the t threshold by $(t - 4\mu^2) \sim \mu^2$ then gives for the numerical coefficient c of the logarithmic term in (6.16)

$$c = \frac{\tau}{4\pi} \cdot \left[(t - 4\mu^2)F_1(t)\right] \sim \frac{\tau}{4\pi} \simeq \frac{1}{4\pi}\frac{1}{16\pi} \sim \frac{1}{600}.$$

This means that though the radius *has* to grow, it may do so very slowly: the formal contradiction starts to be really important only at fantastically high energies, $\ln s \sim 600$.

6.2.3 Modelling a growing radius

Let us attempt to model a growing radius. We wrote $A_1 = s\,F(t)$ for the black-disc picture and failed. Try

$$A_1(s,t) = s^{\alpha(t)}F(t), \tag{6.20}$$

such that

$$\alpha(0) = 1, \qquad \alpha(t) < 1 \quad \text{for} \;\; t < 0.$$

For a finite t we then have approximately

$$A_1(s,t) \simeq s\,\mathrm{e}^{\alpha' t \ln s}F(t) = s\,\mathrm{e}^{-\alpha' \mathbf{q}^2 \ln s}F(-\mathbf{q}^2). \tag{6.21}$$

The essential momentum transfer $\mathbf{q}$ in (6.21) is

$$|\mathbf{q}| \simeq \frac{1}{\sqrt{\alpha' \ln s}},$$

which immediately translates into the energy-dependent radius

$$\rho_0(s) \;\simeq\; \sqrt{\alpha' \ln s}. \tag{6.22}$$

What will change in the t-channel unitarity condition? Examine the r.h.s. of (6.11a):

$$A_1(z_1,t\,)A_1^*(z_2,t) \;\propto\; z_1^{\alpha(t)} z_2^{\alpha^*(t)}.$$

Above the threshold, $t > 4\mu^2$, both $F(t)$ and $\alpha(t)$ in (6.20) will become complex in general:

$$\Longrightarrow\; z_1^{\alpha_1 + i\alpha_2} z_2^{\alpha_1 - i\alpha_2} = (z_1 z_2)^{\alpha_1} \exp\left\{ i\alpha_2 \ln\frac{z_1}{z_2} \right\}, \qquad \alpha_2 = \operatorname{Im}\alpha(t).$$

Recall that the logarithmic s-dependence occurred due to fact that the integrand depended solely on the product $z_1 z_2$. Now, on the contrary, we

https://doi.org/10.1017/9781009290227.007 Published online by Cambridge University Press

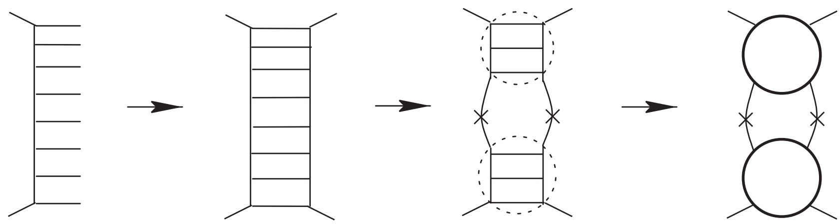


Fig. 6.4 Relation of ladder-like inelastic processes to t-channel unitarity.

have an oscillating function of the ratio z_1/z_2 which will force the integral to converge and produce

$$\int d\ln\frac{z_1}{z_2}\ \exp\left\{i\alpha_2\ln\frac{z_1}{z_2}\right\}\sim\frac{1}{\alpha_2}$$

in place of $\ln s$. Our consistency condition (6.16) will turn into

$$\operatorname{Im} F_1(t)\simeq\frac{c}{\mu^2}\cdot\frac{1}{\alpha_2}+\frac{1}{\mu^2}\cdot\mathcal{O}(1)\,. \tag{6.23}$$

The formal contradiction is gone. Moreover since c is numerically small, we may have $\alpha' t \ll 1$ and $\alpha(t)\simeq 1$ in a broad region of momentum transfer t.

A comment is in order. In Section 5.6 we saw how to 'construct' a growing radius. To do so we allowed a fast incident particle to slow down before hitting the target, by emitting a whole 'comb' of virtual particles on the way.

Imagine that inelastic processes have indeed the structure of a 'comb' as shown in Fig. 6.4(a).

Then, by s-channel unitarity, squaring the amplitude (a) we get the forward scattering amplitude as a 'ladder' of Fig. 6.4(b). A remarkable thing about this picture, based on *repetitions* in the t-channel, is that it directly solves t-channel unitarity!

Indeed, by taking discontinuity in t somewhere along the graph, Fig. 6.4(c), the upper and the lower parts of the 'ladder' will sum up into the full interaction amplitudes as shown by blocks in Fig. 6.4(d). Hence, the *necessity* (as well as *opportunity*) of having the radius grow with energy is related to the possibility of *repetitions* in the t-channel which are the key to the t-channel unitarity.

https://doi.org/10.1017/9781009290227.007 Published online by Cambridge University Press

To move further we need to investigate which solutions are reasonable, what are realistic strong interaction amplitudes in the deep asymptotic regime. One could continue along the lines of this lecture and study the restrictions imposed by cross-channel unitarity conditions.

It turns out, however, that there is a more elegant way to find asymptotics of relativistic amplitudes by establishing a transparent link with the old non-relativistic theory.

https://doi.org/10.1017/9781009290227.007 Published online by Cambridge University Press