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The Remainder Theorem.

No theorem of elementary algebra has given rise to more discussion than the Remainder Theorem :

If a polynomial $f(x)$ *be divided by* $(x - a)$, *the remainder is* $f(a)$.

It is hoped that this note may be the means of settling questions connected with it once for all, and may aid in bringing about correct methods in teaching.

Here is first of all an erroneous proof on which comment is made later :—

Let $f(x)$ be divided by $(x - a)$, and let the quotient be Q and the remainder R.

$$\text{Then } \frac{f(x)}{x-a} = Q + \frac{R}{x-a}. \qquad \ldots.(1)$$

$$\text{Hence } f(x) = (x-a)Q + R. \qquad \ldots.(2)$$

$$\text{When } x = a \text{ this gives } f(a) = (a-a)Q + R = R. \qquad \ldots.(3)$$

The first thing to which exception might be taken is the use of the single letter Q to denote the quotient. The quotient, as it appears in (1), (2) is a function of x ; in (3) the quantity denoted by Q is the value of this function for $x = a$. We must therefore write $Q(x)$ in (1), (2), $Q(a)$ in (3). This error appears in *Charles Smith's Treatise on Algebra*, and is often repeated by those who use or have used that book.

The second error is more serious: Line (1) clearly demands that $x \neq a$; when $x = a$ it has no meaning. But $x = a$ is the value we require in line (3). And so we have a fallacy. This is an error into which even so great a logician as Chrystal has fallen.[1] It is an error

[1] *Text-Book of Algebra.* Part I., p. 96.

natural to a mathematician, who has ideas of continuity lurking at the back of his mind. But continuity is not a method of elementary algebra, and it certainly cannot be taught to classes at the Remainder Theorem stage.

The difficulty seems to arise out of a wrong conception of the meaning of the word "division" in the statement of the theorem. The reference is to the process of "long division," generally tabulated thus :—

$$\begin{array}{l|c|l} x-a & f(x) & Q(x) \\ & \underline{(x-a)Q(x)} & \\ & R & \end{array}$$

A glance at this table will make clear what it does : it enables us to find a polynomial $Q(x)$ such that, if $(x-a)Q(x)$ be subtracted from $f(x)$, the remainder R is of degree lower than the divisor $x-a$. If we desire to express the result in the form of an equation, that equation is :

$$f(x)-(x-a)Q(x)=R$$

or its exact equivalent

$$f(x)=(x-a)Q(x)+R \qquad \ldots.(4)$$

together with the inequality

Degree of $R <$ degree of divisor.

The table does *not* inform us at once that

$$\frac{f(x)}{x-a}=Q(x)+\frac{R}{x-a}. \qquad \ldots.(5)$$

This equation is a deduction from (4), not conversely,[1] and is valid only on the express understanding that $x-a$ is not zero.

The fact is that the word *division* is used in two slightly different senses :

(1) Division, properly speaking, is the operation which is the inverse of multiplication, and is often introduced by the symbol $\div$. To divide $f(x)$ by $(x-a)$ means, by definition,[2] to find a quantity F such that $(x-a)F=f(x)$. $\ldots.(6)$

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[1] See *Chrystal. loc. cit.*, p. 88 (at foot).

[2] *Chrystal*, p. 14.

The theory of division goes on to show that if $x \neq a$, the quantity F in (6), is well defined; under this stipulation the notation is introduced $F = \frac{f(x)}{x - a}$.

(2) The term "division" is later applied to the process previously described. The first use, to which this process was applied, was the transformation—Equation (5)—of a fraction like F above. Hence its name. But the process has other uses, for example in the theory of the Highest Common Factor,[1] and as a rule it is an equation like (4), not (5), which we require of it. In any case the important point is that the fundamental equation which expresses the result of the process is (4), not (5); and that even in the operation of division itself, the fundamental equation of definition is a similar equation, namely equation (6).

In the Remainder Theorem, the use of the word "remainder" should be sufficient to inform us that we have to do with the process rather than the operation. The proof of the Theorem should therefore be preceded by a discussion of the process, beginning from the definition:—

Definition: The process of division of one polynomial $f(x)$ by a second polynomial $D(x)$ consists in finding two polynomials $Q(x)$, the Quotient, and $R(x)$, the Remainder, with the properties:

(i) $f(x) \equiv D(x) \times Q(x) + R(x)$.

(ii) *$R(x)$ is of lower degree than $D(x)$.*

It should be noted that for a complete definition both (i), (ii) are necessary; both must therefore be used, not to say mentioned, in any complete proof of the Remainder Theorem. This fact points to a third weakness in the "proof" first given, namely its incompleteness.

And now to prove the Theorem :—

Let $f(x)$ be divided by $(x - a)$, and let the quotient be $Q(x)$, the remainder $R(x)$.

By definition, $R(x)$ is of lower degree than $(x - a)$, and so is a constant, R, say and $f(x) \equiv (x - a)Q(x) + R$.

[1] *Chrystal*, p. 114.

Now let $x = a$, and we have :

$$f(a) = (a - a)Q(a) + R = R.$$

The question is often asked : Is it not possible to avoid the difficulty, by adopting the alternative proof, depending on the fact that $x - a$ is a factor of $x^n - a^n$? It is not. There is and can be no method of proving a *remainder* theorem which does not require a clear knowledge of the meaning of the term.

Further, the proof given above has an advantage over the alternative proof, in addition to that of brevity. To complete our definition of the process of division, we require to prove its validity. That is, we must prove, *inter alia*, the existence of a remainder with the properties stated; and the enunciation of the Theorem presupposes that this has already been done. But the definition does not guarantee the existence of a unique remainder, and we have, apart from proof, no reason to expect that the remainder is unique. The alternative proof shows that division can be carried out in such a way that $f(a)$ is the remainder, but it leaves us in doubt as to whether there might not be another mode of division leading to a different result. The method above leaves no such doubt. Suppose the division carried out in any way, then the remainder is $f(a)$.

JAMES HYSLOP.

Linear Transformations and Geometry.

The following note suggests certain connections between the theory of linear transformations and quadratic forms on the one hand, and the geometry of second degree surfaces on the other. It is hoped that the note may prove useful to those who may have to teach either theory to students who already possess an elementary knowledge of the other. The general ideas may be such as may well have occurred to anyone familiar with both theories, but the examples given may be new to readers.

In the geometry Cartesian co-ordinates are used throughout, and the axes of reference are rectangular, unless the contrary is stated ; it may however be noted that many of the results, including those of § 1, are valid also when the axes are oblique. For simplicity the discussion is restricted to three dimensions, but the results hold in a space of any number of dimensions.