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ON COMPLEX HOMOGENEOUS SPACES WITH TOP HOMOLOGY IN CODIMENSION TWO

D. N. AKHIEZER AND B. GILLIGAN

ABSTRACT. Define d_X to be the codimension of the top nonvanishing homology group of the manifold X with coefficients in $\mathbb{Z}_2$. We investigate homogeneous spaces $X := G/H$, where G is a connected complex Lie group and H is a closed complex subgroup for which $d_X = 1, 2$ and $\mathcal{O}(X) \neq \mathbb{C}$. There exists a fibration $\pi: G/H \longrightarrow G/U$ such that G/U is holomorphically separable and $\pi^*(\mathcal{O}(G/U)) = \mathcal{O}(G/H)$, see [11]. We prove the following. If $d_X = 1$, then $F := U/H$ is compact and connected and $Y := G/U$ is an affine cone with its vertex removed. If $d_X = 2$, then either F is connected with $d_F = 1$ and Y is an affine cone with its vertex removed, or F is compact and connected and $d_Y = 2$, where Y is $\mathbb{C}$, the affine quadric Q_2, $\mathbb{P}_2 - Q$ (with Q a quadric curve) or a homogeneous holomorphic $\mathbb{C}^*$-bundle over an affine cone minus its vertex which is itself an algebraic principal bundle or which admits a two-to-one covering that is.

1. **Introduction.** A classical approach to understanding Lie groups and homogeneous spaces of Lie groups is to reduce to related compact objects. The fundamental theorem of E. Cartan-Malcev-Iwasawa says that a connected Lie group G is homeomorphic to the product of a maximal compact subgroup $K \subset G$ and a euclidean space. For coset spaces G/H with $|H/H^\circ| < \infty$ one has an analogue of this theorem. Namely, G/H fibers as a vector bundle over a minimal K-orbit in G/H, see [17], [15] and also [7]. The dimension $d = d_{G/H}$ of the fiber of this vector bundle is an important topological invariant of the space.

In the above setting it is clear that G/H is retractable onto some compact submanifold of codimension exactly $d_{G/H}$. Thus $d_{G/H}$ can also be defined as the codimension of the top nonvanishing homology group with coefficients in $\mathbb{Z}_2$. This definition is then applicable to any homogeneous space (independent of whether $|H/H^\circ| < \infty$ or not).

H. Abels [1] introduced an invariant $n_c(X; k)$ for any locally compact topological space X and non-zero abelian group k. He also studied some of its properties, particularly in the context of proper group actions. For X a manifold it is easy to see by means of Poincaré duality that $n_c(X; \mathbb{Z}_2) = d_X$.

In this paper we investigate homogeneous manifolds of complex Lie groups for which this invariant is small. Some cases with $d_{G/H}$ small have been studied earlier. In particular, for G and H linear algebraic groups, see [2] and [3]. It turns out that holomorphically separable homogeneous complex manifolds are surprisingly related to

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the algebraic category. More generally, we consider in this paper manifolds X satisfying $\mathcal{O}(X) \neq \mathbb{C}$.

We recall that given a homogeneous complex manifold G/H then there exists a closed complex subgroup U of G containing H such that G/U is holomorphically separable and if $\pi\colon G/H \longrightarrow G/U$ denotes the natural projection, then $\pi^*\big(\mathcal{O}(G/U)\big) \cong \mathcal{O}(G/H)$, see [11]. The map π is called the *holomorphic reduction of* X. If the fibers of the holomorphic reduction are discrete, then X is said to satisfy the maximal rank condition.

In the sequel we make use of the following construction. Consider an equivariant imbedding of a homogeneous projective rational manifold in some $\mathbb{P}_m$. The affine cone in $\mathbb{C}^{m+1}$ over the image of such an imbedding is an almost homogeneous space with two orbits. The closed orbit is the origin. In this paper its complement, *i.e.* the open orbit, is called an *affine cone minus its vertex*. Note that this orbit is a quasi-affine algebraic manifold with $d = 1$.

THEOREM. *Suppose G is a connected complex Lie group and H is a closed complex subgroup such that $X := G/H$ satisfies $\mathcal{O}(X) \neq \mathbb{C}$ and $d_X \leq 2$. Let $Y := G/U$ be the base of the holomorphic reduction of X and $F := U/H$ be its fiber.*

a) If $d_X = 1$, then F is compact and connected and Y is an affine cone minus its vertex.

b) If $d_X = 2$, then one of the following two cases occurs:

b_1) The fiber F is connected and satisfies $d_F = 1$ and the base Y is an affine cone minus its vertex.

b_2) The fiber F is compact and connected and $d_Y = 2$; moreover, Y is one of the following manifolds:

1) The complex line $\mathbb{C}$;

2) The affine quadric Q_2;

3) $\mathbb{P}_2 - Q$, where Q is a quadric curve;

4) A homogeneous holomorphic $\mathbb{C}^*$-bundle over an affine cone with its vertex removed which is either itself an algebraic principal $\mathbb{C}^*$-bundle or is covered two-to-one by such.

A short outline of the organization of the paper is as follows. In the second section we define the invariant d and present some of its properties. The algebraic case is treated in the third section. The fourth section deals with discrete isotropy. After this the remaining step is to use the normalizer fibration to show that we can reduce to the algebraic category. This is done in the fifth section.

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2. **Topological preliminaries.**

DEFINITION. Given any manifold X define

$$h_X := \min\{r \mid H_k(X, \mathbb{Z}_2) = 0, \text{ for all } k > r\}$$

and set

$$d_X := \dim X - h_X.$$

Any manifold has a triangulation and one can use it to compute homology. In the noncompact case there are no cycles in dimension $k = \dim X$. In the compact case the (finite) sum of all simplices of highest dimension is a cycle (at least mod 2). Therefore $d_X > 0$ for a noncompact manifold and $d_X = 0$ for a compact manifold.

Suppose we have a locally trivial fiber bundle $X \xrightarrow{F} B$, where F, X and B are connected manifolds.

LEMMA 1. *If the fiber bundle $X \xrightarrow{F} B$ is orientable, then*

$$d_X = d_F + d_B.$$

In particular, this is true if the base B is simply connected.

PROOF. Consider the homological spectral sequence of the given fiber bundle, taking $\mathbb{Z}_2$ for the coefficient group. In the orientable case we have

$$E^2_{rs} \simeq H_r(B, \mathbb{Z}_2) \otimes H_s(F, \mathbb{Z}_2).$$

Thus

$$E^2_{rs} = \{0\} \quad \text{if } r > h_B \text{ or } s > h_F$$

and

$$E^2_{rs} \neq \{0\} \quad \text{if } r = h_B \text{ and } s = h_F.$$

Clearly these relations remain valid for E^∞, and our claim follows. ∎

We also need some information in the case when it is false or unknown that the fiber bundle is orientable. Recall that every manifold has the homotopy type of a CW-complex, see [16]. In the compact case the dimension of this CW-complex is equal to the dimension of the manifold. In the noncompact case one can always find a CW-complex of smaller dimension, see [19].

LEMMA 2. *Assume that B has the homotopy type of a* CW-*complex of dimension q. Then*

$$d_X \geq d_F + (\dim B - q).$$

If B is homotopy equivalent to a compact manifold, then

$$d_X \geq d_F + d_B.$$

PROOF. First we claim that
$$h_X \leq h_F + q.$$
In the spectral sequence we have
$$E^1_{rs} \simeq C_r(B) \otimes H_s(F, \mathbb{Z}_2),$$
where $C_r(B)$ is a free abelian group generated by the cells of B. It follows that $E^1_{rs} = \{0\}$ if $r > q$ or $s > h_F$. Therefore $H_k(X, \mathbb{Z}_2) = \{0\}$ for all $k > h_F + q$ showing that $h_X \leq h_F + q$.

Now $d_X = \dim X - h_X \geq \dim F + \dim B - (h_F + q) = (\dim F - h_F) + (\dim B - q) = d_F + (\dim B - q)$ and this proves the first inequality.

In order to obtain the second one denote by M a compact manifold, which is homotopy equivalent to B. Since M has the homotopy type of a CW-complex of the same dimension, one can take $q = \dim M$. Observe that $\dim M = h_M$ and $h_B = h_M$. Therefore we can replace $\dim B - q$ by $\dim B - h_B = d_B$. ■

In this paper we use the expression Y is retractable onto M instead of saying that M is a strong deformation retract of Y.

LEMMA 3. *Let X, Y be connected manifolds, $\pi: X \longrightarrow Y$ an unramified covering, and $M \subset Y$ a compact submanifold. Assume that Y is retractable onto M. Then*
$$d_X = d_Y + d_{\pi^{-1}(M)}.$$
In particular, $d_X > d_Y$ if π is infinite.

PROOF. Since X is obviously retractable onto $\pi^{-1}(M)$, we have
$$h_{\pi^{-1}(M)} = h_X.$$
Also since Y is retractable onto M and M is compact,
$$h_Y = h_M = \dim M.$$
Therefore
$$\begin{aligned} d_X &= \dim X - h_X = \dim X - h_{\pi^{-1}(M)} \\ &= \dim X - \big(\dim \pi^{-1}(M) - d_{\pi^{-1}(M)}\big) \\ &= (\dim Y - \dim M) + d_{\pi^{-1}(M)} \\ &= d_Y + d_{\pi^{-1}(M)}. \end{aligned}$$
■

3. **Algebraic groups.** The following proposition is more or less known.

PROPOSITION 1. *Let G be a connected linear algebraic group, $H \subset G$ an algebraic subgroup, and $X := G/H$. The following statements are equivalent:*

(i) X has two ends;

(ii) $d_X = 1$;

(iii) the normalizer $N = N(H^\circ)$ is a parabolic subgroup of G and $N/H^\circ = \mathbb{C}^*$;

(iv) H is the kernel of a nontrivial character of some parabolic subgroup of G.

PROOF. For (i) $\Leftrightarrow$ (iv) see [2], (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) are clear. In order to prove the remaining implication (ii) $\Rightarrow$ (iii), consider the covering manifold G/H°. Since X is homotopy equivalent to a compact submanifold of codimension one, the same is true for the finite covering. Thus $d_{G/H^\circ} = 1$. Furthermore, since the isotropy subgroup H° is connected, the Mostow-Karpelevich fibering for G/H° not only has $\mathbb{R}$ as fiber, but also is topologically trivial. Therefore G/H° has two ends. Applying (i) $\Leftrightarrow$ (iv) we see that $H^\circ = \ker\varphi$, where $\varphi: P \longrightarrow \mathbb{C}^*$ is a nontrivial character of a parabolic subgroup $P \subset G$. It is clear that $P \subset N = N(H^\circ)$. Since the manifold $N/P = (N/H^\circ)/(P/H^\circ)$ is complete, P/H° is a parabolic subgroup of N/H°. This subgroup is isomorphic to $\mathbb{C}^*$. Therefore N/H° is also isomorphic to $\mathbb{C}^*$ and $N = P$. ∎

It is convenient to recall here the definition of an observable algebraic subgroup (see [6]). Let G be a complex linear algebraic group. An algebraic subgroup of $H \subset G$ is called *observable* if the following equivalent conditions are fulfilled:

(i) each rational H-module is an H-submodule of a rational G-module;

(ii) there exists a rational G-module V and a vector $v \in V$ such that

$$H = \{g \in G \mid gv = v\};$$

(iii) G/H is a quasi-affine algebraic manifold.

If G is connected, then H is observable in G if and only if H° is observable.

The structure of connected observable subgroups is given by the main theorem of [21]. The following lemma is an easy consequence of this result.

LEMMA 4. *Let G be a connected reductive linear algebraic group over $\mathbb{C}$, $P \subset G$ a parabolic subgroup, and $H \subset G$ an algebraic subgroup such that $P' \subset H \subset P$. Then H is observable if and only if there exists an irreducible G-module V and a vector $v \in V$ such that $P = \{g \in G \mid gv \in \mathbb{C} \cdot v\}$ and $H \subset \ker\varphi$, where $\varphi: P \longrightarrow \mathbb{C}^*$ is the character defined by*

$$pv = \varphi(p) \cdot v, \quad p \in P.$$

PROOF. Since $G/\ker\varphi$ is the orbit of the highest weight vector, the subgroup $I := \ker\varphi$ is observable. Assume that $H \subset I$. Then I/H is the product of an algebraic torus with a finite group. In particular, I/H is an affine manifold. Now, since I is observable in G and H is observable in I, it follows from (i) of the definition that H is observable in G.

We now prove the converse. It is enough to consider the case of H connected. For, along with H, the connected component H° is also observable. On the other hand, if it is known that $H^\circ \subset \ker\varphi$ with φ of the above type, then $H \subset \ker(\varphi^k)$ for some positive integer k. Thus, it remains to replace V by the irreducible component of $V^{\otimes k}$ containing $v \otimes \cdots \otimes v$.

For H connected one can apply the main result of [21]. Denote by $\mathrm{Rad}_u(H)$ the unipotent radical of an algebraic group H. According to [21], there exists a parabolic subgroup $Q \subset G$ and a character $\psi: Q \longrightarrow \mathbb{C}^*$ such that:

(a) $\ker\psi$ is the isotropy subgroup of the highest weight vector in an irreducible representation space of G;

(b) $H \subset \ker \psi$;
(c) $\mathrm{Rad}_u(P) = \mathrm{Rad}_u(H) \subset \mathrm{Rad}_u(Q)$.

We only have to show that $P = Q$. For this it suffices to prove the following:

(*) if P and Q are two parabolic subgroups in G and $P' \subset Q$, then $P \subset Q$; if, in addition, $\mathrm{Rad}_u(P) \subset \mathrm{Rad}_u(Q)$ then $P = Q$.

In order to prove $(*)$ pick a Borel subgroup $B \subset P$. Then $B' \subset P' \subset Q$. Thus, there exists a Borel subgroup $B_1 \subset Q$, which contains B'. But then $B_1 = N(B') = B$ so that $B \subset Q$. In this situation the description of parabolic subalgebras in terms of roots shows that $\mathrm{Rad}_u(P) \supset \mathrm{Rad}_u(Q)$. Since B is an arbitrary Borel subgroup in P, it follows that $P \subset Q$. The opposite inclusion is possible only if $P = Q$. ■

Our next lemma is certainly known.

LEMMA 5. *Let G be a connected reductive linear algebraic group over $\mathbb{C}$ and $H \subset G$ an algebraic subgroup containing a maximal torus $T \subset G$. Then the following statements are equivalent:*

(i) H is an observable subgroup;
(ii) the root system of H is symmetric;
(iii) H is reductive.

PROOF. By the Matsushima-Onishchik theorem (iii) implies that G/H is an affine manifold. In particular, (iii) implies (i). The equivalence of (ii) and (iii) is known and easily seen. It remains to prove (i) $\Rightarrow$ (ii). Let $\mathfrak{h}$ (resp. $\mathfrak{t}$) be the Lie algebra of H (resp. T). Assume that $\mathrm{E}_\alpha \in \mathfrak{h}$ but $\mathrm{E}_{-\alpha} \notin \mathfrak{h}$ for some root α, where E_α denotes the root vector corresponding to α. Let S_α be the simple three-dimensional subgroup with Lie algebra

$$\mathfrak{s}_\alpha = \mathbb{C} \cdot \mathrm{E}_\alpha + \mathbb{C} \cdot \mathrm{E}_{-\alpha} + \mathbb{C} \cdot [\mathrm{E}_\alpha, \mathrm{E}_{-\alpha}].$$

Since $[\mathrm{E}_\alpha, \mathrm{E}_{-\alpha}] \in \mathfrak{t} \subset \mathfrak{h}$, the intersection $S_\alpha \cap H$ is a Borel subgroup in S_α. Thus the orbit $S_\alpha \cdot (eH) \subset G/H$ is a complete curve, contradicting (i). ■

PROPOSITION 2. *Let G be a connected linear algebraic group and $H \subset G$ an algebraic subgroup such that $X := G/H$ satisfies the maximal rank condition and $d_X = 2$. Then X is one of the following manifolds:*

1) The complex line $\mathbb{C}$;
2) The affine quadric Q_2;
3) $\mathbb{P}_2 - Q$, where Q is a quadric curve;
4) A homogeneous algebraic principal $\mathbb{C}^*$-bundle over Y, where Y is an affine cone with its vertex removed.

Any X from this list is in fact a quasi-affine algebraic manifold with $d_X = 2$.

PROOF. We start by proving the last assertion, which is non-trivial only in Case 4). Here we a have a triple of algebraic groups $H \lhd I \subset G$ such that $X = G/H$, $Y = G/I$, and $I/H = \mathbb{C}^*$. Since H is observable in I and I is observable in G, we conclude that H is observable in G and X is quasi-affine (we use the equivalent definitions (i) and (iii) of an observable subgroup).

To see that $d_X = 2$ it suffices to represent X as a $\mathbb{C}^* \times \mathbb{C}^*$-principal bundle over a homogeneous projective rational manifold and then apply Lemma 1. The following argument shows that such a representation is possible.

Since Y is an affine cone minus its vertex, there exists a (connected) algebraic subgroup $J \subset G$ such that $I \lhd J$, $J/I = \mathbb{C}^*$, and G/J is projective rational. It is enough to show that $H \lhd J$. Now, $H = \ker\tau$, where $\tau: I \longrightarrow \mathbb{C}^*$ is some character. Since the character group of I is discrete, the connected group J acts on it trivially. In other words,

$$\tau(xyx^{-1}) = \tau(y) \quad \text{for all } x \in J,\ y \in I.$$

It follows that $H \lhd J$.

We now turn to the proof of completeness of our list. For this we need the following result (see [3]):

Any X subject to our assumptions can be represented as a homogeneous bundle $X = L \times_P F$, where L is a maximal reductive subgroup of G, $P \subset L$ a parabolic subgroup, and F is one of the following algebraic P-manifolds:

a) $F = \mathbb{C}$, P acts by affine transforms;

b) $F = \mathbb{P}_2 - Q$, P acts via a homomorphism $P \longrightarrow \mathrm{SO}(3, \mathbb{C})$;

c) $F = \mathbb{C}^* \times \mathbb{C}^*$, P acts by group translations.

Note that a proper subgroup of $\mathrm{SO}(3, \mathbb{C})$ cannot be transitive on $F = \mathbb{P}_2 - Q$. The homomorphism in b) is in fact defined on a possibly bigger group $\hat{P}$, on which it is surjective by the construction of [3]. Namely, $\hat{P}$ (resp. P) is the isotropy subgroup in G (resp. in L) of some point of the base of the bundle. Since any Levi subgroup of P is a Levi subgroup of $\hat{P}$, it follows that the homomorphism in b) is surjective.

Let us now see what happens if X satisfies the maximal rank condition.

a) If P acts on F trivially then we have an imbedding $L/P \hookrightarrow X$, showing that in fact $P = L$ and $X = F = \mathbb{C}$. Otherwise, P acts on F transitively so that X is L-homogeneous. In particular, $X = L/(H \cap L)$ is a quasi-affine manifold by [5]. The isotropy subgroup $H \cap L$ is contained in P and has codimension one in P. Since a torus acting on $\mathbb{C}$ always has a fixed point, $H \cap L$ is a subgroup of full rank. We are in a position to apply Lemma 5. This lemma shows that $H \cap L$ is reductive. Since this subgroup at the same time has codimension one in a parabolic subgroup, it follows that $L/P = \mathbb{P}_1$ and $X = L/(H \cap L) = Q_2$.

b) We know already that P acts transitively on F so that X is L-homogeneous and, in particular, quasi-affine. Now, $H \cap L$ is a subgroup of P (of codimension two). Since a torus in $\mathrm{SO}(3, \mathbb{C})$ has a fixed point in $\mathbb{P}_2 - Q$, the subgroup $H \cap L$ is of full rank. By Lemma 5 we see that $H \cap L$ is reductive. On the other hand, the image of the unipotent radical $\mathrm{Rad}_u(P)$ in $\mathrm{SO}(3, \mathbb{C})$ must be trivial. Therefore $\mathrm{Rad}_u(P)$ is contained in $H \cap L$. It follows that P is reductive. But then $P = L$ and $X = F = \mathbb{P}_2 - Q$.

c) We can replace G by $G \times (\mathbb{C}^* \times \mathbb{C}^*)$, where the second factor acts on X as the structure group of the principal bundle. After this the homomorphism $P \longrightarrow \mathbb{C}^* \times \mathbb{C}^*$ defining the action of P on F becomes surjective. Therefore we may assume that P is transitive on F and, correspondingly, L is transitive on X. In particular, X is quasi-affine. Since $P' \subset H \cap L \subset P$, we are in a position to apply Lemma 4. As a result we obtain an algebraic subgroup I such that $H \cap L \subset I \subset P$, $P/I = \mathbb{C}^*$, and $Y := L/I$ is an affine cone with its vertex removed. Replacing I, if necessary, by a subgroup of finite index, we may assume that $I/(H \cap L)$ is connected. By a dimension argument we then have $I/(H \cap L) = \mathbb{C}^*$. The fibering

$$X = L/(H \cap L) \longrightarrow Y = L/I$$

is the desired one. ∎

4. **Parallelizable manifolds.** The aim of this section is to present the classification in the case of discrete isotropy and in order to do this we will reduce to the setting where we have a solvable group acting transitively. It was observed in [10] that if $X = G/H$, where G is a connected solvable complex Lie group and H is a closed complex subgroup, with $d_X \leq 2$ and $\mathcal{O}(X)$ having maximal rank, then $\dim X \leq 2$. Note that there is one nontrivial $\mathbb{C}^*$-bundle over $\mathbb{C}^*$ which is a homogeneous Stein surface and we will call this the complex Klein bottle, see [13]. Using the above remark about the dimension a check of homogeneous spaces of dimension one and two which satisfy the required conditions leads to the observation:

If $X = G/H$, where G is a connected solvable complex Lie group and H is a closed complex subgroup, with $d_X \leq 2$ and $\mathcal{O}(X)$ having maximal rank, then X is biholomorphic to $\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{C}^* \times \mathbb{C}^*$ or the complex Klein bottle.

In order to handle the situation with discrete subgroups we will need a couple of preliminary results. These are of a somewhat technical nature. The first allows us to handle the situation when the group is a product of its radical with a maximal semisimple subgroup. The second is useful when the group is a nontrivial semidirect product.

LEMMA 6. *Suppose G is a complex Lie group whose Levi decomposition is a direct product $S \times R$. Let Γ be a discrete subgroup of G such that $S \cap \Gamma$ is finite. Then Γ is contained in a subgroup of the form $A \times R$, where A is an algebraic subgroup of S such that its connected component A° is solvable.*

PROOF. Call π_1 and π_2 the projections from G to S and R respectively. By a theorem of Tits [22, Theorem 1] the linear group $\pi_1(\Gamma)$ contains either a solvable subgroup of finite index or a free subgroup with (at least) two generators. The second case cannot occur for the following reason. Pick any two generators for such a subgroup and let a and b denote their preimages in Γ. Then a and b generate a free subgroup. Now $\pi_2(a)$ and $\pi_2(b)$ are in a solvable group and so there is a word on $\pi_2(a)$ and $\pi_2(b)$ equal to one in R. Therefore there is a word on a and b in G which belongs to the kernel of π_2 and so belongs to $S \cap \Gamma$. Since this group is finite, some power of this word is equal to one in G.

Let A be the Zariski closure of $\pi_1(\Gamma)$ in S. Since $\pi_1(\Gamma)$ contains a solvable subgroup of finite index, A has the same property. ∎

LEMMA 7. *Let $G = S \ltimes R$, where R is a vector group and the action of S on R is given by a linear representation. Assume that a generic element of S has no nonzero invariant vector in R. Then the union of all conjugates to S in G contains a Zariski open set of* *the form $\pi_1^{-1}(S - Z)$, where $\pi_1 : G \longrightarrow S$ is the projection mapping and $Z \neq S$ is a Zariski closed subset in S.*

PROOF. The elements of G are pairs (s, V) and the multiplication is given by $(s_1, V_1) \cdot (s_2, V_2) = \big(s_1 s_2, \rho(s_2)^{-1} V_1 + V_2\big)$, where $s_i \in S$, $V_i \in R$ and ρ is the representation. Given a pair (s_0, V_0), let's try to find an element $V \in R$ such that the following holds:

$$(1, V) \cdot (s_0, V_0) \cdot (1, -V) = (s_0, 0).$$

Using the multiplication law we rewrite this as $\rho(s_0)^{-1} V + V_0 - V = 0$. If the eigenvalues of the operator $\rho(s_0)$ are all different from one, then this equation has a unique solution. Let

$$Z := \big\{ s \in S \mid \det\big(\rho(s) - \mathrm{Id}_R\big) = 0 \big\}.$$

According to the assumption, $Z \neq S$ and our claim follows. ∎

Next we describe the holomorphic reduction and set up some notation. Suppose we are given a complex Lie group G and a closed complex subgroup H. Let $X = G/H$ and assume $\mathcal{O}(X) \neq \mathbb{C}$. Then there exists a closed complex subgroup U of G containing H such that G/U is holomorphically separable and if $\pi : G/H \longrightarrow G/U$ is the natural map, then $\pi^*\big(\mathcal{O}(G/U)\big) = \mathcal{O}(G/H)$. The map π is called the *holomorphic reduction of* X. Let $\tilde{U}$ be the union of the connected components of U which meet H, *i.e.*, $\tilde{U} = H \cdot U^\circ$. Then the fibration $G/H \longrightarrow G/\tilde{U}$ has connected fiber $\tilde{U}/H$ and its base $G/\tilde{U}$ satisfies the maximal rank condition. Let $N := N_G(U^\circ)$, $\tilde{N} := \tilde{U} \cdot N^\circ$, and denote by

$$\eta : \tilde{N} \longrightarrow \tilde{N}/U^\circ$$

the canonical map. We put $\Gamma := \eta(\tilde{U}) = \tilde{U}/U^\circ$.

Suppose Q is any closed complex subgroup of $\tilde{N}$ which contains $\tilde{U}$. Let $\tilde{Q} := \tilde{U} \cdot Q^\circ$ and $L := \tilde{Q}/U^\circ = \eta(\tilde{Q})$. Then

$$\tilde{Q}/\tilde{U} = L/\Gamma = L^\circ/\Gamma \cap L^\circ.$$

We now have the double fibration with connected fibers

$$X = G/H \longrightarrow G/\tilde{U} \xrightarrow{L/\Gamma} G/\tilde{Q}.$$

Before proving that, if in this setting one also has $d_{G/H} \leq 2$, then L is solvable, we first use the previous two lemmas to show that the part of the discrete subgroup Γ lying in L° is contained in a proper closed complex subgroup A of a very special form.

LEMMA 8. *Let G be a complex Lie group and H a closed complex subgroup of G such that $X := G/H$ satisfies $d_X \leq 2$ and $\mathcal{O}(X) \neq \mathbb{C}$. Suppose Q is any closed complex subgroup of $\tilde{N}$ containing $\tilde{U}$ with* $\dim Q > \dim U$. *Construct L and Γ as above. Assume also that the radical R of L has dimension at most 2. Let $\sigma : \hat{L} \longrightarrow L^\circ$ be the universal covering, fix a Levi decomposition $\hat{L} = \hat{S} \ltimes \hat{R}$ and set $\hat{\Gamma} := \sigma^{-1}(\Gamma \cap L^\circ)$. Then there is a proper algebraic subgroup $\hat{A}$ of $\hat{L}$ containing $\hat{\Gamma}$. In particular, if we set $A := \sigma(\hat{A})$, then $\Gamma \cap L^\circ \subset A$ and*

$$L^\circ/A = \hat{L}/\hat{A}$$

is the quotient of algebraic groups.

PROOF. The proof splits into two cases, depending on whether the adjoint action of $S := \sigma(\hat{S})$ on the radical R is trivial or not.

CASE 1: THE ADJOINT ACTION OF S ON R IS NONTRIVIAL. Assume that R is not abelian. Then the commutator subgroup of R is invariant under S and one-dimensional. Therefore the semisimple group S acts on it trivially and since the action of S on the Lie algebra of R is completely reducible and $\dim R \leq 2$, this shows that the action on R is trivial. This is a contradiction. Therefore R is abelian. Note also that the action of S on R is irreducible, since otherwise by the complete reducibility it would follow that the action is trivial.

A semisimple group having an irreducible two-dimensional representation decomposes as a locally direct product $S_1 \cdot S_2$, where S_2 is acting trivially and S_1 is isomorphic to $\mathrm{SL}(2,\mathbb{C})$, with the action of S_1 being (isomorphic to) the usual action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{C}^2$. Any element of the form $s_1 \cdot s_2$, where s_1 has different eigenvalues, has no nonzero invariant vector in the representation space. Therefore, Lemma 7 applies to $\hat{L}$.

Note that $\hat{L}$ has the unique structure of a linear algebraic group. We claim that $\hat{\Gamma}$ is contained in a proper algebraic subgroup $\hat{A}$ of $\hat{L}$. Assume the contrary and consider the projection $\pi_1 : \hat{L} \longrightarrow \hat{S}$. Then $\pi_1(\hat{\Gamma})$ is Zariski dense in $\hat{S}$. By using a theorem of Tits [22, Theorem 3] we see that $\pi_1(\hat{\Gamma})$ contains a free subgroup with two generators which is also Zariski dense in $\hat{S}$. Pick the π_1-preimages of any two of these generators in $\hat{\Gamma}$ and let $\hat{\Gamma}_1$ denote the (free) subgroup of $\hat{\Gamma}$ generated by the chosen elements. By a theorem of Selberg, see [20, Corollary 6.13], there is a subgroup of finite index $\hat{\Gamma}_2 \subset \hat{\Gamma}_1$ which is without torsion. Clearly $\pi_1(\hat{\Gamma}_2)$ is also Zariski dense in $\hat{S}$. The intersection of $\hat{\Gamma}_2$ with any conjugate subgroup $g\hat{S}g^{-1}$ is a finite group. (Since $L/\Gamma = L^\circ/\Gamma \cap L^\circ = \hat{L}/\hat{\Gamma}$ satisfies the maximal rank condition, this is even true for the bigger group $\hat{\Gamma}$ [5].) It follows that $\hat{\Gamma}_2 \cap g\hat{S}g^{-1} = \{e\}$. Therefore

$$\hat{\Gamma}_2 - \{e\} \subset \hat{L} - \bigcup_{g \in \hat{L}} g\hat{S}g^{-1} \subset \hat{L} - \pi_1^{-1}(\hat{S} - Z) = \pi_1^{-1}(Z),$$

where Z is defined as in Lemma 7. It follows that $\pi_1(\hat{\Gamma}_2) \subset Z$ contradicting the density of $\pi_1(\hat{\Gamma}_2)$ in $\hat{S}$. Thus $\hat{\Gamma} \subset \hat{A}$, where $\hat{A}$ is a proper algebraic subgroup of $\hat{L}$. Let $A := \sigma(\hat{A})$. Since $\ker \sigma$ is contained in $\hat{\Gamma} \subset \hat{A}$, we see that A is closed in L° and

$$L^\circ/A = \hat{L}/\hat{A}.$$

Thus L°/A may be written as the quotient of algebraic groups.

CASE 2: THE ADJOINT ACTION OF S ON R IS TRIVIAL. In this case $\hat{L} = \hat{S} \times \hat{R}$. By Lemma 6 we have that $\hat{\Gamma}$ is contained in a subgroup of the form $\hat{A} := C \times \hat{R}$, where C is a proper algebraic subgroup of $\hat{S}$ with solvable C°. Let $A := \sigma(\hat{A})$. Then, as in Case 1,

$$L^\circ/A = \hat{L}/\hat{A} = (\hat{S} \times \hat{R})/(C \times \hat{R}) = \hat{S}/C,$$

so again L°/A may be written as the quotient of algebraic groups. ∎

PROPOSITION 3. *Let G be a complex Lie group and H a closed complex subgroup of G such that $X := G/H$ satisfies $d_X \leq 2$ and $\mathcal{O}(X) \neq \mathbb{C}$. Suppose Q is any closed complex subgroup of $\tilde{N}$ containing $\tilde{U}$ with* $\dim Q > \dim U$. *Construct L and Γ as above. Then L° is solvable and L/Γ is biholomorphic to $\mathbb{C}^*$, $\mathbb{C}$, $\mathbb{C}^* \times \mathbb{C}^*$ or the complex Klein bottle. In particular,* $\dim L \leq 2$.

PROOF. The case when L° is solvable is handled in the following way. Applying Lemma 2 to the fibration $G/H \longrightarrow G/\tilde{Q}$ we get $d_{\tilde{Q}/H} \leq d_X \leq 2$. Now since L/Γ is a solv-manifold, it admits a fibering as a vector bundle over a compact manifold, see [4] or [18]. Applying the second inequality of Lemma 2 to the fibration $\tilde{Q}/H \longrightarrow \tilde{Q}/\tilde{U}$ yields $d_{L/\Gamma} = d_{\tilde{Q}/\tilde{U}} \leq d_{\tilde{Q}/H}$. Thus $d_{L/\Gamma} \leq 2$. Because L/Γ satisfies the maximal rank condition the classification follows from the observation at the beginning of this section.

We will prove by induction on $\dim Q$ that L° is solvable. Assume that the assertion is true for any closed complex subgroup Q of $\tilde{N}$ of smaller dimension which contains $\tilde{U}$. First we claim that it is not possible for L° to be semisimple. For, in this case the fact that $L^\circ/\Gamma \cap L^\circ$ satisfies the maximal rank condition implies that $\Gamma \cap L^\circ$ is algebraic, see [5]. Thus $\Gamma \cap L^\circ$ is finite and so $L^\circ/\Gamma \cap L^\circ$ is retractable onto a compact submanifold of half the dimension (the minimal orbit of a maximal compact subgroup of L°). Again from the second inequality of Lemma 2 one has $3 \leq \dim_{\mathbb{C}} L^\circ = d_{L/\Gamma} \leq d_X$, which is a contradiction.

Assume now that L° is mixed, *i.e.*, its Levi decomposition is of the form $L^\circ = S \cdot R$ with S and R both not trivial. We first show that $\dim R \leq 2$ and then derive a contradiction. In order to do this we note that there exists a proper closed complex subgroup J of L which contains both Γ and R. In the connected case this follows from a result in [9]. If L is not connected, we can find a proper closed complex subgroup $J_1 \subset L^\circ$ containing both $\Gamma \cap L^\circ$ and R. Taking J_1 minimal with these properties we see that

$$J_1 = \bigcap_{\gamma \in \Gamma} \gamma J_1 \gamma^{-1}.$$

Thus we may take $J := \Gamma \cdot J_1$ in L. The only property that one has to check is the closedness of J, but this follows immediately from $J \cap L^\circ = J_1$. Let $\tilde{J} := \Gamma \cdot J^\circ$ and $M := \eta^{-1}(\tilde{J})$. Then $M/U^\circ = \tilde{J}$ and $M/\tilde{U} = M/U^\circ/\tilde{U}/U^\circ = \tilde{J}/\Gamma$. Consider the double fibration

$$G/H \longrightarrow G/\tilde{U} \xrightarrow{\tilde{J}/\Gamma} G/M.$$

Now $\dim \tilde{J} < \dim L$, by construction, *i.e.*, $\dim M < \dim Q$. It follows by induction that J° is solvable and the fiber $\tilde{J}/\Gamma$ belongs to the list given above. In particular, $\dim J^\circ \leq 2$ and since $J^\circ \supset R$, we obtain that $\dim R \leq 2$.

As in Lemma 8 we denote by $\sigma: \hat{L} \longrightarrow L^\circ$ the universal covering, fix a Levi decomposition $\hat{L} = \hat{S} \ltimes \hat{R}$ so that $\sigma(\hat{S}) = S$, and set $\hat{\Gamma} := \sigma^{-1}(\Gamma \cap L^\circ)$.

In both of the cases in Lemma 8 we have the following inclusions

$$\Gamma \cap L^\circ \subset A \subset L^\circ.$$

Observe that these inclusions imply $\Gamma \cap A = \Gamma \cap L^{\circ}$. Replacing A if necessary by

$$\bigcap_{\gamma \in \Gamma} \gamma A \gamma^{-1}$$

we may assume that Γ normalizes A. Since the union of all connected components of A which meet $\Gamma \cap L^{\circ}$ is also Γ-invariant, we may replace A by this union and reduce to the case when $A/\Gamma \cap A$ is connected. Assume that this has been done and observe that the changes we have made do not violate the algebraicity of $\hat{A} = \sigma^{-1}(A)$ in Case 1 and of C in Case 2 of Lemma 8. Therefore, $L^{\circ}/A = \hat{L}/\hat{A}$ is a quotient of algebraic groups. Now let $B := A \cdot \Gamma \subset L$. Since $\Gamma \cap A = \Gamma \cap L^{\circ}$, one has $B/\Gamma = A/\Gamma \cap A = A/\Gamma \cap L^{\circ}$. Note also that $B \cap L^{\circ} = A$ showing that B is closed and $L/B = L^{\circ}/A$. Let $I := \eta^{-1}(B)$ so $B = I/U^{\circ}$. Then $B/\Gamma = I/U^{\circ}/\tilde{U}/U^{\circ} = I/\tilde{U}$ and we have the double fibration with connected fibers

$$G/H \longrightarrow G/\tilde{U} \xrightarrow{B/\Gamma} G/I.$$

Note that $\dim B < \dim L$, so $\dim I < \dim Q$. As above it follows by induction that B° is solvable and $\dim B \leq 2$.

Now we have the sequence of fibrations

$$G/H \longrightarrow G/\tilde{U} \longrightarrow G/I \longrightarrow G/\tilde{Q}.$$

Applying Lemma 2 to the fibration $G/H \longrightarrow G/\tilde{Q}$ we get $d_{G/H} \geq d_{\tilde{Q}/H}$. Next consider the fibration $\tilde{Q}/H \longrightarrow \tilde{Q}/I$. Since $\tilde{Q}/I = (\tilde{Q}/U^{\circ})/(I/U^{\circ}) = L/B = \hat{L}/\hat{A}$ with the latter being a quotient of algebraic groups, the base of this bundle is retractable onto a compact submanifold (see [17]) and from Lemma 2 one has $d_{\tilde{Q}/H} \geq d_{I/H}+d_{L/B}$. Finally the bundle $I/H \longrightarrow I/\tilde{U}$ has base $I/\tilde{U} = (I/U^{\circ})/(\tilde{U}/U^{\circ}) = B/\Gamma$ which, as a solv-manifold, is also retractable onto a compact submanifold. By Lemma 2 again one has $d_{I/H} \geq d_{\tilde{U}/H}+d_{B/\Gamma}$. Putting these inequalities together one gets

$$d_{G/H} \geq d_{\tilde{U}/H} + d_{B/\Gamma} + d_{L/B}.$$

Now if $d_{L/B} = 0$, *i.e.*, if $L/B = \hat{L}/\hat{A}$ were compact, then $\hat{A}$ would be parabolic. But $A^{\circ} = B^{\circ}$ is solvable. Since $\hat{A}$ is connected, $\hat{A}$ is also solvable. Thus $\hat{A}$ is a Borel subgroup in $\hat{L}$ and $\dim \hat{A} \leq 2$. But this is a contradiction if $\dim R > 0$, because then $\dim(\hat{A} \cap \hat{S}) < 2$, *i.e.*, $\hat{A}$ cannot be a Borel subgroup in $\hat{L}$. It follows that L° cannot be mixed.

Next assume $d_{L/B} = 1$. Then the above inequality implies $d_{B/\Gamma} \leq 1$. Note that $B/\Gamma \subset G/\tilde{U}$ and therefore B/Γ satisfies the maximal rank condition. Since B is solvable, it is clear that $\dim B \leq 1$. Because $L/B = \hat{L}/\hat{A}$ is the quotient of algebraic groups, it follows from Proposition 1 that $\hat{L}/\hat{A}$ has two ends. By [2] there exists a parabolic subgroup P in $\hat{L}$ containing $\hat{A}$ with $P/\hat{A} = \mathbb{C}^*$. Thus $\dim P = \dim \hat{A} + 1 = \dim B + 1 \leq 2$ and L° cannot be mixed by the arguments of the previous paragraph.

Finally assume that $d_{L/B} = 2$. Again by using the above inequality we have $d_{B/\Gamma} = 0$, *i.e.*, B/Γ is compact. Since B/Γ satisfies the maximal rank condition, this means that

B is discrete. But then A and also $\hat{A}$ are discrete. In Case 2 this is impossible because $\hat{A} \supset \hat{R}$ and $\hat{A}$ is finite in Case 1. Then Lemma 3 implies $d_{\hat{L}} = d_{\hat{L}/\hat{A}} = d_{L/B} \leq 2$. Hence $\hat{L}$ is abelian by [10] and this contradicts the assumption that L° is mixed.

Thus the assumption that L° is mixed leads to a contradiction and the assertions of the proposition are proved. ∎

COROLLARY 1. *With the above notation assume again that $d_X \leq 2$ and $\mathcal{O}(X) \neq \mathbb{C}$. Then N° / U° is solvable and $N^\circ / \tilde{U} \cap N^\circ$ is biholomorphic to a point, $\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{C}^* \times \mathbb{C}^*$ or the complex Klein bottle. In particular,* $\dim N^\circ / U^\circ \leq 2$.

PROOF. It is enough to apply the proposition to the case $Q = \tilde{N}$. ∎

COROLLARY 2. *Suppose G is a connected complex Lie group and Γ is a discrete subgroup such that the coset space $X := G/\Gamma$ satisfies the maximal rank condition and $d_{G/\Gamma} \leq 2$. Then G is solvable and X is biholomorphic to a point, $\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{C}^* \times \mathbb{C}^*$ or the complex Klein bottle.*

PROOF. In this situation $\tilde{U} = H = \Gamma$, $\tilde{N} = N = G$ and for $Q := G$ one has $L = \tilde{Q} = G$. ∎

5. **Reduction to the algebraic category.** Given a connected complex Lie group G and a closed complex subgroup H, we let $G/H \to G/U$ be the holomorphic reduction of G/H. As in the previous section, let $\tilde{U} := H \cdot U^\circ$, $N := N_G(U^\circ)$ and $\tilde{N} := \tilde{U} \cdot N^\circ$. Now, G/N is an orbit in some $\mathbb{P}_m$ of a connected Lie subgroup of $\mathrm{GL}(m+1, \mathbb{C})$. Therefore, by Chevalley's Theorem [8], the commutator subgroup G' acts on G/N as an algebraic subgroup of $\mathrm{GL}(m+1, \mathbb{C})$ and thus the orbits of G' are closed in G/N. Because $G/\tilde{N} \to G/N$ is a covering, the orbits of G' are closed in $G/\tilde{N}$ as well. Therefore, we may consider the commutator fibrations $G/N \to G/NG'$ and $G/\tilde{N} \to G/\tilde{N}G'$. Note that their bases are Stein abelian groups, see [12, p. 168]. We display this in the diagram

$$\begin{array}{ccccccc} G/H & \longrightarrow & G/\tilde{U} & \longrightarrow & G/\tilde{N} & \longrightarrow & G/N \\ & & & & \downarrow & & \downarrow \\ & & & & G/\tilde{N}G' & \longrightarrow & G/NG' \end{array}$$

The radical of any connected Lie group L is now denoted by R_L. We let S denote a maximal semisimple subgroup of G.

LEMMA 9. *With the above notation assume further that $\tilde{N}/\tilde{U}$ has the homotopy type of a* CW*-complex of dimension q. Then*

$$d_{G/H} \geq d_{\tilde{U}/H} + (\dim \tilde{N}/\tilde{U} - q) + d_{R_{G'}/R_{G'} \cap \tilde{N}} + d_{S/S \cap \tilde{N} R_{G'}} + d_{G/\tilde{N}G'}.$$

Moreover, if $d_{G/H} \leq 2$ and $\mathcal{O}(G/H) \neq \mathbb{C}$, then

$$d_{G/H} \geq d_{\tilde{U}/H} + d_{\tilde{N}/\tilde{U}} + d_{R_{G'}/R_{G'} \cap \tilde{N}} + d_{S/S \cap \tilde{N} R_{G'}} + d_{G/\tilde{N}G'}, \tag{1}$$

where $\tilde{N}/\tilde{U}$ is a point, $\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{C}^* \times \mathbb{C}^*$, or the complex Klein bottle. Furthermore, $d_{R_{G'}/R_{G'} \cap \tilde{N}} = 0$ or 2.

PROOF. Since $G/\tilde{N}G'$ is a connected Lie group, it is retractable onto its maximal compact subgroup, and so by Lemma 2 one has

$$d_{G/H} \geq d_{\tilde{N}G'/H} + d_{G/\tilde{N}G'}.$$

Since G' acts algebraically on G/N, the $R_{G'}$-orbits are closed in $G'/G' \cap N$ and so consequently in $G'/G' \cap \tilde{N}$. One has the following diagram

$$\begin{array}{ccc} G'/G' \cap \tilde{N} & \longrightarrow & G'/G' \cap N \\ {\scriptstyle R_{G'}/R_{G'}\cap\tilde{N}}\big\downarrow & & \big\downarrow{\scriptstyle R_{G'}/R_{G'}\cap N} \\ S/S \cap \tilde{N}R_{G'} & \longrightarrow & S/S \cap NR_{G'} \end{array} \tag{2}$$

The subgroup $M := S \cap NR_{G'} \subset S$ is algebraic. Since $\tilde{M} := S \cap \tilde{N}R_{G'}$ consists of some of the connected components of M, it follows that $\tilde{M}$ is also algebraic. Now consider the fibration $\tilde{N}G'/H \longrightarrow \tilde{N}G'/\tilde{N}R_{G'} = S/\tilde{M}$. Since the base is the quotient of algebraic groups, it is retractable onto a compact submanifold [17]. Thus by Lemma 2 one has $d_{\tilde{N}G'/H} \geq d_{\tilde{N}R_{G'}/H} + d_{S/\tilde{M}}$. Next we look at the fibration $\tilde{N}R_{G'}/H \longrightarrow \tilde{N}R_{G'}/\tilde{N} = R_{G'}/R_{G'} \cap \tilde{N}$. The base, being a solv-manifold, is retractable onto a compact submanifold ([4] or [18]) and by Lemma 2 one has $d_{\tilde{N}R_{G'}/H} \geq d_{\tilde{N}/H} + d_{R_{G'}/R_{G'}\cap\tilde{N}}$.

Finally consider the fibration

$$\tilde{N}/H \longrightarrow \tilde{N}/\tilde{U},$$

where $\tilde{N}/\tilde{U} = (\tilde{N}/U^\circ)/(\tilde{U}/U^\circ)$ is parallelizable. Assume $\tilde{N}/\tilde{U}$ is homotopy equivalent to a CW-complex of dimension q. Then by Lemma 2 we have

$$d_{\tilde{N}/H} \geq d_{\tilde{U}/H} + (\dim \tilde{N}/\tilde{U} - q).$$

Combining the previous inequalities we arrive at

$$d_{G/H} \geq d_{\tilde{U}/H} + (\dim \tilde{N}/\tilde{U} - q) + d_{R_{G'}/R_{G'}\cap\tilde{N}} + d_{S/S\cap\tilde{N}R_{G'}} + d_{G/\tilde{N}G'}.$$

Now assume that $d_{G/H} \leq 2$ and $\mathcal{O}(G/H) \neq \mathbb{C}$. Then we obtain from Corollary 1 to Proposition 3 that $\tilde{N}/\tilde{U}$ is of dimension less than or equal to two and is biholomorphic to a point, $\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{C}^* \times \mathbb{C}^*$ or the complex Klein bottle. Thus we can replace $\dim \tilde{N}/\tilde{U} - q$ by $d_{\tilde{N}/\tilde{U}}$. Therefore, we have

$$d_{G/H} \geq d_{\tilde{U}/H} + d_{\tilde{N}/\tilde{U}} + d_{R_{G'}/R_{G'}\cap\tilde{N}} + d_{S/S\cap\tilde{N}R_{G'}} + d_{G/\tilde{N}G'}.$$

The last observation follows from the fact that the $R_{G'}$-orbits, as the orbits of a unipotent group, are biholomorphic to $\mathbb{C}^k$ and thus $k = 0$ or 1 in this setting. ∎

LEMMA 10. *Suppose G is a connected complex Lie group and H is a closed complex subgroup of G such that $X := G/H$ satisfies the maximal rank condition. Let $N := N_G(H^\circ)$, $\tilde{N} := H \cdot N^\circ$ and $\pi\colon G/H \longrightarrow G/\tilde{N}$ be the natural map. Assume further that*

$$1 = d_{\tilde{N}/H} + d_{R_{G'}/R_{G'}\cap\tilde{N}} + d_{S/S\cap\tilde{N}R_{G'}}. \tag{3}$$

Then $\tilde{N}/H = \mathbb{C}^*$ and $G'/G' \cap \tilde{N}$ is compact and thus is a projective rational manifold. In particular, $Z := \pi^{-1}(G'/G' \cap \tilde{N}) \subset G/H$ has two ends.

PROOF. The group $\tilde{N}G'$ leaves Z invariant and acts transitively on Z. Moreover, Z fibers in the following way

$$Z = \tilde{N}G'/H \xrightarrow{\tilde{N}/H} \tilde{N}G'/\tilde{N} = G'/G' \cap \tilde{N} \xrightarrow{R_{G'}/R_{G'}\cap\tilde{N}} S/S \cap \tilde{N}R_{G'}.$$

We will show $\tilde{N}/H = \mathbb{C}^*$ and $G'/G' \cap \tilde{N}$ is a homogeneous projective rational manifold.

Assume first $d_{\tilde{N}/H} = 0$. Since G/H satisfies the maximal rank condition and $\tilde{N}/H$ is compact, $\tilde{N} = H$ and so we have a fibration

$$Z = \tilde{N}G'/H = \tilde{N}G'/\tilde{N} = G'/\tilde{N} \cap G' \xrightarrow{R_{G'}/R_{G'}\cap\tilde{N}} S/S \cap \tilde{N}R_{G'}.$$

Note that $R_{G'}/R_{G'} \cap \tilde{N}$, as the orbit of a unipotent group, is biholomorphic to $\mathbb{C}^k$ and so $d_{R_{G'}/R_{G'}\cap\tilde{N}} = 2k$. From (3) it follows that $k = 0$ and S is transitive on Z. Thus S is also transitive on the base of the holomorphic reduction of Z. This space is holomorphically separable and so its isotropy subgroup is algebraic [5]. Therefore the isotropy for the S-action on Z is also algebraic. Hence Z has two ends by Proposition 1. But any homogeneous space which has more than one end has a normalizer fibration with positive dimensional fiber, see [14, Corollary 9, p. 78]. This contradiction shows that this case is not possible.

It follows that $d_{\tilde{N}/H} = 1$ and the other two terms in (3) are zero. Then $G'/G' \cap N$ is compact, and thus is a projective rational manifold. Since $d_{\tilde{N}/H} = 1$, we obtain from Corollary 1 to Proposition 3 that $\tilde{N}/H = \mathbb{C}^*$ as a complex manifold. A well-known argument shows that Z has two ends, *e.g.*, see the proof of [9, Lemma 2, p. 549]. ∎

In the next proposition we are interested in the case when G/H satisfies the maximal rank condition, *i.e.*, $H = \tilde{U}$, and the following condition holds:

$$1 = d_{\tilde{N}/H} + d_{R_{G'}/R_{G'}\cap\tilde{N}} + d_{S/S\cap\tilde{N}R_{G'}} + d_{G/\tilde{N}G'} \tag{4}$$

PROPOSITION 4. *Suppose G is a connected complex Lie group and $H \subset G$ a closed complex subgroup such that $X := G/H$ satisfies the maximal rank condition and equation (4) is fulfilled. Then G/H has two ends.*

PROOF. Since $G/\tilde{N}G'$ is a Stein abelian group, it is of the form $G/\tilde{N}G' = \mathbb{C}^k \times (\mathbb{C}^*)^l$ and $d_{G/\tilde{N}G'} = 2k+l$. Therefore, we see from (4) that if $\dim G/\tilde{N}G'$ is positive, then $k = 0$, $l = 1$ and this group is $\mathbb{C}^*$ and we have the three equalities $d_{\tilde{N}/H} = 0$, $d_{R_{G'}/R_{G'}\cap\tilde{N}} = 0$ and $d_{S/S\cap\tilde{N}R_{G'}} = 0$. The first implies that $\tilde{N}/H$ is compact. Since G/H satisfies the maximal rank condition, $\tilde{N} = H$ and $G/\tilde{N} = G/H$. The second implies that the $R_{G'}$-orbits in $G/\tilde{N}$ are compact. Thus these orbits are points and $R_{G'} \subset \tilde{N}$. Hence S is transitive on the G'-orbit in $G/\tilde{N}$ and since $G/\tilde{N}$ satisfies the maximal rank condition, the third of these equalities implies $S \subset \tilde{N}R_{G'}$. These two inclusions yield $G' \subset \tilde{N} = H$ and so $G' \subset H^\circ$. Thus $N = G$ which is a contradiction.

In fact, the above argument shows $G = NG'$ and thus $G/\tilde{N} = G'/\tilde{N} \cap G'$. Hence the term $d_{G/\tilde{N}G'}$ in (4) does not appear and this equation becomes (3). Since $G/H = \pi^{-1}(G'/\tilde{N} \cap G')$, the assertion is now a consequence of Lemma 10. ∎

SPECIAL ASSUMPTIONS. G is a connected complex Lie group, H is a closed complex subgroup such that the quotient $X := G/H$ satisfies $\mathcal{O}(X) \neq \mathbb{C}$ and

$$2 = d_{\tilde{N}/\tilde{U}} + d_{R_{G'}/R_{G'}\cap\tilde{N}} + d_{S/S\cap\tilde{N}R_{G'}} + d_{G/\tilde{N}G'} \tag{5}$$

REMARK. For technical reasons it is inconvenient to replace (5) by the equation $d_{G/\tilde{U}} = 2$ (resp. (4) by $d_{G/H} = 1$, *etc.*).

The proof of the theorem stated in the introduction will follow from a number of propositions. Note that if Y is a homogeneous space of a reductive algebraic group which satisfies the maximal rank condition, then the isotropy subgroup for this action is also algebraic. (Since the holomorphic reduction has discrete fibers and the isotropy for the base is algebraic by [5], the isotropy for the total space is also algebraic.) This means that Proposition 2 applies and Y belongs to the list given in part b_2) of the theorem. Thus, under the special assumptions stated above, we would like to show that, with some obvious exceptions, there is a reductive algebraic group which is acting transitively either on $G/\tilde{U}$ or on a two-to-one covering $G/\hat{U}$ of $G/\tilde{U}$.

Before we look at the case when G/N is compact, we prove the following lemma.

LEMMA 11. *Let L be a connected two-dimensional complex Lie group and $\Gamma \subset L$ a discrete subgroup such that $Z := L/\Gamma$ is biholomorphic to $\mathbb{C}^* \times \mathbb{C}^*$. Assume that $\mathbb{C}^*$ acts on Z via an imbedding $\mathbb{C}^* \hookrightarrow L$. Then this action extends to a transitive action of $\mathbb{C}^* \times \mathbb{C}^*$ on Z whose restriction to the first factor is the given action.*

PROOF. If L is abelian, then Z is a group isomorphic $\mathbb{C}^* \times \mathbb{C}^*$. The imbedding $\mathbb{C}^* \hookrightarrow L$ gives rise to a homomorphism $\mathbb{C}^* \longrightarrow Z$ with finite kernel. It is clear that this homomorphism extends to an epimorphism $\mathbb{C}^* \times \mathbb{C}^* \longrightarrow Z$.

Assume now that L is non-abelian. Then $L = \mathbb{C}^* \ltimes \mathbb{C}$, with the $\mathbb{C}^*$-action on $\mathbb{C}$ given by

$$w \xrightarrow{z} z^m \cdot w, \quad \text{where } w \in \mathbb{C},\ z \in \mathbb{C}^*,\ m \text{ is a fixed positive integer.}$$

The multiplication in L is given by

$$(z_1, w_1) \cdot (z_2, w_2) = (z_1 z_2, z_2^{-m} w_1 + w_2), \quad \text{where } z_1,\ z_2 \in \mathbb{C}^*,\ w_1, w_2 \in \mathbb{C}.$$

The imbedding $\mathbb{C}^* \hookrightarrow L$ is unique up to conjugation. (In fact, L is linear algebraic and such an imbedding defines a maximal torus in L.) We may assume that the imbedding $\mathbb{C}^* \hookrightarrow L$ coincides with the identity isomorphism onto the first factor of the semidirect decomposition.

Let $a \in \mathbb{C}^*$, $b \in \mathbb{C}$. Suppose that (a, b) is an element of infinite order contained in Γ. We claim that $a^m = 1$.

Assume the contrary. Then one can determine w from the equation

$$(1 - a^{-m}) \cdot w = b.$$

Consequently

$$(1, w) \cdot (a, b) \cdot (1, -w) = (a, 0).$$

Since Γ is discrete and $(a,0)$ generates an infinite group, we have $|a| \neq 1$. But then we obtain an elliptic curve $\mathbb{C}^*/\langle a\rangle$, admitting a nontrivial holomorphic mapping

$$\mathbb{C}^*/\langle a\rangle \longrightarrow L/\langle (a,0)\rangle \simeq L/\langle (a,b)\rangle \longrightarrow L/\Gamma.$$

The contradiction thus obtained shows that $a^m = 1$.

It is clear that $b \neq 0$. A straightforward calculation shows that an element $(z,w) \in L$ commutes with (a,b) if and only if $z^m = 1$. Since Γ is abelian, it follows that

$$\Gamma \subset \Gamma_m \ltimes \mathbb{C} \simeq \mathbb{Z}_m \times \mathbb{C},$$

where

$$\Gamma_m := \{z \in \mathbb{C}^* \mid z^m = 1\}.$$

Therefore we have a principal fibering

$$Z = L/\Gamma \longrightarrow L/(\Gamma_m \ltimes \mathbb{C}).$$

The action of its structure group $A = (\Gamma_m \ltimes \mathbb{C})/\Gamma$ commutes with the given action of $\mathbb{C}^*$. Since the identity component of A is isomorphic to $\mathbb{C}^*$, we obtain the required action of $\mathbb{C}^* \times \mathbb{C}^*$. ∎

PROPOSITION 5. *Under the special assumptions, assume further that the base G/N of the normalizer fibration is a projective rational manifold. Let $Y = G/U$ be the base of the holomorphic reduction of X and set $\tilde{Y} := G/\tilde{U}$. Then $Y = \tilde{Y} = \mathbb{C}$, $Y = \tilde{Y} = \mathbb{C}^* \times \mathbb{C}^*$ or $Y = \tilde{Y}$ is the complex Klein bottle or else one can define a transitive holomorphic action of $S \times \mathbb{C}^*$, where S is a maximal semisimple subgroup of G, on Y and $\tilde{Y}$ or on a two-to-one covering $\hat{Y} = G/\hat{U}$ of $\tilde{Y}$.*

PROOF. By our assumption $P := S \cap N$ is a parabolic subgroup in S and $G/N = S/P$. Note that, since $\pi_1(S/P) = 1$, $N = \tilde{N}$. Let $L := N/U^\circ$, $\Gamma := \tilde{U}/U^\circ$, and $F = N/\tilde{U} = L/\Gamma$. Since $G/\tilde{N} = G/N$ is a projective rational manifold, the last three terms in (5) vanish and we have $d_F = 2$. Hence Proposition 3 implies that F is biholomorphic to $\mathbb{C}$, $\mathbb{C}^* \times \mathbb{C}^*$ or the complex Klein bottle.

a) $F = \mathbb{C}$.

If P is not transitive on F, then P has a fixed point in F and S/P can be imbedded in $\tilde{Y}$. Since $\tilde{Y}$ satifies the maximal rank condition, it follows that $S = P$ and $\tilde{Y} = F = \mathbb{C}$. Then, of course, we also have $Y = \mathbb{C}$. If P is transitive on F, then S is transitive on $\tilde{Y}$ and Y and our assertion is clear.

b) $F = \mathbb{C}^* \times \mathbb{C}^*$ or the complex Klein bottle.

Consider the quotient group $P/(P \cap U^\circ)$ and denote its dimension by k. If $k = 0$, then P has a fixed point in F and the same argument as above shows that $\tilde{Y} = F = \mathbb{C}^* \times \mathbb{C}^*$ or the complex Klein bottle and $Y = \tilde{Y}$. If $k = 2$, then P is transitive on F and S is transitive on $\tilde{Y}$. In this case the assertion is clear by Proposition 2. Finally, if $k = 1$, then $P/(P \cap U^\circ)$ is isomorphic to $\mathbb{C}^*$ (note that P/P' is an algebraic torus) and so we can apply Lemma 11

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to F or a two-to-one covering $\hat{F}$ of F. As a result there exists a transitive action of $P \times \mathbb{C}^*$ on F or on $\hat{F}$, extending the initial action of P. Therefore there exists a transitive action of $S \times \mathbb{C}^*$ either on $\tilde{Y} = S \times_P F$ and, as a consequence, on Y, or else on the two-to-one covering $\hat{Y} := S \times_P \hat{F}$ of $\tilde{Y}$.

If F is the complex Klein bottle, then the two-to-one covering $\hat{F} = \mathbb{C}^* \times \mathbb{C}^* \longrightarrow F$ is equivariant with respect to any two-dimensional group transitive on F. Therefore L and, as a consequence, N acts on $\hat{F}$ and one can write $\hat{Y} = G \times_N \hat{F}$ in the form $\hat{Y} = G/\hat{U}$. ∎

The next observation is very useful. Recall $N := N_G(U^\circ)$ and $\tilde{N} := \tilde{U} \cdot N^\circ$.

LEMMA 12. *Suppose the fibration*

(6) $\quad G/\tilde{U} \longrightarrow G/\tilde{N}$ *satisfies* $\tilde{N}/\tilde{U} = \mathbb{C}^*$, $G/\tilde{N}$ *has two ends and* $\dim G/\tilde{N} > 1$.

Then a group of the form $S \times \mathbb{C}^*$, where S is a maximal semisimple subgroup of G, acts transitively either on $G/\tilde{U}$ or on a two-to-one covering $G/\hat{U}$ of $G/\tilde{U}$.

PROOF. We distinguish two cases depending on whether S acts transitively on $G/\tilde{N}$ (Case A) or not (Case B).

CASE A. The bundle in (6) is defined by a representation $\rho: \tilde{N} \longrightarrow \operatorname{Aut}(\mathbb{C}^*)$. Let $\hat{N} := \{n \in \tilde{N} \mid \rho(n) \in \operatorname{Aut}(\mathbb{C}^*)^\circ\}$ and set $\hat{U} := \tilde{U} \cap \hat{N}$. Then the bundle $G/\hat{U} \longrightarrow G/\hat{N}$ is principal. Since $\operatorname{Aut}(\mathbb{C}^*)$ consists of two components, it follows that either $\hat{U} = \tilde{U}$ and this bundle is the original one or $\hat{U}$ is a subgroup of index two in $\tilde{U}$ and the map $G/\hat{U} \longrightarrow G/\tilde{U}$ is a two-to-one covering. Now since S is transitive on $G/\tilde{N}$, by assumption, and therefore also on $G/\hat{N}$, we can just add the right $\mathbb{C}^*$-action to obtain a transitive action of $S \times \mathbb{C}^*$ on $G/\hat{U}$.

CASE B. Recall that G/N and, as a consequence $G/\tilde{N}$, admits a G-equivariant fibration $G/\tilde{N} \xrightarrow{P/\tilde{N}} G/P =: D$, where D is a homogeneous projective rational manifold and $P/\tilde{N} = \mathbb{C}^*$; see [14, Proposition 8, pp. 75–7]. Now, since S is transitive on D, but not on $G/\tilde{N}$, it follows that $G/\tilde{N}$ is isomorphic to $D \times \mathbb{C}^*$. In particular, the holomorphic reduction of $G/\tilde{N}$ is given by $G/\tilde{N} \longrightarrow G/I = \mathbb{C}^*$. Since I is a one-codimensional normal subgroup of G containing S, there is a one-dimensional S-stable subgroup of G acting transitively on G/I. This gives us an action of $S \times \mathbb{C}$ on $G/\tilde{N}$ which lifts to $G/\tilde{U}$. The S-orbits in $G/\tilde{U}$ are one-codimensional, because these orbits lie over the S-orbits in $G/\tilde{N}$ and $G/\tilde{U}$ satisfies the maximal rank condition. It follows from this that the above action of $S \times \mathbb{C}$ is transitive on $G/\tilde{U}$.

Let $\tilde{Y} = G/\tilde{U}$ and $\tilde{Z} = G/\tilde{N}$. We now change groups and let $G_1 := S \times \mathbb{C}$, so that the radical R_1 of G_1 is isomorphic to $\mathbb{C}$. Since S is not transitive on $\tilde{Z}$ and thus is not transitive on $\tilde{Y} = G_1/H_1$, it follows that $R_1 \not\subset H_1$. Now let $N_1 := N_{G_1}(H_1^\circ)$. Because R_1 is central in G_1, it follows that $R_1 \subset N_1$. In particular, $\dim N_1 \geq \dim H_1 + 1$.

If $\dim N_1 = \dim H_1 + 1$, then for dimension reasons, $N_2 := R_1 \cdot H_1$ is an open subgroup in N_1 and thus is a closed subgroup of G_1. Hence $R_1/H_1 \cap R_1 = N_2/H_1$, as the fiber of the map $G_1/H_1 \longrightarrow G_1/N_2$, is closed. Because $\tilde{Y}$ satisfies the maximal rank condition,

$N_2/H_1 = \mathbb{C}$ or $\mathbb{C}^*$. First assume $N_2/H_1 = \mathbb{C}$. Since $P/\tilde{U}$ fibers as a $\mathbb{C}^*$-bundle over $\mathbb{C}^*$, it follows that $d_{P/\tilde{U}} = 2$. Because D is compact and simply connected, it follows from Lemma 1 that $d_{\tilde{Y}} = 2$. Hence G_1/N_2 is compact, since G_1/N_2 cannot have the homotopy type of a CW-complex of dimension smaller than its dimension by Lemma 2. As in the proof of Proposition 5, part a), it follows that S is transitive on $\tilde{Y}$, contrary to the initial assumption in Case B. Otherwise, $N_2/H_1 = R_1/R_1 \cap H_1 = \mathbb{C}^*$ and since the subgroup $R_1 \cap H_1$ is central, the action of $G_1 = S \times \mathbb{C}$ factors to an action of the group $G_1/R_1 \cap H_1 = S \times \mathbb{C}^*$ on $\tilde{Y}$.

Now assume $\dim N_1 \geq \dim H_1 + 2$. Because there is a fibration $S/S \cap H_1 \xrightarrow{\mathbb{C}^*} D$ with D compact and simply connected, we have $d_{S/S\cap H_1} = 1$ by Lemma 1. Hence $Q := N_S\big((S\cap H_1)^\circ\big)$ is a parabolic subgroup of S and $Q/(S\cap H_1)^\circ = \mathbb{C}^*$ by Proposition 1. The same fibration also shows that $S \cap P$ normalizes $S \cap H_1$. Thus $S \cap P \subset Q$ and, for dimension reasons, $S \cap P = Q$ so that $D = S/Q$. Note that $N_1 \cap S \subset Q$. Consequently,

$$\dim H_1 + 1 \leq \dim N_1 - 1 = \dim N_1 \cap S \leq \dim Q = \dim H_1 \cap S + 1 \leq \dim H_1 + 1,$$

showing that $H_1^\circ \subset S$ and $N_1 = Q \cdot R_1$. Therefore, $G_1/N_1 = S/Q = D = G/P$ and $N_1/H_1 = P/\tilde{U}$. As noted above, $P/\tilde{U}$ fibers as a $\mathbb{C}^*$-bundle over $\mathbb{C}^*$. Since $\dim \tilde{Y} > 2$, the proof of Proposition 5, part b), yields a transitive action of $S \times \mathbb{C}^*$ on $\tilde{Y}$ or on a two-to-one covering of $\tilde{Y}$. ∎

We now consider the situation when G/N is not compact. There are two cases depending on whether G' is transitive on G/N or not, with the latter case being handled first.

PROPOSITION 6. *Under the special assumptions, assume further that G' is not transitive on G/N. Then a group of the form $S \times \mathbb{C}^*$, where S is a maximal semisimple subgroup of G, acts transitively on $G/\tilde{U}$ or on a two-to-one covering $G/\hat{U}$ of $G/\tilde{U}$ or else $G/\tilde{U}$ is biholomorphic to $\mathbb{C}^* \times \mathbb{C}^*$ or to the complex Klein bottle.*

PROOF. Consider the fibration

$$G/\tilde{U} \xrightarrow{\tilde{N}G'/\tilde{U}} G/\tilde{N}G'.$$

Its base $G/\tilde{N}G'$ is a group isomorphic to $\mathbb{C}^k \times (\mathbb{C}^*)^l$, where $k + l > 0$, and the fiber $F := \tilde{N}G'/\tilde{U}$ is connected. Moreover, F fibers in the following way

$$F = \tilde{N}G'/\tilde{U} \xrightarrow{\tilde{N}/\tilde{U}} \tilde{N}G'/\tilde{N} = G'/G' \cap \tilde{N} \xrightarrow{R_{G'}/R_{G'}\cap\tilde{N}} S/S \cap \tilde{N}R_{G'}.$$

We claim that we can use Lemma 10 in order to show $\tilde{N}/\tilde{U}$ is biholomorphic to $\mathbb{C}^*$ and $G'/G' \cap \tilde{N}$ is a projective rational manifold. From (5) we have

$$2 = d_{\tilde{N}/\tilde{U}} + d_{R_{G'}/R_{G'}\cap\tilde{N}} + d_{S/S\cap\tilde{N}R_{G'}} + d_{G/\tilde{N}G'},$$

where $d_{G/\tilde{N}G'} = 2k + l$. Observe that the sum of the first three terms on the right hand side of this equation cannot be equal to zero. For, if this were so, then repeating the argument in the first paragraph of Proposition 4 would yield $G' \subset U^\circ$ and $N = G$. Since this contradicts the assumption that G' is not transitive on G/N, the sum of these three terms must be positive. Then $k = 0$, $l = 1$ and

$$1 = d_{\tilde{N}/\tilde{U}} + d_{R_{G'}/R_{G'}\cap\tilde{N}} + d_{S/S\cap\tilde{N}R_{G'}}.$$

This is condition (3) with H replaced by $\tilde{U}$. Hence from Lemma 10 it follows that $\tilde{N}/\tilde{U} = \mathbb{C}^*$ and $G'/G' \cap \tilde{N}$ is compact.

Now consider the fibration $G/\tilde{N} \longrightarrow G/\tilde{N}G' = \mathbb{C}^*$. Since the base $\mathbb{C}^*$ is homeomorphic to $S^1 \times \mathbb{R}$ and the fiber $\tilde{N}G'/\tilde{N} = G'/G' \cap \tilde{N}$ is compact, it follows from the general theory of fiber bundles that $G/\tilde{N}$ is homeomorphic to $M \times \mathbb{R}$, where M is a connected compact manifold. Hence $G/\tilde{N}$ has two ends.

The result follows from Lemma 12, provided $\dim G/\tilde{N} > 1$. If $\dim G/\tilde{N} = 1$, then $G/\tilde{U}$ fibers as a $\mathbb{C}^*$-bundle over $\mathbb{C}^*$. In this case $G/\tilde{U}$ is biholomorphic to the direct product $\mathbb{C}^* \times \mathbb{C}^*$ or to the complex Klein bottle. ∎

PROPOSITION 7. *Under the special assumptions, suppose further that G' is transitive on G/N and G/N is not compact. Then either there is a group of the form $S \times \mathbb{C}^*$, where S is a maximal semisimple subgroup of G, acting transitively on $\tilde{Y} = G/\tilde{U}$ or on a two-to-one covering $\hat{Y}$ of $\tilde{Y}$ or else $\tilde{Y} = G/\tilde{U}$ is one of the manifolds 1)–4) in Proposition 2.*

PROOF. Since G' is transitive on G/N, and thus also on $G/\tilde{N}$, one has $d_{G/\tilde{N}G'} = 0$. Substitution into (5) yields

$$2 = d_{\tilde{N}/\tilde{U}} + d_{R_{G'}/R_{G'}\cap\tilde{N}} + d_{S/S\cap\tilde{N}R_{G'}}. \tag{7}$$

Since G/N is not compact, by assumption, $G/\tilde{N}$ is also not compact and thus

$$\delta := d_{R_{G'}/R_{G'}\cap\tilde{N}} + d_{S/S\cap\tilde{N}R_{G'}} \neq 0.$$

Hence $\delta = 1$ or 2. We look at these two cases separately.

First assume $\delta = 1$. It then follows from equation (7) that $d_{\tilde{N}/\tilde{U}} = 1$. Thus by Corollary 1 of Proposition 3 one has

$$\tilde{N}/\tilde{U} = N^\circ/\tilde{U} \cap N^\circ = (N^\circ/U^\circ)/(\tilde{U} \cap N^\circ/U^\circ) = \mathbb{C}^*,$$

as a complex manifold. We also claim that in this situation $G/\tilde{N}$ has two ends. Since $R_{G'}$ is a unipotent group, the orbits of $R_{G'}$ in $G/\tilde{N}$ are biholomorphic to $\mathbb{C}^k$. Thus $d_{R_{G'}/R_{G'}\cap\tilde{N}} = 2k$. By assumption $\delta = 1$ and so $k = 0$, *i.e.*, $R_{G'}$ acts trivially on $G/\tilde{N}$. Thus S is transitive on $G/\tilde{N}$ and $d_{G/\tilde{N}} = d_{S/S\cap\tilde{N}} = 1$. Since $S \cap \tilde{N}$ is an algebraic subgroup of S, by Proposition 1 we see that $G/\tilde{N}$ has two ends. Note that if $\dim G/\tilde{N} = 1$, then this would imply $G/\tilde{N} = \mathbb{C}^*$. But this would contradict the fact that S is transitive on $G/\tilde{N}$. Hence $\dim G/\tilde{N} > 1$. It now follows from Lemma 12 that a group of the form $S \times \mathbb{C}^*$ is transitive on $\tilde{Y}$ or on a two-to-one covering $\hat{Y}$ of $\tilde{Y}$.

Next assume $\delta = 2$. Then $d_{\tilde{N}/\tilde{U}} = 0$, so that $\tilde{N}/\tilde{U}$ is compact. Since $G/\tilde{U}$ satisfies the maximal rank condition, this implies $\tilde{N} = \tilde{U}$. Now we claim that the covering $G/\tilde{N} = G'/G' \cap \tilde{N} \longrightarrow G'/G' \cap N = G/N$ is finite. (We keep the same notation as in the proof of Lemma 9; in particular, see (2).) The $R_{G'}$-orbits in G/N and $G/\tilde{N}$ are the orbits of a unipotent group. Hence $R_{G'}/R_{G'} \cap \tilde{N} = R_{G'}/R_{G'} \cap N = \mathbb{C}^k$. Also the covering $S/\tilde{M} \longrightarrow S/M$ is finite, since $\tilde{M} \subset M$ are algebraic subgroups of S. It follows that the covering $G/\tilde{N} \longrightarrow G/N$ is also finite and so its fiber $N/\tilde{N}$ is finite. Since $N \supset U \supset \tilde{U}$, the map $G/U \longrightarrow G/N$ is also a finite covering. By definition, G/U is holomorphically separable. It is easy to see that the base of a finite covering is holomorphically separable if its total space has this property. Therefore $G/N = G'/G' \cap N$ is one of the manifolds of Proposition 2. In Case 1) there are no coverings and so $Y = G/U = G/N$. In the remaining cases one can find a reductive algebraic group acting transitively on G/N. Since we can easily show that a reductive algebraic group also acts on a finite covering space of G/N, it follows from the result of [5] and from Proposition 2 that $\tilde{Y} = G/\tilde{U}$ is one of the manifolds 2)–4) in the list. ∎

PROOF OF THE THEOREM. Assume $X = G/H$ is given with $\mathcal{O}(X) \neq \mathbb{C}$ and let $G/H \longrightarrow G/U$ be the holomorphic reduction of X. Recall equation (1) from Lemma 9

$$d_{G/H} \geq d_{\tilde{U}/H} + d_{\tilde{N}/\tilde{U}} + d_{R_{G'}/R_{G'} \cap \tilde{N}} + d_{S/S \cap \tilde{N}R_{G'}} + d_{G/\tilde{N}G'}.$$

First note that

$$d_{\tilde{N}/\tilde{U}} + d_{R_{G'}/R_{G'} \cap \tilde{N}} + d_{S/S \cap \tilde{N}R_{G'}} + d_{G/\tilde{N}G'} > 0. \tag{8}$$

For, if this sum were zero, then $G/\tilde{U}$ would be compact and thus G/U would be a point. But then $\mathcal{O}(G/H) = \mathbb{C}$, contrary to our assumptions.

Now assume $d_{G/H} = 1$. From (8) it is clear that $d_{\tilde{U}/H} = 0$ and thus

$$1 = d_{\tilde{N}/\tilde{U}} + d_{R_{G'}/R_{G'} \cap \tilde{N}} + d_{S/S \cap \tilde{N}R_{G'}} + d_{G/\tilde{N}G'}.$$

This is (4), with H replaced by $\tilde{U}$. It follows from Proposition 4 that $G/\tilde{U}$ has two ends. Since $\tilde{U}/H$ is compact and connected, G/H also has two ends. The base of the holomorphic reduction of G/H is an affine cone minus its vertex, see [9] or [14] and the fiber is compact and connected. This is the situation described in part a) of the theorem.

In the rest of the proof we suppose $d_{G/H} = 2$. Assume first $d_{\tilde{U}/H} = 1$. Hence the sum on the left hand side of (8) is equal to one, *i.e.*, (4) holds with H replaced by $\tilde{U}$. Then $G/\tilde{U}$ has two ends by Proposition 4 and is thus an affine cone minus its vertex. Since such a manifold is holomorphically separable, $U = \tilde{U}$ and the fiber of the holomorphic reduction of G/H is connected. This is the situation described in part b_1) of the theorem.

We claim that it is not possible that $d_{\tilde{U}/H} = 0$ and the sum in (8) is equal to one. For, if it were so, then $\tilde{U}/H$ would be compact and connected and $G/\tilde{U}$ would be an affine cone minus its vertex, again by Proposition 4. Therefore $G/\tilde{U}$ is homeomorphic to $M \times \mathbb{R}$, where M is a connected compact manifold. From the general theory of fiber

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bundles it is clear that the total space G/H is homeomorphic to $M' \times \mathbb{R}$, where M' is another connected compact manifold. Thus $d_{G/H} = 1$, contrary to our assumptions.

The remaining case occurs when $d_{\tilde{U}/H} = 0$ and the sum in (8) equals two. This is equation (5) and the conditions of the special assumptions hold. Then Propositions 5, 6 and 7 imply that either the base $\tilde{Y} := G/\tilde{U}$ of the map $G/H \longrightarrow G/\tilde{U}$ is one of the manifolds in b_2) or a group of the form $S \times \mathbb{C}^*$ acts transitively on $\tilde{Y}$ (resp. on a two-to-one covering $\hat{Y} = G/\hat{U}$ of $\tilde{Y}$). In the latter case Proposition 2 along with [5] shows that $\tilde{Y}$ (resp. $\hat{Y}$) is in the list. In this situation the fiber of the holomorphic reduction of G/H is compact and connected and its base is one of the manifolds described in part b_2) of the theorem. By Proposition 2 every manifold listed in b_2) is a quasi-affine algebraic manifold with $d = 2$ or is covered two-to-one by such. Thus the proof is complete. ∎

6. **Concluding remarks.** 1) The proof of the theorem shows that the inequality in (1) is, in fact, an equality.

2) Lemma 3 generalizes to all locally trivial fiber bundles and the proof is essentially the same. Since the fiber F of the holomorphic reduction in Case a) (resp. b_2)) is compact and the base Y is retractable onto a compact submanifold, the equality $d_X = 1$ (resp. $d_X = 2$) follows from the description of Y, given in the theorem.

3) In b_2) we have $\mathcal{O}(F) = \mathbb{C}$, because of the compactness of F. This is generally no longer true in the setting of b_1), for here the fiber can even be Stein. A simple example can be found in [5]. Namely, let

$$G := \mathrm{SL}(2,\mathbb{C}), \quad H := \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \,\middle|\, n \in \mathbb{Z} \right\}, \quad U := \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \,\middle|\, z \in \mathbb{C} \right\},$$

where $U = \bar{H}$ is the Zariski closure of H in G. Then $G/U = \mathbb{C}^2 - \{0\}$ is the base of the holomorphic reduction of $X := G/H$ and its fiber U/H is biholomorphic to $\mathbb{C}^*$.

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B. Spasskaya ul. 33, kv. 33
129010 Moscow
Russia
e-mail: shelf@adonis.iasnet.com

Department of Mathematics and Statistics
University of Regina
Regina, Saskatchewan
S4S 0A2
e-mail: gilligan@max.cc.uregina.ca