A FILTER DESCRIPTION OF THE HOMOMORPHISMS OF H^∞

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1. Introduction. The algebra of bounded analytic functions on the open unit disk D, usually written H^∞, is a commutative Banach algebra under the supremum norm. Since its compact maximal ideal space $\mathbf{M}$ (space of complex homomorphisms) is an extension space of the unit disk, there must be a continuous mapping form βD, the Stone–Čech compactification of D, onto $\mathbf{M}$. R. C. Buck has remarked **(4)**, that this mapping fails to be one–one, in the light of a classical theorem of Pick. If the points of βD are represented by filters of subsets of D, we can identify those filters which are sufficiently close in terms of the hyperbolic metric on D in an attempt to get a one–one correspondence between filters and points of $\mathbf{M}$. This attempt is successful in certain instances. In §3, it is shown that any homomorphism in the closure of an interpolating sequence corresponds to only one such filter. In §§4 and 5, respectively, two subsets of $\mathbf{M}$ are defined which can be approached in a natural fashion by two different types of non-Euclidean rigid motions on D. The correspondence is one–one on each of these. In §7 examples are constructed of homomorphisms for which this correspondence fails to be one–one. These include all Silov boundary homomorphisms, as well as an example off the boundary. In §6 we construct analytic mappings ψ from D into the maximal ideal space, generalizing the discussion of I. J. Schark **(13)**. Professor Kenneth Hoffman has kindly shown us some of his recent work with similar mappings, and has related this work to the mappings ψ, giving a result quoted at the end of §6. This result applies to earlier sections, showing that it is exactly the homomorphisms in the closure of an interpolating sequence which correspond to a unique filter of the type mentioned above.

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2. Star filters. Because the evaluation mappings, $z \to f(z)$ for $f \in H^\infty$, give complex homomorphisms for each $z \in D$, we have a copy of D inside $\mathbf{M}$. This is called Δ, and is topologically equivalent to D. For a discussion of this and of other properties of H^∞, the reader is referred to the last chapter of Hoffman **(10)**. Carleson **(7)** has shown that Δ is dense in $\mathbf{M}$. Hence there must

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be a continuous mapping θ from βD onto $\mathbf{M}$, which is a homeomorphism onto Δ when restricted to D:

$$\theta\colon \beta D \to \mathbf{M}.$$

The points of βD are in 1–1 correspondence with maximal closed filters on D. By maximal closed filters we mean filters that are closed (generated by closed sets) and are maximal among such filters. We see this correspondence by comparison with **(8)**, where Gillman and Jerison demonstrate, for completely regular X, a 1–1 correspondence between points of βX and maximal filters of zero sets on X. In the metric space D, every closed set is a zero set, so Gillman's filters are just the restrictions of maximal closed filters to the lattice of closed subsets of D.

H^∞ is a closed subalgebra of $C(D)$, the algebra of all bounded continuous functions on D with supremum norm. The maximal ideal space of $C(D)$ is βD, and each complex homomorphism ϕ in βD is an evaluation $\phi\colon f \to \lim f(\mathbf{U})$ for $f \in C(D)$, where $\mathbf{U}$ is the maximal closed filter corresponding to ϕ. θ is the mapping between the two maximal ideal spaces induced by the injection of H^∞ in $C(D)$, and is just the restriction of H^∞ of homomorphisms of the larger algebra. Hence every homomorphism of H^∞ has the form

$$f \to \lim f(\mathbf{U}), \qquad f \in H^\infty, \tag{1}$$

where $\mathbf{U}$ is some maximal closed filter on D.

The hyperbolic distance $h(z, w)$ and the pseudo-hyperbolic distance $\psi(z, w)$ between points z and w of D are defined by

$$\psi(z, w) = \left|\frac{z - w}{1 - z\bar{w}}\right| = \tanh(h(z, w));$$

see **(5)** for a discussion of the hyperbolic metric, and **(9)** for the following classical result.

THEOREM OF PICK. *Suppose f is a holomorphic function mapping the unit disk D into itself. Then for arbitrary z and w in D.*

$$h(f(z), f(w)) \leqslant h(z, w).$$

This means that θ will not be 1–1, for any two maximal closed filters whose sets are arbitrarily close in the hyperbolic metric must give the same homomorphism of H^∞.

The following identification suggests itself. For any maximal closed filter $\mathbf{U}$ on D, we define a $*$-filter $\mathbf{U}^*$ by taking as basis all hyperbolic ϵ-neighbourhoods of the sets in $\mathbf{U}$. If we define for any subset A of D

$$N(A, \epsilon) = \{z \in D | \psi(z, w) < \epsilon \text{ for some } w \in A\},$$

then $\mathbf{U}^*$ is the filter generated by the collection

$$\{N(A, \epsilon) | A \in \mathbf{U}, \epsilon > 0\}.$$

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PROPOSITION 1. *For each maximal closed filter* $\mathbf{U}$ *on* D, *and each* $f \in H^\infty$, $\lim f(\mathbf{U}^*)$ *exists.*

Proof. We can restrict the proof to functions f with $\|f\| \leqslant 1$, since every function in H^∞ is a scalar multiple of such a function. Let $\lim f(\mathbf{U}) = \alpha$, and take any positive number ϵ. Choose a set $A \in \mathbf{U}$ with $|f(z) - \alpha| < \epsilon/2$ for $z \in A$. For any point $w \in N(A, \epsilon/2)$, there exists a point $z \in A$ with $h(w, z) < \epsilon/2$. Now

$$|f(w) - \alpha| \leqslant |f(w) - f(z)| + |f(z) - \alpha| < \epsilon$$

since, by Pick's theorem applied to f,

$$|f(w) - f(z)| \leqslant h(f(w), f(z)) \leqslant h(w, z) < \epsilon/2.$$

It therefore follows that $\lim f(\mathbf{U}^*)$ exists and equals $\lim f(\mathbf{U})$.

COROLLARY. *The filter* $\mathbf{U}^*$ *determines a homomorphism of* H^∞ *given by* $f \to f(\mathbf{U}^*)$ *for* $f \in H^\infty$. *Every homomorphism of* H^∞ *can be represented in this way.*

We exhibit in later sections subsets of $\mathbf{M}$ on which each homomorphism is determined by exactly one of these $*$-filters.

PROPOSITION 2. *Each maximal closed filter* $\mathbf{U}$ *contains one and only one* $*$*-filter* $\mathbf{U}^*$. *If* $\mathbf{U}^*$ *and* $\mathbf{V}^*$ *are distinct* $*$*-filters, then there exist a positive constant* ϵ *and sets* $A \in \mathbf{U}$ *and* $B \in \mathbf{V}$ *with* $\psi(a, b) \geqslant \epsilon$ *whenever* $a \in A$, $b \in B$.

Proof. The proof is given in a series of steps; it depends largely on the simple property (i) of $*$-filters. We write $\mathrm{Cl}(A)$ for the closure of a set A. The filters $\mathbf{U}$ and $\mathbf{V}$ are always assumed to be maximal closed.

(i) For any $A \in \mathbf{U}$, and arbitrary ϵ, $N(A, \epsilon/2)$ and $\mathrm{Cl}(N(A, \epsilon/2))$ belong to $\mathbf{U}^*$.

(ii) $\mathbf{U}^* \subset \mathbf{U}$. This holds because $\mathbf{U}^*$ is generated by sets $N(A, \epsilon)$ where $A \in \mathbf{U}$, and $A \subset N(A, \epsilon)$ always holds.

(iii) If $\mathbf{U}^* \subset \mathbf{V}$, then $\mathbf{U}^* \subset \mathbf{V}^*$. Any set B in $\mathbf{U}^*$ contains a basic set $N(A, \epsilon)$ from $\mathbf{U}$ with $A \in \mathbf{U}$. By (i), $C = N(A, \epsilon/2)$ also belongs to $\mathbf{U}^*$ and must by hypothesis belong to $\mathbf{V}$. This means that $N(C, \epsilon/2) \in \mathbf{V}^*$. But

$$N(C, \epsilon/2) \subset N(A, \epsilon) \subset B,$$

and therefore $B \in \mathbf{V}^*$, as required.

(iv) If $\mathbf{U}^* \subset \mathbf{V}$, then $\mathbf{V}^* \subset \mathbf{U}^*$. We show, by indirect proof, that $\mathbf{V}^* \subset \mathbf{U}$, and then the result is obtained by repeating the proof of (iii). Suppose $\mathbf{V}^* \not\subset \mathbf{U}$, and choose a set $B \in \mathbf{V}^* - \mathbf{U}$. We can take a basic set $N(A, \epsilon)$ with $A \in \mathbf{V}$, having the same property. Indeed the smaller closed set $\mathrm{Cl}[N(A, \epsilon/2)]$ is also in $\mathbf{V}^* - \mathbf{U}$. Using the fact that $\mathbf{U}$ is a maximal closed filter, we can find a set $C \in \mathbf{U}$, with $\mathrm{Cl}[N(A, \epsilon/2)] \cap C = \emptyset$. The disjointness of these last sets easily implies $N(A, \epsilon/8) \cap N(C, \epsilon/8) = \emptyset$. The first of these is in $\mathbf{V}^*$, and the second is in $\mathbf{U}^*$, a contradiction.

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The first assertion of Proposition 2 follows from (ii) and from the combination of (iii) and (iv). The second follows from (v) and (vi).

(v) If $\mathbf{U}^* \neq \mathbf{V}^*$, then $\mathbf{U}^*$ and $\mathbf{V}^*$ are incompatible. Suppose this is false; that is, suppose distinct filters $\mathbf{U}^*$ and $\mathbf{V}^*$ exist, but that any two sets from these have a non-empty intersection. It will then be possible to choose an ultrafilter $\mathbf{F}_1$ above both $\mathbf{U}^*$ and $\mathbf{V}^*$. If we form the closed filter $\mathbf{F}_2$ by taking as basis the closures of all the sets in $\mathbf{F}_1$, then this filter will still contain both $*$-filters. This follows from (i), which says that every set in a $*$-filter contains a closed member of the filter. Take now any maximal closed filter $\mathbf{F}$ above $\mathbf{F}_2$. We have $\mathbf{U}^* \subset \mathbf{F}$, and by (iii) and (iv) this means that $\mathbf{U}^* = \mathbf{F}^*$. Likewise $\mathbf{V}^* = \mathbf{F}^*$, and this gives the contradiction $\mathbf{U}^* = \mathbf{V}^*$.

(vi) If $\mathbf{U}^*$ and $\mathbf{V}^*$ are incompatible, then we can find sets $A \in \mathbf{U}$ and $B \in \mathbf{V}$, and positive ϵ, such that $\psi(a, b) \geqslant \epsilon$ whenever $a \in A$ and $b \in B$. We can let the disjoint sets from $\mathbf{U}^*$ and $\mathbf{V}^*$, guaranteed by imcompatibility, be basic sets of the form $N(A, \epsilon)$ and $N(B, \epsilon)$ for sufficiently small ϵ, where $A \in \mathbf{U}$ and $B \in \mathbf{V}$.

It is clear that we can choose the sets A and B of Proposition 2 to lie in $\mathbf{U}^*$ and $\mathbf{V}^*$ respectively for some $\epsilon > 0$.

We make a few remarks on terminology. For any maximal closed filter $\mathbf{U}$, we say that the homomorphism

$$\phi_{\mathbf{U}} : f \rightarrow \lim f(\mathbf{U}), \qquad f \in H^\infty,$$

is *determined by* $\mathbf{U}$, or *corresponds to* $\mathbf{U}$. We frequently use the fact that $\phi(f)$ must be in the closure of $f(A)$ if A is a set in $\mathbf{U}$. We say that we can *separate* two $*$-filters $\mathbf{U}^*$ and $\mathbf{V}^*$ if we can find $f \in H^\infty$ with $\phi_{\mathbf{U}}(f) \neq \phi_{\mathbf{V}}(f)$, which is possible if and only if the two filters determine different homomorphisms.

The set $\mathbf{M}$ can be divided into *fibres*

$$\mathbf{M}_\alpha = \{\phi \in \mathbf{M} : \phi(z) = \alpha\}$$

of homomorphisms "above" any point α in the closed unit disk $\bar{D}$. Each maximal closed filter $\mathbf{U}$ on D must converge to some point α in $\bar{D}$, and this means that $\phi_{\mathbf{U}} \in \mathbf{M}_\alpha$. Since $\mathbf{M}_\alpha$ has just one point for $|\alpha| < 1$, interest centers on the homeomorphic space $\mathbf{M}_\alpha$ for $|\alpha| = 1$.

Let $\mathbf{M}^*$ be the set of all $*$-filters in the quotient topology of βD. This can be characterized as the largest compact Hausdorff extension space of D in which the embedding $D \rightarrow \mathbf{M}^*$ is uniformly continuous in the hyperbolic metric; for a discussion of such matters, see **(1** and **12)**.

3. Interpolating homomorphisms. A sequence (z_n) of distinct points in D is called an *interpolating sequence* if there is a function in H^∞ to interpolate the value w_n at z_n, $n = 1, 2, 3, \ldots$, for an arbitrary bounded sequence (w_n).

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The condition

$$(C) \qquad \prod_{\substack{n=1 \\ n\neq m}}^{\infty} \left|\frac{z_n - z_m}{1 - \bar{z}_n z_m}\right| \geqslant \delta > 0, \qquad m = 1, 2, 3, \ldots,$$

where δ is a constant independent of m, is necessary and sufficient for (z_n) to be an interpolating sequence.

We say that a filter is *interpolating* if the set of points in some interpolating sequence belongs to it. If a homomorphism is determined by an interpolating maximal closed filter, it is said to be an *interpolating homomorphism.*

Any sequence (z_n) of points in D satisfying the condition

$$\sum (1 - |z_n|) < \infty$$

determines a bounded analytic function B, called a Blaschke product:

$$B(z) = \prod_{n=1}^{\infty} \frac{\bar{z}_n}{|z_n|}\cdot\frac{z_n - z}{1 - \bar{z}_n z}.$$

These notions are discussed fully in **(10).**

LEMMA 1. *If $B(z)$ is the Blaschke product of a sequence (z_n) satisfying the condition (C), and if $\epsilon > 0$ is given, there exists a constant δ_0 depending only on δ and ϵ such that $|B(z)| \geqslant \delta_0$ whenever $\psi(z, z_n) \geqslant \epsilon$ for all n.*

Proof. We first prove the following property of D, where all distances $\psi(z, z')$ referred to are pseudo-hyperbolic. An integer N exists, depending only on ϵ, such that from any infinite sequence of points of D distant at least ϵ from the origin, one can extract a finite subset with at most N members, such that every other point of the sequence is at least as close to one of these as to the origin. To do this it is sufficient to find an absolute constant $\theta_0 > 0$ such that for any point A in D with modulus a, we have $\psi(P, (A) \leqslant \psi(P, O)$ for any point P lying outside the circle

$$(1) \qquad x^2 + y^2 = a^2$$

and satisfying $|\arg(P) - \arg(A)| \leqslant \theta_0$.

To show the existence of θ_0, we assume A is the point $(a, 0)$ and draw the neighbourhood $\{P' \in D: \psi(P', A) \leqslant a\}$, a circle with centre $(a/(1 + a^2), 0)$ and radius $a/(1 + a^2)$. We intersect this circle, whose equation is

$$(2) \qquad x^2 - \frac{2ax}{1 + a^2} + y^2 = 0,$$

with (1) to obtain points R and Q. The co-ordinates (x_0, y_0) of R satisfy

$$\frac{y_0}{x_0} = \frac{\sqrt{[(5 + a^2)(3 - a^2)]}}{\sqrt{[(1 + a^2)(1 + a^2)]}},$$

which is bounded away from zero when a is allowed to range. Hence the angle $ROQ \geqslant 2\theta_0$ for some constant θ_0.

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To prove that θ_0 has the required property, take any point P outside (1) and within the angle $2\theta_0$ about OA. The circle through P, centred at the origin has the equation $x^2 + y^2 = b^2$. Let B be the point $(b, 0)$ and let C be a point where the circle $x^2 + y^2 = b^2$ meets the circle $\{P' \in D: \psi(P', B) = b\}$. To show that $\psi(P, A) \leqslant \psi(P, O)$, it is sufficient to show that $\psi(C, A) \leqslant \psi(C, O)$. However,

$$b = \psi(C, O) = \psi(C, B) \geqslant \psi(C, A),$$

where the concluding inequality is evident when we draw the neighbourhood $\{P' \in D; \psi(P', C) \leqslant b\}$ and observe that A lies inside it.

We next take any point $z_0 \in D$ with $\psi(z_0, z_n) \geqslant \epsilon$, $n = 1, 2, 3, \ldots$, and by a suitable non-Euclidean rigid motion T move z_0 to the origin. Denote by z_n' the point Tz_n for $n = 1, 2, \ldots$ and let B' be the Blaschke product on the points z_n'. Since $B(z) = B'(Tz)$ holds in general, we must show that $|B'(0)| \geqslant \delta_0$. Because T is a rigid motion, we have that z_n' is ψ-distant from the origin by at least ϵ for each n, and also that (C) is satisfied by the sequence (z_n'). Thus

$$\prod_{n \neq m} \left| \frac{z_m' - z_n'}{1 - \bar{z}_n' z_m'} \right| = \prod_{n \neq m} \psi(z_m', z_n') \geqslant \delta.$$

If the sequence is rearranged so that every member is as close to one of $z_1', z_2', \ldots, z_N'$ as to the origin, then we can make a partition of the set of integers $\{N + 1, N + 2, N + 3, \ldots\}$ into sets $A_1, A_2, \ldots, A_N$ satisfying the condition: $n \in A_i$ implies $\psi(z_n', z_i') \leqslant \psi(z_n', 0)$. We obtain

$$\begin{aligned} |B'(0)| &\geqslant \epsilon^N \prod_{n \in A_1} \psi(z_n', z_i') \ldots \prod_{n \in A_n} \psi(z_n', z_N') \\ &\geqslant (\epsilon\delta)^N = \delta_0. \end{aligned}$$

Note that δ_0 does not depend on the sequence (z_n) of points given, but only on the value of δ used in (C).

THEOREM 1. *Each interpolating homomorphism of H^∞ is determined by a unique $*$-filter.*

Proof. Suppose the maximal closed filter $\mathbf{U}$ contains the set $A = \{z_1, z_2, z_3, \ldots\}$, where (z_n) is an interpolating sequence. We construct the Blaschke product $F \in H^\infty$ with zeros z_n, $n = 1, 2, \ldots$. Take any $\epsilon > 0$ and construct hyperbolic ϵ-neighbourhoods $N_1, N_2, N_3, \ldots$ about the points $z_1, z_2, \ldots$. By Lemma 1, F is bounded away from zero on the set

$$D - \bigcup_{i=1}^{\infty} N_i.$$

This will be sufficient to prove the theorem, for if $\mathbf{V}^*$ is a $*$-filter different from $\mathbf{U}^*$, we can, for some ϵ, find sets $A' \in \mathbf{U}$ and $B \in \mathbf{V}$ with $\psi(z, b) \geqslant \epsilon$ whenever $a \in A'$ and $b \in B$. We can, by intersection with A if necessary, have $A' \subset A$. The function F will separate the filters; we know that $\phi_{\mathbf{U}}(F) = 0$ and we have $\phi_{\mathbf{V}}(F) \neq 0$ because F is bounded away from zero on the set $B \in \mathbf{V}$.

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4. Non-tangential homomorphisms. A hypercycle is the restriction to D of some circle which meets the unit circle Γ in exactly two points α and β. In the hyperbolic geometry, any two hypercycles through α and β are curves at a constant distance from each other.

We say that a filter on D converging to $\alpha \in \Gamma$ is *non-tangential* if and only if two hypercycles through α and $-\alpha$ can be chosen, along with a set from the filter which lies between them. Any $\phi \in \mathbf{M}_\alpha$ is called a *non-tangential homomorphism* if and only if $\phi = \phi_{\mathbf{U}}$ for some non-tangential maximal closed filter $\mathbf{U}$ converging to α.

Let $\mathbf{N}_\alpha$ be the subset of $\mathbf{M}_\alpha$ consisting of all these non-tangential homomorphisms, and let $\mathbf{N} = \bigcup_{\alpha \in \Gamma} \mathbf{N}_\alpha$. The results in this section lead to the following theorem about $\mathbf{N}$.

THEOREM 2. *Each non-tangential homomorphism is determined by a unique $*$-filter.*

It is sufficient to prove results for a particular value of α, say $\alpha = 1$. Let $\mathbf{C}$ be the family of hypercycles through -1 and 1. The terms "above" and "below" are used in the obvious geometric sense.

PROPOSITION 3. *If the filter $\mathbf{U}^*$ is non-tangential and converges to $\alpha = 1$, there exists $C_0 \in \mathbf{C}$ with the following property: For completely arbitrary $C_1 \in \mathbf{C}$ above C_0, and $C_2 \in \mathbf{C}$ below C_0, some set from $\mathbf{U}^*$ lies between C_1 and C_2.*

Proof. $\mathbf{U}$ is a maximal closed filter. Take $F \in \mathbf{U}$ and $C \in \mathbf{C}$. Let F_1 be the closed set of those points of F which lie above or on C, and F_2 the closed set of those points below or on C. $F = F_1 \cup F_2$, and by the maximality of the filter, either $F_1 \in \mathbf{U}$ or $F_2 \in \mathbf{U}$. In the former case, we say the filter lies above, in the latter case below, the circle C. One of these must hold. If a filter lies above [below] any member of $\mathbf{C}$, then it is in the same relation to any lower [higher] member of $\mathbf{C}$. Since $\mathbf{U}^*$ is non-tangential, it and therefore $\mathbf{U}$ will lie below some member of $\mathbf{C}$ and above some member of $\mathbf{C}$. From these properties we infer the existence of some $C_0 \in \mathbf{C}$ such that $\mathbf{U}$ lies below any circle which is above C_0, and above any circle which lies below C_0. Because of the relation between the family C and the hyperbolic metric, we can make the same assertion about $\mathbf{U}^*$.

We see in the above proof that a filter $\mathbf{U}$ is non-tangential if and only if $\mathbf{U}^*$ is non-tangential. With each circle $C \in \mathbf{C}$, we associate the positive angle k between the upward vertical and the inward-directed tangent to C, both taken at $\alpha = 1$. We have $0 < k < \pi$. For each non-tangential $\mathbf{U}^*$ converging to $\alpha = 1$, we let the *angle of approach* to $\alpha = 1$ be the k of its circle C_0. The remaining $*$-filters converging to $\alpha = 1$ must either lie above every member of $\mathbf{C}$, or below every member of $\mathbf{C}$; we assign the values $k = 0$ and $k = \pi$ to the two cases respectively.

PROPOSITION 4. *If $\mathbf{U}^*$ and $\mathbf{V}^*$ have different angles of approach to $\alpha = 1$, then $\phi_{\mathbf{U}} \neq \phi_{\mathbf{V}}$.*

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Proof. It is sufficient to construct a function $h \in H^\infty$ such that $|\phi_{\mathbf{U}}(h)| = e^k$, if the filter $\mathbf{U}^*$ has angle of approach k to $\alpha = 1$. Take the harmonic function $(\pi/2) - \arg w$ in the right half w-plane, and map conformally onto D with $z = (w-1)/(w+1)$. The image function u is constant on each member of $\mathbf{C}$; the value on any circle C is the k associated with the circle. Let v be the harmonic conjugate of u, and set $h = \exp(u + iv)$. Then $h \in H^\infty$, and on the circle C, $|h| = e^k$. If $\mathbf{U}^*$ is non-tangential and is associated with the circle C in the manner of Proposition 3, then it is easily seen that $|\phi_{\mathbf{U}}(h)| = e^k$, since the sets of $\mathbf{U}$ can be made to lie between circles arbitrarily close to C. For filters $\mathbf{U}^*$ with $k = 0$, we can for each $\epsilon > 0$ choose $A \in \mathbf{U}$ above a high enough circle of C to make $u < \epsilon$ on A, which means that $\phi_{\mathbf{U}}(h)$ has modulus 1. A similar argument holds for $k = \pi$.

From this proposition, it is obvious that non-tangential homomorphisms are determined *only* by maximal closed filters which are non-tangential. We also have

COROLLARY. $\mathbf{N}_\alpha$ *is an open subset of* $\mathbf{M}_\alpha$.

Proof. For the function h used in the proposition,

$$\mathbf{N}_\alpha = \{\phi \in \mathbf{M}_\alpha : 1 < |\hat{h}(\phi)| < e^\pi\}$$

must be an open subset of $\mathbf{M}_\alpha$. Here $\hat{h}$ is the image of h under the Gelfand representation, with $\hat{h}(\phi) = \phi(h)$; it is just the extension of h to all of $\mathbf{M}$, if we identify D and Δ.

We consider now the case of two different $*$-filters, $\mathbf{U}^*$ and $\mathbf{V}^*$, both of which are non-tangential and approach $\alpha = 1$ along the same hypercycle C_0. There exist sets $A \in \mathbf{U}^*$, $B \in \mathbf{V}^*$, along with a positive constant which we take to be

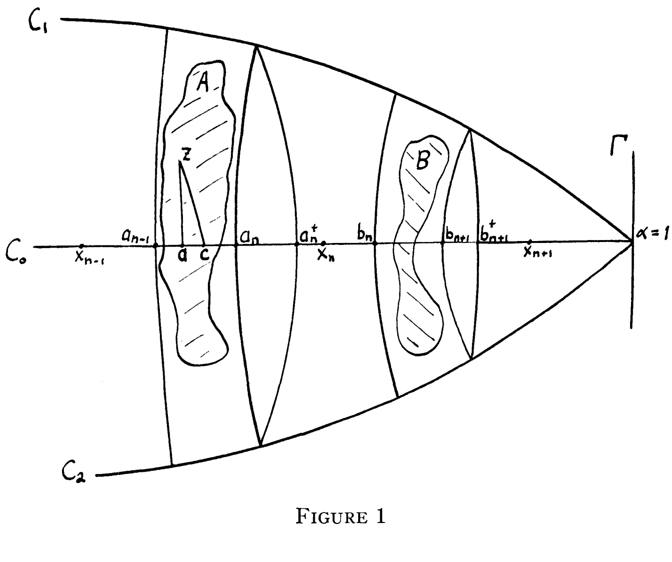

FIGURE 1

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8ϵ, such that $\psi(a, b) \geqslant 8\epsilon$ whenever $a \in A$, $b \in B$. Take two circles C_1 and C_2 from the family $\mathbf{C}$ which are equal pseudo-hyperbolic distances above and below C_0; this distance is assumed to be less than 2ϵ. A and B can be made to lie between C_1 and C_2. We also restrict A and B to the right half of D.

It is next shown that the points of A and B can be separated into a sequence of blocks, which consist alternately of points of A and B, tending to $\alpha = 1$; see Figure 1, in which this is illustrated for the special case where C_0 is the real axis. Take the family $\mathbf{B}$ of circles orthogonal to the circles of $\mathbf{C}$. In other words $\mathbf{B}$ is the family of circles with centre on the real axis, and orthogonal to the unit circle Γ. The restrictions of these circles to D are straight lines in the hyperbolic geometry, and are orthogonal to the members of $\mathbf{C}$. Each point of A lies on a circle from $\mathbf{B}$ which determines some point of intersection on C_0. The set A_0 of these points of intersection is called the trace of A on C_0. A similar set B_0 is defined, and we readily see that the two sets are separated in the pseudo-hyperbolic metric. Thus

$$\psi(\alpha, \beta) \geqslant 4\epsilon \qquad \text{for } \alpha \in A_0, \beta \in B_0.$$

We introduce an order relation on C_0: a is greater than b if it is closer to $\alpha = 1$. In the diagram, this is the natural ordering of the reals. We use the appropriate notation: $[\alpha, \beta]$ denotes the closed subarc of C_0 joining α and β, and we take infima and suprema of subsets of C_0 using the given ordering. Start from the left and move towards $\alpha = 1$. Suppose without loss of generality that $b_1 = \inf\{\beta : \beta \in B_0\}$ is smaller than any $\alpha \in A_0$. If b_2 is the supremum of the set of points in B_0 which are less than each point of A_0, then $b_1 \leqslant b_2$. Let $a_2 = \inf\{\alpha : \alpha \in A_0\}$. Then $b_2 \leqslant a_2$. In fact we can choose points $\beta \in B_0$ and $\alpha \in A_0$ arbitrarily close to b_2 and a_2, which means that $\psi(b_2, a_2) \geqslant 4\epsilon$ and $b_2 < a_2$. Define a_3 to be the supremum of those points of A_0 less than each point in B_0, excluding those in $[b_1, b_2]$. We continue in this way; b_3 is the infimum of B_0 after $[b_1, b_2]$ has been removed, and b_4 is the supremum of those $\beta \in B_0$ which are less than any $\alpha \in A_0$ with $\alpha > b_3$. We obtain the following:

$$b_1 \leqslant b_2 < a_2 \leqslant a_3 < b_3 \leqslant b_4 < a_4 \ldots,$$

$$\psi(a_n, b_n) > 4\epsilon \qquad \text{for } n = 2, 3, 4, \ldots,$$

$$B_0 \subset [b_1, b_2] \cup [b_3, b_4] \cup [b_5, b_6] \cup \ldots,$$

$$A_0 \subset [a_2, a_3] \cup [a_4, a_5] \cup [a_6, a_7] \cup \ldots.$$

We shall make use of this decomposition to find a function in H^∞ which separates the filters $\mathbf{U}^*$ and $\mathbf{V}^*$.

We assume first of all that $\mathbf{U}^*$ and $\mathbf{V}^*$ have angle of approach $k = \pi/2$ to the point $\alpha = 1$. Define a sequence (x_n), where x_n bisects the gap between a_n and b_n. Thus

$$x_n \geqslant 0, \qquad \psi(a_n, x_n) = \psi(b_n, x_n), \qquad n = 2, 3, 4, \ldots.$$

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We have the inequalities

$$\psi(x_n, a_n) > 2\epsilon, \qquad \psi(x_n, b_n) > 2\epsilon, \qquad n = 2, 3, 4, \ldots,$$

Since

$$4\epsilon < \psi(a_n, b_n) \leqslant \psi(a_n, x_n) + \psi(b_n, x_n) = 2\psi(a_n, x_n) = 2\psi(b_n, x_n).$$

Define, in addition, $x_1 < b_1$ with

$$\psi(x_1, b_1) = 2\epsilon.$$

We start with circles C_1 and C_2, and sets $A \in \mathbf{U}^*$, $B \in \mathbf{V}^*$ lying between the two circles. At various stages, restrictions will be placed on C_1 and C_2 which bring them closer to C_0. When this happens, we restrict the initial A and B to the region between the new circles. The restrictions, which we still denote by A and B, must also belong to the filters $\mathbf{U}^*$ and $\mathbf{V}^*$ respectively. Finally sets A and B are obtained which are separated by the function $F \in H^\infty$:

$$F(z) = \prod_{n=1}^{\infty} \frac{x_n - z}{1 - x_n z}.$$

This function takes on real values on the real axis, and is positive on A_0, negative on B_0. It will be shown that F is separated away from zero on these sets, and indeed that it separates the larger sets A and B. From this, we obtain

PROPOSITION 5. *Two different $*$-filters with angle of approach $k = \pi/2$ to the point $\alpha = 1$ give different homomorphisms.*

The proof follows from a sequence of lemmas.

Take an individual block enclosed by C_1, C_2, and the circles of $\mathbf{B}$ through a_{n-1} and a_n (n odd); see Figure 1. We draw a circle centred at the origin and passing through the right-hand corner points of this block. Let a_n^+ be the point where this circle meets the real line. In like fashion we obtain b_n^+ for n even.

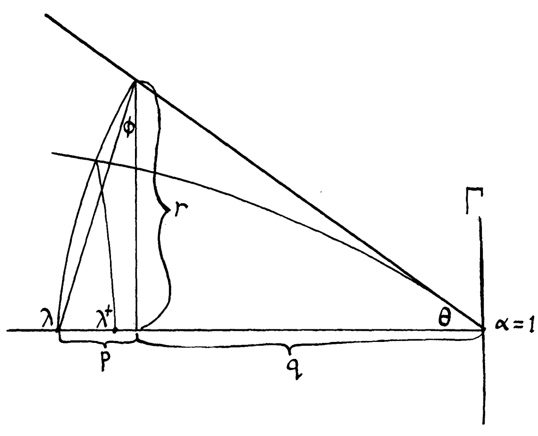

FIGURE 2

LEMMA 2. *Circles C_1 and C_2 can be chosen such that*

$$\psi(a_n, a_n^+) < \epsilon, \qquad n = 3, 5, 7, \ldots,$$
$$\psi(b_n, b_n^+) < \epsilon, \qquad n = 2, 4, 6, \ldots.$$

Proof. See Figure 2. Suppose C_1 and C_2 are given. Corresponding to each λ, $0 < \lambda < 1$, is a point $\lambda^+ > \lambda$ obtained as a_n^+ is obtained from a_n. We show that for suitable C_1 and C_2, $\psi(\lambda, \lambda^+)$ can be made less than an arbitrary positive number δ independent of λ.

The tangent to C_1 at $\alpha = 1$ intersects the circle through λ at a point from which a perpendicular is dropped to the real axis. Let the distance from the foot of the perpendicular to λ and to $\alpha = 1$ be p and q respectively, and let θ be the angle between the real line and the tangent at $\alpha = 1$.

As $\lambda \to 1$, the radius of the circle through λ^+ approaches 0. At the point where the two radii are equal, it is clear that $p > (\lambda^+ - \lambda)/2$; and this inequality holds for all larger values of λ. Thus there exists λ_0 such that

$$\lambda^+ - \lambda < 2p \qquad \text{for } \lambda \geqslant \lambda_0.$$

Since the angle at the top vertex of the large triangle is always less than $\pi/2$, the angle ϕ (marked on the diagram) must be less than θ. Thus

$$p = r \tan\phi = q \tan\theta \tan\phi < q \tan^2\theta,$$
$$\lambda^+ - \lambda < 2q \tan^2\theta \qquad \text{for } \lambda \geqslant \lambda_0.$$

Hence

$$\psi(\lambda, \lambda^+) = \frac{\lambda^+ - \lambda}{1 - \lambda\lambda^+} < \frac{2q \tan^2\theta}{1 - \lambda} < 2\tan^2\theta \qquad \text{for } \lambda \geqslant \lambda_0.$$

This means that, by choosing C_1 and C_2 to give sufficiently small θ, we can make $\psi(\lambda, \lambda^+) < \delta$ for $\lambda \geqslant \lambda_0$. On the remaining segment, $0 \leqslant \lambda \leqslant \lambda_0$, the ratio of pseudo-hyperbolic to Euclidean metric is bounded, and since $\lambda^+ - \lambda$ can obviously be made uniformly small on the interval, we can obtain here also the inequality $\psi(\lambda, \lambda^+) < \delta$.

The lemma follows for the value $\delta = \epsilon$, and we assume henceforth this restriction on C_1 and C_2.

In Figure 1, we have enclosed certain points of A in a closed block bounded by two circles of **C**, one circle of **B**, and one circle centred at the origin. We denote by $\tilde{A}$ the union of all such blocks. In addition we define a set

$$A_1 = [a_2, a_3^+] \cup [a_4, a_5^+] \cup [a_6, a_7^+] \cup \ldots.$$

Similar sets $\tilde{B}$ and B_1 are defined. We easily obtain

COROLLARY 1. *For each $z \in \tilde{A} \cup \tilde{B}$, $\psi(z, x_n) \geqslant \epsilon$, $n = 1, 2, 3, \ldots$.*

COROLLARY 2. *Given $\eta > 0$, circles C_1 and C_2 can be chosen such that to any $z \in \tilde{A}$ $[z \in \tilde{B}]$ corresponds $c \in A_1$ $[c \in B_1]$ with $|z| = c$ and $\psi(z, c) < \eta$.*

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LEMMA 3. *The sequence* (x_n) *is an interpolating sequence.*

Proof. For each n, $\psi(x_n, x_{n+1}) \geqslant \epsilon > 0$. We show in general that any increasing sequence of reals with this property approaches the boundary exponentially:

$$\psi(x_n, x_{n+1}) = \frac{x_{n+1} - x_n}{1 - x_{n+1} x_n} \geqslant \epsilon,$$

$$1 - x_{n+1} \leqslant \frac{(1-\epsilon)(1-x_n)}{1 + x_n \epsilon}.$$

Hence $(1 - x_{n+1})/(1 - x_n) < 1 - \epsilon$ for all n, and the sequence (x_n) is said to approach the boundary exponentially. Such a sequence is necessarily an interpolating sequence (**10**, p. 203). This guarantees that F is a convergent Blaschke product, and hence defines a function in H^∞.

In addition, if we combine Lemma 1 with Corollary 1 of Lemma 2, we find that there must exist a positive constant ρ such that

$$|F(z)| \geqslant \rho \qquad \text{for } z \in \tilde{A} \cup \tilde{B}.$$

On the real axis, this gives

LEMMA 4. $F(x) \geqslant \rho$ *for* $x \in A_1$, $F(x) \leqslant -\rho$ *for* $x \in B_1$.

The next lemma is sufficient to give Proposition 5 since it shows, in combination with Lemma 4, that F separates the sets A and B.

LEMMA 5. *Circles* C_1 *and* C_2 *in* $\mathbf{C}$ *can be so chosen that to every* z *in* $\tilde{A}$ [*in* $\tilde{B}$] *there corresponds* c *in* A_1 [*in* B_1] *such that*

$$|F(z) - F(c)| < \rho/2.$$

Proof. Set $\eta = \rho/16$. We give the proof for $\tilde{A}$. Circles C_1 and C_2 corresponding to this value of η can be selected, according to Lemma 2, Corollary 2, to give for each $z \in A$, a point $c \in A_1$ with

$$\psi(z, c) < \eta \quad \text{and} \quad |z| = c.$$

We integrate along the straight line L from c to z, obtaining

$$F(z) - F(c) = \int_L F'(\xi)\, d\xi.$$

We can apply the following inequality to F', since any Blaschke product has modulus less than one inside D (**5**, p. 18):

$$|F'(\xi)| \leqslant \frac{1}{1 - |\xi|} \leqslant \frac{1}{1 - c}$$

for points ξ on the line segment L.

Because we have

$$\psi(z, c) = \left|\frac{z - c}{1 - cz}\right| < \eta,$$

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it is sufficient to show that

$$(1) \qquad |1 - cz| < 8(1 - c),$$

for then

$$|F(z) - F(c)| < \frac{1}{1-c} \cdot \eta \cdot |1 - cz| < \frac{\rho}{2}.$$

We see (1) from Figure 3. Take the base P of the perpendicular from the point cz to the real axis, and the intersection Q with the real axis of the circle,

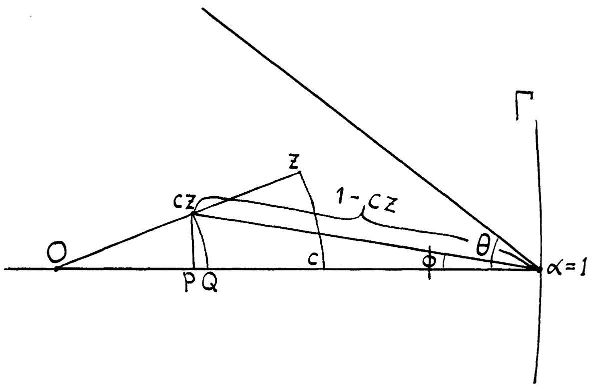

FIGURE 3

centred at 0, which passes through cz. The angle ϕ is less than θ, and as long as $\theta < \pi/3$, we have $\sec \phi < \sec \theta < 2$. $PR < 2QR$ always holds, and in fact as $z \to 1$, $PR/QR \to 1$. Hence

$$|1 - zc| = PR \sec \phi < 2 \cdot \sec \theta < 2(1 - c^2) \cdot 2 < 8(1 - c).$$

We must now extend Proposition 5 to an arbitrary non-tangential approach, in order to complete the proof of Theorem 2.

PROPOSITION 6. *Two different $*$-filters approaching $\alpha = 1$ along the same circle of approach C_0 give different homomorphisms.*

Proof. Two different non-tangential $*$-filters, $\mathbf{U}^*$ and $\mathbf{V}^*$, are taken. Sets $A \in \mathbf{U}^*$ and $B \in \mathbf{V}^*$ are obtained, and traces on C_0 of these sets, which give a sequence

$$b_1 \leqslant b_2 < a_2 \leqslant a_3 < b_3 \ldots$$

of points on C_0 satisfying $\psi(a_n, b_n) > 4\epsilon$, for $n = 2, 3, \ldots$, for some ϵ. Recall that the sets A and B can be cut down to lie between arbitrary members of $\mathbf{C}$ above and below C_0.

Consider the conformal map L:

$$(2) \qquad z' = L(z) = \frac{-z + ci}{ciz - 1}, \qquad -1 < c < 1,$$

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where ci is the intercept of C_0 on the imaginary axis. L leaves -1 and $+1$ invariant, and maps C_0 to the line segment between these points. Furthermore, individual points move along a circle of the family **B**, and this means that the sets A and B map into sets A' and B' similar to the sets which are separated in Proposition 5. We have a sequence of points

$$b_1' \leqslant b_2' < a_2' \leqslant a_3' < b_3' \ldots,$$

the images of the points in the previously mentioned sequence. We are able to use the argument of Proposition 5, defining a sequence (x_n) of points and a function F that separates A' and B'. The argument requires that A' and B' be restricted to the region between two circles of **C** above and below the real axis; but this means only an appropriate selection of C_1 and C_2 around C_0, because every circle through -1 and $+1$ is carried into another with the same property.

To make this method of proof valid, two points must be verified. The function F cannot properly be defined unless the points of the sequence satisfy

$$\psi(a_n', b_n') \geqslant \epsilon', \qquad n = 2, 3, 4, \ldots,$$

for some positive number ϵ'. This is shown in Lemma 6, and gives a Blaschke product F which separates the sets A' and B'. However, we are using the composite function $F \circ L$ to separate A and B, and we show in Lemma 7 that F is bounded and analytic on the set $L(D)$, which is the interior of a finite circle passing through -1 and $+1$.

LEMMA 6. *Let C_0 be the portion within the open unit disk of some circle passing through $z = -1$ and $z = 1$. If L is the bilinear function leaving these two points fixed, and mapping C_0 to the real axis, then*

$$\psi(L(z_1), L(z_2)) > k\psi(z_1, z_2) \textit{ for arbitrary } z_1, z_2 \in C_0,$$

where k is a constant depending on C_0.

Proof. We have only to substitute in the formulae. If ci is the intersection of C_0 with the imaginary axis, we have (2). For $z_1, z_2 \in C_0$, the numbers $w_1 = L(z_1)$ and $w_2 = L(z_2)$ are real and

$$z_n = \frac{w_n + ci}{ciw_n + 1}, \qquad n = 1, 2;$$

$$\begin{aligned}\psi(z_1, z_2) &= \frac{1+c^2}{1-c^2}\cdot\left|\frac{w_2 - w_1}{1 - w_1 w_2 + [2ci/(1-c^2)](w_1 - w_2)}\right| \\ &< \frac{1+c^2}{1-c^2}\cdot\left|\frac{w_2 - w_1}{1 - w_1 w_2}\right| = \frac{1}{k}\,\psi(w_1, w_2)\end{aligned}$$

where $k = (1 - c^2)/(1 + c^2)$.

In particular, for the given sequences, we have on C_0 that

$$\psi(a_n, b_n) > 4\epsilon \qquad \text{when } n = 2, 3, 4, \ldots.$$

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Setting $\epsilon' = 4\epsilon k$, we obtain

$$\psi(a_n', b_n') > \epsilon' \qquad \text{when } n = 2, 3, 4, \ldots .$$

LEMMA 7. *Given a Blaschke product*

$$F(z) = \prod_{n=1}^{\infty} \frac{x_n - z}{1 - x_n z}$$

whose zeros form an increasing sequence of reals which is an interpolating sequence, then F is bounded and analytic in the interior of any circle passing through -1 and $+1$.

Proof. The function F is defined and analytic everywhere in the whole plane except for poles at points $1/x_n$, $n = 1, 2, 3, \ldots$, and for an essential singularity at the limit point of these poles, namely at $\alpha = 1$. This follows from the Schwarz reflection principle, since F is analytic with $|F| = 1$ everywhere on the unit circle except at $\alpha = 1$.

Given a circle C through -1 and $+1$, the function is clearly bounded inside C except possibly in the neighbourhood of $\alpha = 1$. We draw some neighbourhood N of this point. The boundedness of F is in doubt only in that part of the interior of C which lies outside $D \cup \Gamma$, which we call R. Let R' be the reflection of this region, and C' the reflection of C in the circle Γ. The values of $|F|$ inside R are just the values of $|1/F|$ in R', and hence we must prove that F is bounded away from zero on the set R'. However, this is an immediate consequence of Lemma 1. The pseudo-hyperbolic distances from points in R' to points of the sequence (x_n) are certainly bounded away from zero. If we draw the hypercycle H through -1 and $+1$ which is tangent to C' and $\alpha = 1$, then the above distances are at least as great as the constant pseudo-hyperbolic distance between H and the real axis.

5. Oricycle homomorphisms. A second subset of $\mathbf{M}_\alpha$ is here considered, consisting of those homomorphisms whose filters approach the point α in a fashion similar to the oricycles at α. A circle inside the unit disk is called an oricycle if it touches the unit circle at one point α. The family of oricycles through a given α, considered in the non-Euclidean geometry, is a family of curves equidistant from each other, like $\mathbf{C}$ in the previous section.

A filter on D converging to some α with $|\alpha| = 1$ is called an *oricycle filter* if some member of it lies between two oricycles at α. A homomorphism $\phi \in \mathbf{M}_\alpha$, $|\alpha| = 1$, is called an *oricycle homomorphism* if there exists a maximal closed filter $\mathbf{U}$ on D, which is an oricycle filter, such that $\phi = \phi_{\mathbf{U}}$. Such homomorphisms form a subset of $\mathbf{M}_\alpha$ which we call $\mathbf{O}_\alpha$. Let $\mathbf{O} = \bigcup_{\alpha \in \Gamma} \mathbf{O}_\alpha$. We prove for $\mathbf{O}$ the analogue of Theorem 2. The arguments are very similar, with oricycles replacing hypercycles.

THEOREM 3. *Any oricycle homomorphism of H^∞ is determined by a unique $*$-filter.*

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The proof follows from Propositions 7 and 8 of this section. We restrict our attention to the fibre at $\alpha = 1$. Any maximal closed oricycle filter must have either a set in the upper half of D or a set in the lower half of D; this follows from the maximality of the filter. We say that $\mathbf{U}$ approaches the point $\alpha = 1$ from above or from below respectively. In either case, we can argue with the family $\mathbf{E}$ of oricycles at $\alpha = 1$ exactly as we did with the family $\mathbf{C}$. A particular circle C_0 of $\mathbf{E}$ must exist, such that for arbitrary C_1 and C_2 of $\mathbf{E}$ inside and outside C_0, some set of $\mathbf{U}$ will lie between C_1 and C_2. Because the circles of $\mathbf{E}$ are separated by constant hyperbolic distance, we can argue, as before, that the filter $\mathbf{U}^*$ must also have this property. If we assume that $\mathbf{U}$ approaches $\alpha = 1$ from above, this set can be assumed to lie in the upper half of D. The filter is said to approach $\alpha = 1$ along C_0.

Proposition 7. *If* $\mathbf{U}$ *is a maximum closed filter approaching* $\alpha = 1$ *from above along any oricycle* C_0, *then any maximal closed filter* $\mathbf{V}$ *with* $\phi_{\mathbf{V}} = \phi_{\mathbf{U}}$ *also has this property. The same holds if* $\mathbf{U}$ *approaches* $\alpha = 1$ *from below.*

Proof. The function g in H^∞ defined by $g(z) = \exp((z+1)/(z-1))$ is constant in modulus on the circles of $\mathbf{E}$. Following the argument of Proposition 4, we use this function to separate $\mathbf{U}^*$ from any $*$-filter approaching $\alpha = 1$ along a different oricycle, or from any non-oricycle $*$-filter. Also the function h of Proposition 4 can be used to separate $\mathbf{U}^*$ from any $*$-filter approaching $\alpha = 1$ along the same oricycle, but from below. As before, we have:

Corollary. $\mathbf{O}_\alpha$ *is an open subset of* $\mathbf{M}_\alpha$.

The remaining problem is to separate different $*$-filters $\mathbf{U}^*$ and $\mathbf{V}^*$ approaching $\alpha = 1$ from above along the same oricycle C_0. We can proceed as in the previous section, this time using the family $\mathbf{E}$ of oricycles through $\alpha = 1$ and the family $\mathbf{F}$ of circles through $\alpha = 1$ orthogonal to the oricycles. These are related to the hyperbolic metric just as were $\mathbf{C}$ and $\mathbf{B}$, and we construct similar sets A and B, lying between two oricycles about C_0, which must be separated by an H^∞ function.

To find a separating function, we map D onto the right half of the w plane with $z = (w-1)/(w+1)$. By the argument of Proposition 4, it is sufficient to construct a bounded harmonic function on $\operatorname{Re}(w) > 0$, which is positive and bounded away from zero on the image of A, and is negative on the image of B. Because circles from $\mathbf{E}$ and $\mathbf{F}$ map into vertical and horizontal lines respectively, the sets which must be separated are as follows, where

$$b_1 \leqslant b_2 < a_2 \leqslant a_3 < b_3 \ldots$$

and where u_0 is positive, and $\eta > 0$ is arbitrarily small:

$$A' = \{u + iv \mid |u - u_0| < \eta, v \in [a_2, a_3] \cup [a_4, a_5] \cup [a_6, a_7] \cup \ldots\},$$
$$B' = \{u + iv \mid |u - u_0| < \eta, v \in [b_1, b_2] \cup [b_3, b_4] \cup \ldots\}.$$

On any one oricycle, equal hyperbolic distances become equal Euclidean

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distances on the vertical line which is its image. This means that for all $n \geqslant 2$, we have $|a_n - b_n|$ greater than some positive constant. If we let x_n bisect the interval between a_n and b_n, we obtain

$$|x_n - a_n| \geqslant Ku_0 |x_n - b_n| \geqslant Ku_0, \qquad n = 2, 3, 4, \ldots,$$

for suitable K. Define $x_1 = b_1 - Ku_0$.

We use the Poisson formula

$$h(u + iv) = \frac{1}{\pi} \int_{-\infty}^{\infty} H(t) \frac{u}{u^2 + (t - v)^2} \, dt$$

to obtain the separating function, and to establish

PROPOSITION 8. *If* $\mathbf{U}$ *and* $\mathbf{V}$ *are maximal closed filters approaching* $\alpha = 1$ *from above along the same oricycle, and if* $\mathbf{U}^* \neq \mathbf{V}^*$, *then* $\phi_{\mathbf{U}} \neq \phi_{\mathbf{V}}$.

Proof. We first assume that the sequence $(|x_{n+1} - x_n|)$, $n = 1, 2, 3, \ldots$, is bounded. Then the function H defined on the V-axis by

$$\begin{aligned} H(t) &= (-1)^n && \text{for } x_n < t < x_n + Ku_0,\ n \text{ even},\\ &= (-1)^n && \text{for } x_n - Ku_0 < t < x_n,\ n \text{ odd},\\ &= 0 && \text{otherwise} \end{aligned}$$

will give a harmonic h with the required properties.

To prove this statement, take a point $u_0 + iv \in A'$, and let P be the point on the V-axis at the same height v. Then P is above $i(x_n + Ku_0)$ and below $i(x_{n+1} - Ku_0)$ for some even value of n. Suppose the points ix_{n+1} and ix_{n+2} are at a distance Lu_0 and $L_1 u_0$ from P respectively. We consider the contribution to h due to the values of H around these two points. If

$$T(x) = (1/\pi) \arctan(x),$$

this is

$$\begin{aligned} &[2T(Lu_0/u) - T((L - K)u_0/u) - T((L + K)u_0/u)]\\ &\quad + [2T(L_1 u_0/u) - T((L_1 - K)u_0/u) - T((L_1 + K)u_0/u)]\\ &\qquad = 2r - 2r_1. \end{aligned}$$

The quantity $2r - 2r_1$ is positive since $L_1 > L$. This follows from the behaviour of the second derivative $T''(x)$, which is negative, and which increases with x. Note also that $2r - 2r_1$ is a continuous function of u for $u > 0$, when the other quantities are left fixed.

All the points of the sequence (ix_n) can be paired just as these two have been, including the even number lying below P. The contribution due to each pair is positive and so $h(u + iv) > 0$. (In the case of B', a similar pairing into negative terms is obtained; there is one term left over at ix_1, but the contribution from this single term is also negative.)

Our hypothesis says that the distance between P and ix_{n+1} is bounded; there exists an absolute constant M such that $L \leqslant M$. We also have

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$L_1 - L \geqslant 2K$, and these two conditions give a minimum positive value for $2r - 2r_1$ for any fixed value of u. For $u = u_0$, let ρ be any positive number that is smaller than this minimum value. This gives us, for any point in A' of the form $u_0 + iv$, the inequality

$$h(u_0 + iv) > 2r - 2r_1 > \rho > 0.$$

Because $2r - 2r_1$ is a continuous function of u at u_0, this yields a positive η such that $h(u + iv) > \rho$ whenever $|u - u_0| < \eta$ and v is as above. Since A' can be restricted to lie between the lines $u = u_0 - \eta$ and $u = u_0 + \eta$, this proves that h is bounded away from zero on A'. It is likewise negative and bounded away from zero on B'.

If we discard the restriction that there is an upper bound for the numbers $|x_{n+1} - x_n|$, $n = 1, 2, 3, \ldots$, the function H which we have defined must be supplemented by inserting values at places where the distances between ix_{n+1} and ix_n become too large. This must be done without disturbing the inequalities already proved.

We continue in the previous setting. Take P between ix_n and ix_{n+1}. For arbitrary M, we can deal with all points P which are below ix_{n+1} by an amount at most Mu_0. Similarly, the previous arguments will give the required conclusion for any points P at most Mu_0 above ix_n. Choose M such that $T(M/2) > 3/8$. It is possible to find a neighbourhood of u_0 with $T(Mu_0/2u) > 3/8$ by the continuity of T at $u = u_0$. Corresponding to this value of M, there will exist $\rho > 0$, as in the first part of the proof. For any pair ix_n and ix_{n+1} separated by more than $2Mu_0$, we introduce an additional amount $(-1)^n\rho$ into the function H between $i(x_n + Mu_0/2)$ and $i(x_{n+1} - Mu_0/2)$.

For P in the interval between ix_n and ix_{n+1} for n even, the total additional contribution of negative values must come from points outside the interval, and is not more in absolute value than

$$\frac{2}{\pi}\int_{Mu_0/2}^{\infty} \frac{\rho u}{(u^2 + s^2)}\, ds = 2\rho[\tfrac{1}{2} - T(Mu_0/2)] < \rho/4.$$

If P lies within a distance Mu_0 of ix_n or ix_{n+1}, this means h must still be more than $3\rho/4$. For the remaining points P between ix_n and ix_{n+1}, if any exist, the additional values extend at least a distance of $Mu_0/2$ in either direction. This gives at least

$$\frac{2}{\pi}\int_{0}^{Mu_0/2} \frac{\rho u}{(u^2 + s^2)}\, ds = 2\rho T(Mu_0/2) > \frac{3\rho}{4}.$$

Thus the values on this part for h are at least $3\rho/4 - \rho/4 = \rho/2$, and in all cases h is at least $\rho/2$ on the set A'.

6. Analytic mappings into the fibres. In the paper of I. J. Schark **(13)** the existence of a non-trivial analytic mapping from the unit disk D into

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$\mathbf{M} - D$ is shown. By analytic mapping, we mean a continuous function $\psi : D \rightarrow M$, such that $\hat{f} \circ \psi$ is analytic for each $f \in H^\infty$ (where $\hat{f}$ is the Gelfand extension of f to $\mathbf{M}$). In Theorem 4 we generalize Schark's example, and give an explicit construction for the mapping ψ.

Fix any α on the unit circle Γ; and for $a \in D$, let S_a be the conformal map of D onto itself which keeps α fixed and sends the origin to a. The function

$$F_z : a \rightarrow S_a(z) \tag{1}$$

is a homeomorphism of D, by inspection of the explicit formula. It need not be analytic, but F_z has the property that each point is mapped an equal hyperbolic distance $h(z, 0)$. Thus

$$h(F_z(a), a) = h(S_a(z), S_a(0)) = h(z, 0),$$

because S_a is a non-Euclidean rigid motion on D; and in fact

$$h(F_z(a), F_w(a)) = h(z, w)$$

where a, z, w are arbitrary points in D.

For any $A \subset D$, and for any filter $\mathbf{A}$ of subsets of D, we define

$$A_z = F_z(A), \qquad \mathbf{A}_z = \{A_z : A \in \mathbf{A}\}.$$

If $\mathbf{A}$ is a maximal closed filter on D, then for each $z \in D$, $\mathbf{A}_z$ will also be maximal closed, and will determine a homomorphism of H^∞

$$\phi_z : f \rightarrow \lim f(\mathbf{A}_z), \qquad f \in H^\infty.$$

THEOREM 4. *Take a homomorphism ϕ_0 in the fibre $\mathbf{M}_\alpha$ for $|\alpha| = 1$, and a maximal closed filter $\mathbf{A}$ which determines ϕ_0. The mapping ψ, defined by $\psi(z) = \phi_z$, is an analytic mapping of D into $\mathbf{M}_\alpha$ which carries the origin to ϕ_0.*

Proof. We first show that ψ is continuous at $z \in D$. Let

$$\{\phi \in \mathbf{M} : |\hat{f}(\phi_z) - \hat{f}(\phi)| < \epsilon\}$$

be an arbitrary sub-basic neighbourhood of ϕ_z, where f in H^∞ can satisfy $||f|| \leqslant 1$. Since $\hat{f}(\phi_z) = \lim f(\mathbf{A}_z)$, there exists $A \in \mathbf{A}$ with

$$|\hat{f}(\phi_z) - f(b)| < \epsilon/4 \qquad \text{for all } b \in A_z. \tag{2}$$

Take the hyperbolic neighbourhood N of z

$$N = \{w \in D : h(z, w) < \epsilon/4\}.$$

For any $w \in N$ and $c \in A_w$, there must exist $b \in A_z$ with $h(b, c) < \epsilon/4$. By Pick's Theorem

$$|f(b) - f(c)| \leqslant h(f(b), f(c)) \leqslant h(b, c) < \epsilon/4.$$

Thus we have $|\hat{f}(\phi_z) - f(c)| < \epsilon/2$ for all $c \in A_w$, which yields

$$|\hat{f}(\phi_z) - \hat{f}(\phi_w)| < \epsilon \qquad \text{for all } w \in N. \tag{3}$$

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Because N is also a neighbourhood of z in the ordinary topology, (3) shows that ψ is continuous.

To show that $\hat{f} \circ \psi$ is analytic, we approximate it uniformly on an arbitrary compact subset K of D. By the first part of the proof (again assuming $||f|| \leqslant 1$), each $z \in K$ has a neighbourhood N such that $w \in N$ implies that $|\hat{f}(\phi_w) - f(c)| < \epsilon$ for all $c \in A_w$, where $A \in \mathbf{A}$. Taking a finite cover $N_1, N_2, \ldots, N_k$ of K, we obtain sets $A_1, A_2, \ldots, A_k$ in the filter $\mathbf{A}$, and defining $A = A_1 \cap \ldots \cap A_k$, we have for all $w \in K$ that

$$|\hat{f}(\phi_w) - f(c)| < \epsilon \qquad \text{for all } c \in A_w. \tag{4}$$

We now let a be an arbitrary point in A, and restate (4) in the form

$$|\hat{f}(\psi(w)) - f(S_a(w))| < \epsilon \qquad \text{for all } w \in K, \tag{5}$$

which approximates $\hat{f} \circ \psi$ by the analytic function $f \circ S_a$.

In the setting of this theorem, the example of I. J. Schark is the following. Take any limiting rotation L on the unit disk with fixed point α. This maps points of D along oricycles touching Γ at the point α. Define the sequence

$$A_z = \{z, L(z), L^2(z), L^4(z), \ldots, L^{2n}(z), \ldots\}, \tag{6}$$

and, with a single ultrafilter on the integers, induce a corresponding ultrafilter $\mathbf{A}_z$ on the sequence A_z for each z. In this special case, there exists $h \in H^\infty$ with the property that $\hat{h} \circ \psi$ is the identity on D. Using this function, Schark deduces that ψ is a homeomorphism, and that its extension to $\mathbf{M}$ is an embedding of $\mathbf{M}$ in $\mathbf{M}_\alpha$.

It is possible to repeat Schark's entire argument using a non-Euclidean translation L, which translates points along circles from $-\alpha$ towards α. However, one must take alternate members of the set A_z defined in (6), because it is not true this time that both $L^m(z)$ and $L^{-m}(z)$ tend to α as m tends to infinity. This case gives analytic disks in the set of non-tangential homomorphisms, just as Schark's analytic disks are in the set of oricycle homomorphisms.

The mapping ψ of Theorem 4 is non-trivial whenever ϕ_0 is a non-tangential (oricycle) homomorphism, because the filters $\mathbf{A}_z$ which approach α along different hypercycles (oricycles) must give rise to different homomorphisms. Similarly ψ is non-trivial if ϕ_0 is in the closure of an interpolating sequence A. If b is the Blaschke product on A, then $\hat{b}(\psi(0)) = \hat{b}(\phi_0) = 0$; but $\hat{b}(\psi(z)) \neq 0$ for non-zero z close to the origin (cf. Lemma 1).

Professor Kenneth Hoffman has shown me some results on H^∞ which treat mappings similar to ψ. These settle many questions about ψ and shed considerable light on the problems raised in earlier sections.

Define an equivalence relation between any two homomorphisms ϕ_1 and ϕ_2 of $\mathbf{M}$:

$\phi_1 \sim \phi_2$ if and only if $||\phi_1 - \phi_2|| < 2$, where the norm $||\phi_1 - \phi_2||$ is taken in the dual space of H^∞; the equivalence classes are called the Gleason parts

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of **M**. Hoffman **(11)** has shown that the Gleason part $P(\phi)$, for $\phi \in \mathbf{M}$, either is a single point or there exists a continuous one–one map π of D onto $P(\phi)$, with $\pi(0) = \phi$, such that $\hat{f} \circ \pi$ is analytic for each $f \in H^\infty$. In other words, every non-trivial part is an analytic disk.

Hoffman has shown me that this recent unpublished work proves the following: The mapping ψ is a homeomorphism onto the analytic disk $P(\phi)$ whenever ϕ is in the closure of an interpolating sequence, and is a constant function otherwise.

Because ψ is non-trivial in the non-tangential and oricycle cases, as we remarked above, this means that every non-tangential and every oricycle homomorphism lies in the closure of an interpolating sequence. Thus Theorem 1 implies Theorems 2 and 3, if we assume this powerful result.

Moreover, the converse of Theorem 1 holds. A homomorphism of H^∞ is determined by a unique $*$-filter if *and only if* it lies in the closure of an interpolating sequence. To prove this, suppose $\phi_0 \in \mathbf{M}$ is determined by a maximal closed filter $\mathbf{A}$ on D. Take an arbitrary neighbourhood N of ϕ_0, which can be written

$$N = \{\phi \in \mathbf{M} : |\hat{f}_i(\phi)| < \epsilon, i = 1, 2, \ldots, n\},$$

where $f_i \in H^\infty$ and $\hat{f}_i(\phi_0) = 0$ for $i = 1, 2, \ldots, n$. If ϕ_0 is not in the closure of an interpolating sequence, the mapping ψ defined in Theorem 4 will map all of D to the single point ϕ_0. We take the compact set

$$K = \{z : |z| \leqslant r < 1\}$$

and can choose $A \in \mathbf{A}$, which gives an approximation (5) simultaneously for the n functions. Thus

$$|f_i(S_a(w)) - \hat{f}_i(\psi(w))| < \epsilon \qquad \text{for } w \in K \text{ and } a \in A.$$

Hence $|f_i| < \epsilon$ on the set $S_a(K)$; in other words, every neighbourhood N of ϕ_0 contains an entire disk whose hyperbolic radius can be made arbitrarily large. This will allow the construction of different $*$-filters which determine the same homomorphism ϕ_0.

It is a conjecture, so far unsettled, that every homomorphism of H^∞ which is not in the closure of an interpolating sequence has a one-point part, which would mean that $\psi(D) = P(\phi)$ in all cases.

7. Silov boundary filters. The Silov boundary of H^∞ was identified in **(13)**. By the classical Fatou Theorem, each $f \in H^\infty$ has a radial limit

$$f(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})$$

for almost all θ in $0 \leqslant \theta < 2\pi$. This defines a function in L^∞ of the unit circle, and the injection of H^∞ into L^∞ induces a homeomorphism τ of the maximal ideal space X of L^∞ onto the Silov boundary of H^∞. X is a Stone space, and the Gelfand representation $f \to \hat{f}$ is an isometric isomorphism of L^∞ onto $C(X)$.

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The open–closed sets in X which generate the topology are of the form

$$(1) \qquad X_E = \{x \in X : \chi_E(x) = 1\},$$

where E is any measurable subset of Γ, and χ_E is its characteristic function.

Because the lattice of all open–closed subsets of X is isomorphic to the lattice L of measurable sets in Γ-modulo sets of zero measure, we can set up a 1–1 correspondence between points x of X and maximal filters $\mathbf{F}_x$ on L. The mapping sends x to its neighbourhood filter of basic open–closed sets. We note that every set E in a filter $\mathbf{F}_x$ has positive measure, and that if E_1 differs from E only on a set of zero measure, then E_1 also belongs to $\mathbf{F}_x$.

LEMMA 8. *For each* $y \in X$,

$$\hat{g}(y) = \lim g(\mathbf{F}_y) \qquad \textit{for } g \in L^\infty.$$

Proof. We show first that the essential values of any g in L^∞ on a measurable subset E of Γ are precisely the values of $\hat{g}$ on the basic open–closed set

$$X_E = \{x \in X : \hat{\chi}_E(x) = 1\} = \{x \in X : \hat{\chi}_{\Gamma - E}(x) = 0\}.$$

It suffices to prove this for the value zero. If k is the essential supremum of g, the function

$$h = 2k\chi_{\Gamma - E} + g$$

agrees with g on the set E, and h has an essential zero on Γ if and only if g has an essential zero on E. This can happen if and only if h is invertible in L^∞, which is equivalent to saying that $\hat{h}$ has a zero on X. This zero must be on X_E, and must be a zero of $\hat{g}$ also.

Take g and y in the statement of the lemma; we can assume $\hat{g}(y) = 0$. For $\epsilon > 0$, we choose an open–closed neighbourhood U of y with $|\hat{g}| < \epsilon$ on U:

$$U = \{x \in X : \chi_E(x) = 1\} = \{x \in X : E \in \mathbf{F}_x\}$$

for some measurable set $E \subset \Gamma$. Hence for each $x \in E$, where $E \in \mathbf{F}_x$, we have $|\hat{g}(x)| < \epsilon$, and the essential values of g on E are all of modulus less than ϵ. Removing a set of measure zero from E, we obtain $|g(\alpha)| \leqslant \epsilon$ for all $\alpha \in E_1$, where $E_1 \in \mathbf{F}_x$, which proves that $\lim g(\mathbf{F}_x) = 0$.

Using the co-ordinate function z, the space X can be partitioned into fibres

$$(2) \qquad X_\beta = \{x \in X : \hat{z}(x) = \beta\}, \qquad |\beta| = 1.$$

If we take any $x \in X$, the filter $\mathbf{F}_x$ on Γ must converge to some point β on Γ by its maximality, and the above lemma tells us that $\hat{z}(x) = \lim z(\mathbf{F}_x) = \beta$. Hence $x \in X_\beta$ if and only if $\mathbf{F}_x$ converges to β.

We agree to identify X and its homeomorphic image $\tau(X)$, considering X itself as the Silov boundary of H^∞. Noting that the fibres $\mathbf{M}_\beta$ of $\mathbf{M}$ are defined in the same manner as X_β in (2), we see that

$$X_\beta = \mathbf{M}_\beta \cap X, \qquad |\beta| = 1.$$

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We now define a filter $\mathbf{G}_x$ on the open disk which determines any given homomorphism x in X. Choose a set E in $\mathbf{F}_x$, and for each $\alpha \in E$ choose a completely arbitrary radial line segment $\{r\alpha : \delta_\alpha < r < 1\}$, where δ_α is strictly between 0 and 1. Take the union of these segments and let $\tilde{E}$ be its closure in D. We now generate a closed filter $\mathbf{G}_x$ by taking all possible $\tilde{E}$ for a given E, and then letting E range over $\mathbf{F}_x$.

THEOREM 5. *For each function f in H^∞, and each point x in the Silov boundary X,*

$$\hat{f}(x) = \lim f(G_x).$$

Proof. We can assume that $\hat{f}(x) = 0$. Let $\epsilon > 0$, and choose a set E in $\mathbf{F}_x$ such that $f(\alpha)$ is defined and satisfies $|f(\alpha)| < \epsilon$ for $\alpha \in E$. By Fatou's Theorem, there must be a segment $\{r\alpha : \delta_\alpha < r < 1\}$ on which $|f| < \epsilon$. Hence there is a set E in $\mathbf{G}_x$ on which $|f| \leqslant \epsilon$.

It is clear that $\mathbf{G}_x$, considered as a filter on the closed disk, must converge to β if $x \in X_\beta$. We next show that many $*$-filters lie above each $\mathbf{G}_x$.

THEOREM 6. *The set of distinct $*$-filters which determine any Silov boundary homomorphism has at least the cardinality of βN, the Čech compactification of the integers.*

Proof. Let $x \in X_\beta$, and pick a sequence $O_1, O_2, \ldots$ of oricycles touching Γ at the point β. Suppose O_{n+1} is larger than O_n, and the fixed hyperbolic distance between them is independent of n, for $n = 1, 2, 3, \ldots$. Take any ultrafilter $\mathbf{U}$ on the set N of positive integers, and the filter $\mathbf{G}_x$ which determines x. For $U \in \mathbf{U}$ and G a basic closed set in $\mathbf{G}_x$, we use the closed sets $\theta(U) \cap G$ to generate a filter on D, where θ is the natural correspondence between integer n and oricycle O_n. If $\mathbf{V}$ is any maximal closed filter above this filter, then $\mathbf{V}^*$ will be a $*$-filter which determines x. Any different ultrafilter on N must give a different $*$-filter, because the points on different oricycles of the sequence are bounded away from each other in the hyperbolic metric.

Hence there is a distinct $*$-filter for each ultrafilter on the integers, which gives the stated cardinality. This cardinal number is actually

$$2^{2^{\aleph_0}}$$

as is shown in **(1)**.

A second example of a homomorphism $\phi \in \mathbf{M}$ not having a unique $*$-filter is now constructed. This is not in X, but is in the Silov boundary of the homeomorphic copy $\psi(\mathbf{M})$ of $\mathbf{M}$ of the type embedded in $\mathbf{M}_\alpha$ by I. J. Schark in **(13)**. As mentioned after the proof of Theorem 4, this mapping ψ sends points of D along oricycles into the fibre $\mathbf{M}_\alpha$, and using ψ to co-ordinatize the embedded disk, we obtain a sequence $\psi(O_n)$, $n = 1, 2, \ldots$, of oricycles in $\psi(D)$. Let ϕ be any Silov boundary homomorphism of $\psi(\mathbf{M})$ which is in the fibre at the point where the oricycles $\psi(O_n)$ meet the boundary of $\psi(D)$. By the method of

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Theorem 6, we can construct different filters $\mathbf{V}$, this time of subsets of the embedded oricycles, which determine ϕ. From each filter $\mathbf{V}$, we can construct a filter on D which also determines ϕ as follows. For each point $v = \psi(z)$ in V, where $V \in \mathbf{V}$, we choose an arbitrary set B in the filter $\mathbf{A}_z$ (discussed in Theorem 4). The union of all such B, where v is allowed to range over V, gives a set C. A new filter $\mathbf{C}$ is generated by taking all sets $V \in \mathbf{V}$, and for each V all possible sets C constructed from it. For each filter $\mathbf{C}$, we have $\lim f(\mathbf{C}) = f(\phi)$, and two sets from any two different filters $\mathbf{C}$ lie on different oricycles of the family O_n, which guarantees that they give rise to different $*$-filters.

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