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ISOMETRIES AND CERTAIN DYNAMICAL SYSTEMS

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It is shown that there exists a metric under which a diffeomorphism f on a Riemannian manifold M becomes an isometry, provided that the dynamical system generated by f is of characteristic $0^{\pm}$ and all its orbits are closed. Furthermore, it is shown that the foliation given by the suspension of f is parallel in this case.

Let f be a diffeomorphism on a Riemannian manifold (M,g), where M is always assumed to be connected, complete, not compact, and differentiable with a Riemannian metric g [3,8]. Then f generates a smooth dynamical system $\Phi : Z \times M \to M$ given by $\Phi(n,x) = f^n(x)$, for each $n \in \mathbb{Z}$ and $x \in M$, where Z is the additive group of integers with the discrete topology [3]. The Riemannian metric g induces a distance function (a metric) d and a norm $\| \ \|$ on M [3,8]. Let TM denote the tangent bundle of M and Tf^n be the differential of f^n [3.8]. Then f is said to be an isometry if it preserves the Riemannian metric g. Equivalently, f is an isometry if $\|Tf^n(v)\| = \|v\|$, for all $v \in TM$ and $n \in \mathbb{Z}$.

The notion of characteristic $0^{\pm}$ was first introduced by Ahmad [1] for continuous flows. Later, Knight [6] made a thorough investigation of such flows. The reader may consult the references cited there for more information about the literature on the notion of characteristic $0^{\pm}$.

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Elaydi and Kaul [4] generalized the above mentioned notion to transformation groups. The definitions and notions here are mostly adopted from the theory of continuous flows [1,2,6] and partly from [4].

The main purpose of this paper is to prove the following two theorems.

THEOREM A. Let Φ be a dynamical system of characteristic $0^{\pm}$ generated by the diffeomorphism f on the Riemannian manifold (M,g), such that every orbit is closed. Then M admits a Riemannian metric with respect to which f is an isometry.

THEOREM B. Let f be a diffeomorphism on a Riemannian manifold (M,g) such that any one of the following is satisfied

(1) Every point in M is periodic.

(2) The dynamical system Φ generated by f is properly discontinuous [3].

(3) The dynamical system Φ generated by f is of characteristic $0^{\pm}$ and every orbit is closed.

Then the suspension of f admits a Riemannian connection [8] such that the foliation is parallel [5].

To prove Theorems A and B we need several results and definitions. Let us start by recalling the following definitions. For a point $x \in M$ we have the following [2]:

(1) The positive orbit of x ; $0^+(x) = \{y \mid f^n(x) = y$ for some $n \in \mathbb{Z}^+\}$.

(2) The set of all α- limit points of x ; $L^+(x) = \{y \mid$ there exists a sequence $\{n_i\}$ in $\mathbb{Z}^+$, with $n_i \to +\infty$, such that $f^{n_i}(x) \to y\}$.

(3) The positive prolongation set of x ; $D^+(x) = \{y \mid$ there exist sequences $\{x_i\}$ in M and $\{n_i\}$ in $\mathbb{Z}^+$ such that $x_i \to x$, $f^{n_i}(x_i) \to y\}$.

(4) The positive prolongation limit set of x ; $J^+(x) = \{y \mid$ there exist sequences $\{x_i\}$ in M and $\{n_i\}$ in $\mathbb{Z}^+$, with $n_i \to +\infty$, such that $x_i \to x$, $f^{n_i}(x_i) \to y\}$.

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Negative versions of the above definitions may be given in the obvious way. The orbit of $x; O(x)$, the limit set of $x; L(x)$, the prolongation limit set of $x; D(x)$ and the prolongation limit set of $x; J(x)$ are defined to be the union of the positive and the negative corresponding sets.

We remark here that the sets $L^+(x)$, $L^-(x)$, $J^+(x)$ and $J^-(x)$ are closed and invariant. The set $D^+(x)$ is closed and positively invariant, while $D^-(x)$ is closed and negatively invariant.

Following [1], a dynamical system Φ is said to be of characteristic $0^{\pm}$ if $D^+(x) = \overline{O^+(x)}$ and $D^-(x) = \overline{O^-(x)}$, for each $x \in M$.

THEOREM 1. *For every point* $x \in M$, x *is periodic iff* $L^+(x) = L^-(x) = O(x)$.

Proof. If x is periodic, then it is clear that $O(x) = L^+(x) = L^-(x)$. Conversely, assume that $O(x) = L^+(x)$. Then $O(x)$ is closed and thus is locally compact. Let $A_y = O(x) - \{y\}$ for each $y \in O(x)$. Since $O(x) = L^+(x)$, $\overline{A_y} = O(x)$. Hence A_y is open and dense in $O(x)$, for each $y \in O(x)$. Furthermore, $\cap\{A_y | y \in O(x)\} = \phi$. This violates the Baire Category theorem, unless $O(x)$ is finite. Therefore, $O(x)$ is finite and x is thus periodic.

THEOREM 2. *If* Φ *is a dynamical system of characteristic* $0^{\pm}$ *on a manifold* M *in which every orbit is closed, then either every point in* M *is periodic or no point in* M *is periodic.*

Proof. Let $A = \{x \in M | x \in L^+(x) \cap L^-(x)\}$. Since $D^+(x) = O^+(x) \cup J^+(x) = \overline{O^+(x)}$ and $D^-(x) = O^-(x) \cup J^-(x) = \overline{O^-(x)}$, it follows that $L^+(x) = J^+(x)$ and $L^-(x) = J^-(x)$. Hence $A = \{x \in M | x \in J^+(x) \cap J^-(x)\}$. Using [4; 1.6], we conclude that A is a closed and invariant set. Now $\overline{O(x)} = O(x)$ is minimal for each $x \in M$ [4; 1.5]. This implies that $O(x) = L^+(x) = L^-(x)$ for each $x \in A$ and consequently, every point in A is periodic (Theorem 1). Therefore,

A consists of all periodic points in M. To show that A is open, let U be a neighborhood of $O(x)$, $x \in A$, such that $\overline{U}$ is compact. We claim that there exists a neighborhood V of x with $O(V) \subset U$. Suppose that no such neighborhood exists. Then there are sequences $\{x_i\}$ in M, $\{n_i\}$ in $\mathbb{Z}$ and $\{V_i\}$ of connected neighborhoods of x, with $V_i \subset U$, such that $x_i \in V_i$, $\cap V_i = \{x\}$, and $f^{n_i}(x_i) \notin U$, for each i. Since V_i is connected, there is $y_i \in V_i$ such that $f^{n_i}(y_i) \in \partial U$. Since ∂U is compact, we may assume that $f^{n_i}(y_i) \to y \in \partial U$. Hence $y \in D(x) = O(x) \subset U$ and we thus have a contradiction. Thus for each $z \in V$, $\overline{O(z)} = O(z) \subset \overline{U}$. This implies that $O(z)$ is compact for each $z \in V$ and consequently, every point in V is periodic. This proves that A is open. Since M is connected, either $A = \phi$ or $A = M$. The proof of the theorem is now complete.

THEOREM 3. *Let f be a diffeomorphism on a Riemannian manifold (M,g). If every point in M is periodic, then M admits a Riemannian metric with respect to which f is an isometry.*

Proof. According to [7] f has a finite period n, that is $f^n(x) = x$ for all $x \in M$. Let g_x denote the bilinear form at x given by g and let g_x^i denote $g_{f^i(x)}$. Now we define a new Riemannian metric $\tilde{g}$ on M by the rule $\tilde{g}_x = \sum_{i=0}^{n-1} \frac{1}{n} g_x^i$. It can be easily seen that $\tilde{g}$ is a Riemannian metric of M. Furthermore, f is an isometry with respect to $\tilde{g}$.

This completes the proof of the theorem.

DEFINITION 4 [3]. The dynamical system Φ is said to be properly discontinuous if the following conditions hold:

(i) If $x,y \in M$ are not in the same orbit, then there are neighborhoods U of x and V of y such that $U \cap O(V) = \phi$.

(ii) Each $x \in M$ has a neighborhood U such that the set $\{n \in Z | f^n(U) \cap U \neq \phi\}$ is finite.

Note that ϕ has no periodic points by (ii).

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THEOREM 5. *If the dynamical system* Φ *is properly discontinuous, then* M *admits a Riemannian metric* $\bar{g}$ *such that* f *is an isometry with respect to* $\bar{g}$.

Proof. If $N = M/Z$ is the quotient space (or the orbit space), then N admits a differentiable structure such that the projection $\pi : M \to N$ is a differentiable covering. Since M is paracompact, so is N . Hence we can find a Riemannian metric h on N . This induces a unique Riemannian metric $\bar{g}$ on M such that the covering π is Riemannian [8]. Thus Z acts on $(M,\bar{g})$ as a group of isometries and f is therefore an isometry of $(M,\bar{g})$.

Now we characterize the notion of proper discontinuity via the prolongation limit sets.

THEOREM 6. *For a dynamical system* Φ *on a manifold* M , *the following are equivalent.*

(1) Φ *is properly discontinuous.*

(2) M/Z *is Hausdorff and* $x \notin J(x)$ *for each* $x \in M$.

(3) $J(x) = \phi$ *for each* $x \in M$.

Proof. (1) $\to$ (2) . Suppose that Φ is properly discontinuous. Then it is well known that this implies that M/Z is Hausdorff [3]. To prove that $x \notin J(x)$ for each $x \in M$, let us assume that for some $y \in M$, $y \in J(y)$. Then there exist sequences $\{y_i\}$ in M , $\{n_i\}$ in Z with $n_i \to +\infty$ such that $y_i \to y$, $f^{n_i}(y_i) \to y$. Let U be a neighborhood of y . Then there exists $n_o \in Z$ such that $y_i \in U$ and $f^{n_i}(y_i) \in U$ for all i , $n_i > n_o$. Hence we have a contradiction.

(2) $\to$ (3). Suppose that condition (2) holds and assume that $J(y) \neq \phi$ for some $y \in M$. Let $z \in J(y)$. Then there are sequences $\{y_i\}$ in M and $\{n_i\}$ in Z with $n_i \to \infty$ such that $y_i \to y$ and $f^{n_i}(y_i) \to z$. Since $\pi(y_i) \to \pi(y)$, $\pi(y_i) = \pi(f^{n_i}(y_i)) \to \pi(z)$ and M/Z is Hausdorff, it follows that $\pi(y) = \pi(z)$. This implies that $z \in O(y)$. Thus $J(y) \subset O(y)$. Since $J(y)$ is invariant, $y \in J(y)$ and thus we have a contradiction.

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(3) → (1). Suppose that $J(x) = \phi$ for each $x \in M$. Then $D(x) = O(x)$ for each $x \in M$. Let $x, y \in M$ be two points not in the same orbit. Assume that there are neighborhood filters $\{U_i\}$ of x, $\{V_i\}$ of y such that $U_i \cap O(V_i) \neq \phi$ for all i. Let $x_i \in U_i \cap O(V_i)$ for each i. Then $x_i \to x$, $f^{n_i}(x_i) \to y$ for some sequence $\{n_i\}$ in Z. It follows that $y \in D(x) = O(x)$ and hence we have a contradiction. This proves part (i) of Definition 7. Assume now that part (ii) of Definition 7 does not hold. Let $\{U_i\}$ be a neighborhood filter of x. Then for each i, the set $\{n^i_j | f^{n^i_j}(U_i) \cap U_i \neq \phi\}$ is infinite. There exists a subsequence $\{n_k\}$ of n^i_j such that either $n_k \to +\infty$ or $n_k \to -\infty$. Let us assume that $n_k \to +\infty$. Let $x_{n_k} \in f^{n_k}(U_{n_k}) \cap U_{n_k}$. Then $x_{n_k} \to x$ and $f^{n_k}(x)_{n_k} \to x$. This implies that $x \in J(x)$ which is a contradiction. This completes the proof of the Theorem.

The following theorem establishes the relationship between the notions of characteristic $0^{\pm}$ and proper discontinuity.

THEOREM 7. *Let* Φ *be a dynamical system of characteristic* $0^{\pm}$ *whose orbits are closed. If* M *has a non-periodic point, then* Φ *is properly discontinuous.*

Proof. Since M has a point which is not periodic, it follows from Theorem 2 that none of the points of M is periodic. Assume that Φ is not properly discontinuous. Then according to Theorem 6, $J(x) \neq \emptyset$ for some $x \in M$. Let $y \in J(x)$. Then either $y \in J^+(x)$ or $y \in J^-(x)$. So assume that $y \in J^+(x)$. Then $y \in L^+(x)$. Since $O(x)$ is closed, $y \in L^+(x) \subset O(x)$. Thus $y = f^m(x)$, for some $m \in \mathbb{Z}$, and consequently $x \in L^+(x)$. Furthermore, $x \in L^+(x) = J^+(x)$, which implies that $x \in J^-(x) = L^-(x)$. This implies that $L^+(x) = L^-(x) = O(x)$. It follows from Theorem 1, that x is periodic and thus we have a contradiction. This completes the proof of the theorem.

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REMARK. The above theorem fails if the orbits of M were not assumed to be closed. The dynamical system Φ defined on the torus $S \times S$, where S is the circle, generated by $f(x,y) = (e^{2\pi i\alpha}x, xy)$, where α is an irrational number between 0 and 1, is of characteristic $0^{\pm}$, not properly discontinuous, and has no periodic points [4; 5.3]. Notice that none of the orbits is closed. It is easy to see that the above mentioned properties are also possessed by the dynamical system generated by an irrational rotation on the circle.

Proof of Theorem A. This follows from Theorems 7 and 3.

Let $X + R \times M$ be the product manifold of M and the real line R. The group of integers Z acts on X by the rule $n(t,x) = (n+t, f^n(x))$.

Proof of Theorem B. This follows from Theorems A,5,3 and [5, Theorem 2].

COROLLARY 8. *If the family of iterates* $\{f^n | n \in \mathbb{Z}\}$ *of* f *is pointwise equicontinuous on* M *and the orbits of* Φ *are closed, then*

(i) M *admits a Riemannian metric under which* f *is an isometry,*

(ii) the suspension of f *admits a Riemannian connection such that the foliation is parallel.*

Proof. According to [4] equicontinuity implies characteristic $0^{\pm}$. The conclusion of the Corollary now follows from Theorems A and B.

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