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On Orbit Closures of Symmetric Subgroups in Flag Varieties

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Abstract. We study K-orbits in G/P where G is a complex connected reductive group, $P \subseteq G$ is a parabolic subgroup, and $K \subseteq G$ is the fixed point subgroup of an involutive automorphism θ. Generalizing work of Springer, we parametrize the (finite) orbit set $K \setminus G/P$ and we determine the isotropy groups. As a consequence, we describe the closed (resp. affine) orbits in terms of θ-stable (resp. θ-split) parabolic subgroups. We also describe the decomposition of any (K, P)-double coset in G into (K, B)-double cosets, where $B \subseteq P$ is a Borel subgroup. Finally, for certain K-orbit closures $X \subseteq G/B$, and for any homogeneous line bundle $\mathcal{L}$ on G/B having nonzero global sections, we show that the restriction map $\mathrm{res}_X\colon H^0(G/B, \mathcal{L}) \to H^0(X, \mathcal{L})$ is surjective and that $H^i(X, \mathcal{L}) = 0$ for $i \geq 1$. Moreover, we describe the K-module $H^0(X, \mathcal{L})$. This gives information on the restriction to K of the simple G-module $H^0(G/B, \mathcal{L})$. Our construction is a geometric analogue of Vogan and Sepanski's approach to extremal K-types.

Introduction

Let G be a connected reductive group over an algebraically closed field k; let $B \subseteq G$ be a Borel subgroup and $K \subseteq G$ a closed subgroup. Assume that K is a *spherical* subgroup of G, that is, the number of K-orbits in the flag variety G/B is finite; equivalently, the set $K \setminus G/B$ of (K, B)-double cosets in G is finite. Then the following problems arise naturally.

1) Parametrize the set $K \setminus G/B$ and, more generally, $K \setminus G/P$ where $P \supseteq B$ is a parabolic subgroup of G.
2) Decompose any (K, P)-double coset into (K, B)-double cosets.
3) For connected K, describe the singularities of closures of double cosets or, equivalently, of K-orbit closures in G/B. Are these closures normal?
4) For such an orbit closure X and a homogeneous line bundle $\mathcal{L}$ on G/B having non-zero global sections, describe the K-module $H^0(X, \mathcal{L})$ and the image of the restriction map $\mathrm{res}_X\colon H^0(G/B, \mathcal{L}) \to H^0(X, \mathcal{L})$. Is res_X surjective?

In the case where $K = B$, the answers to Problems 1 and 2 are well known: by the Bruhat decomposition, each (B, P)-double coset intersects the Weyl group W into a unique coset of W_P, the parabolic subgroup of W associated with P. And for $w \in W$, the double coset BwP is the disjoint union of the $Bw\tau B$ where $\tau \in W_P$. Much is known concerning Problems 3 and 4: the B-orbit closures in G/B are the Schubert varieties; they are normal,

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with rational singularities [12]. The spaces $H^0(X, \mathcal{L})$ are the Demazure modules; their character is given by the Demazure character formula, and the maps res_X are surjective. Moreover, the higher cohomology groups $H^i(X, \mathcal{L})$ vanish for $i \geq 1$. Similar results hold for the diagonal G-action on $G/B \times G/B$ [11].

For general spherical subgroups, no explicit solution of Problem 1 seems to be known; but work of Springer [16] and Richardson-Springer [13], [14] gives detailed information on $K \setminus G/B$ in the case of a *symmetric* subgroup K, that is, K consists of all fixed points of an involutive automorphism θ of G. An example is the diagonal action of G on $G/B \times G/B$, since the diagonal is the fixed point subgroup of the involution of $G \times G$ exchanging both factors. But for arbitrary symmetric subgroup K of G, the K-orbit closures in G/B need not be normal (an example is given in [1, p. 281]), and the maps res_X need not be surjective. This is mentioned in [1]; see 4.3 below for more detailed examples. On the other hand, positive answers to Questions 3 and 4 are obtained in [1] for some singular orbit closures.

In the present paper, we give a solution of Problem 2 for a symmetric subgroup $K = G^\theta$ (1.4), and we describe the isotropy subgroups of G^θ-orbits in G/P (2.2). As a consequence, we characterize the affine (resp. closed) orbits (2.3, 3.2), in relation to θ-split (resp. θ-stable) parabolic subgroups. Then we solve Problem 4 for certain G^θ-orbit closures $X \subseteq G/B$ which we call *induced flag varieties*. They are the pull-backs under the projection $G/B \to G/P$ of closed G^θ-orbits in G/P, where $B \subseteq P$ and both are θ-stable. Of course, each such X is smooth; we show that res_X is surjective, and that the G^θ-module $H^0(X, \mathcal{L})$ is obtained from $H^0(P/B, \mathcal{L})$ by parabolic induction. Furthermore, we obtain vanishing of $H^i(X, \mathcal{L})$ for $i \geq 1$ (4.1). As a consequence, X is projectively normal in the embedding given by any ample line bundle on G/B.

Our proof of these results concerning Problem 4 is only valid in characteristic zero. In positive characteristics, it would be useful to know that the G^θ-module $H^0(G/B, \mathcal{L})$ admits a good filtration (this was conjectured by Brundan [6, Conjecture 4.4 (ii)]). Our analysis of restriction maps gives information on the restriction to G^θ of the simple G-module $H^0(G/B, \mathcal{L})$: all isotypical components which are extremal in a precise sense arise from the quotient $H^0(X, \mathcal{L})$ for some induced flag variety X (4.2).

This is related to work of Sepanski [15] on boundaries of K-types of a $(\mathfrak{g}, K)$-module M. He considered the cohomology of $\mathfrak{u}$ with coefficients in M, where $\mathfrak{u}$ is the nilradical of the Lie algebra of a θ-stable parabolic subgroup P of G, and he studied a "restriction of cohomology" map $\tau\colon H^*(\mathfrak{u}, M) \to H^*(\mathfrak{u}^\theta, M)$ [15, Section 3]. Let X be the pull-back in G/B of the closed orbit $G^\theta/P^\theta \subseteq G/P$; then the map res_X can be seen as a geometric version of τ.

The simplest situation for restricting G-modules to G^θ is the "multiplicity-free" case, considered in detail in [15, Section 4]. In this case, it turns out that all G^θ-orbit closures in G/B are induced flag varieties; in particular, all orbit closures are smooth (4.2). In the general case, most orbit closures are not induced flag varieties, but the latter can be used to construct "short" desingularizations of the former; this will be developed elsewhere.

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Notation

Throughout the paper, the ground field k is algebraically closed of characteristic $\neq 2$. We denote by G a connected reductive group, by B a Borel subgroup of G, and by T a maximal torus of B. The unipotent part of B is denoted by U. We denote by P a parabolic subgroup of G containing B, and by L the Levi subgroup of P which contains T.

Let N be the normalizer of T in G, and let $W = N/T$ be the Weyl group. Let Φ (resp. Φ^+; Φ^-) be the set of roots of (G, T) (resp. of positive roots, that is, roots of (B, T); of negative roots). The set of simple roots is denoted by Δ.

Let $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{t}$, ... be the Lie algebras of G, B, T, ... We have the decomposition $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$; for each $\alpha \in \Phi$, we choose a non-zero root vector $X_\alpha \in \mathfrak{g}_\alpha$.

Let θ be an automorphism of order 2 of G; let $G^\theta \subseteq G$ be the fixed point subgroup. Then G^θ is reductive by [17, Section 8]; let $G^{\theta,0}$ be its connected component containing 1. For the θ-action on $\mathfrak{g}$, the fixed point subspace $\mathfrak{g}^\theta$ is the Lie algebra of G^θ by [2, Corollary 9.2]. Let $\tau\colon G \to G$ be the map $g \mapsto g^{-1}\theta(g)$; observe that $\theta(x) = x^{-1}$ for all $x \in \tau(G)$.

1 First Results on Double Cosets

1.1 Preliminaries

We begin by collecting several lemmas on involutions of reductive groups, to be used later. Although these results are known (see [16] and [9]), we give complete proofs because they are very short, or simpler than existing ones.

Lemma 1 *Let $\Gamma \subset G$ be a θ-stable connected unipotent subgroup. Then:*

(i) The product map $\Gamma^\theta \times \tau(\Gamma) \to \Gamma$ is an isomorphism.
(ii) Γ^θ is connected.
(iii) $\tau(\Gamma) = \{g \in \Gamma \mid \theta(g) = g^{-1}\}$.
(iv) For any subgroup $\Gamma' \subseteq G$ containing Γ, the map $G \to G/\Gamma$ sends Γ'^θ onto $(\Gamma'/\Gamma)^\theta$.

Proof (i) follows from [2, Proposition 9.3], and it implies (ii). For (iii), let $g \in U$ such that $\theta(g) = g^{-1}$. By (i), we can write $g = xy^{-1}\theta(y)$ for a unique $x \in \Gamma^\theta$ and some $y \in \Gamma$. Then

$$x\theta(y)^{-1}y = \theta(y)^{-1}yx^{-1} = x^{-1}\theta(yx^{-1})^{-1}yx^{-1}$$

whence $x = x^{-1}$ by (i) again. Because Γ is unipotent and connected, it follows that $x = 1$. For (iv), let $g \in \Gamma'$ such that $g\Gamma$ is in $(G/\Gamma)^\theta$. Then $g^{-1}\theta(g) \in \Gamma$. By (iii), we can find $\gamma \in \Gamma$ such that $g^{-1}\theta(g) = \gamma^{-1}\theta(\gamma)$; then $g\gamma^{-1}$ is in Γ'^θ. ■

Lemma 2 *Any Borel subgroup $B \subseteq G$ contains a θ-stable maximal torus of G, and any two such tori are conjugate in U^θ.*

Proof Because $\theta(B)$ is a Borel subgroup of G, the group $B \cap \theta(B)$ is connected, solvable and contains a maximal torus of G. Thus, it contains a θ-stable maximal torus, by [17, 7.6]. Let T, T' be two such tori. There exists $g \in U \cap \theta(U)$ such that $T' = gTg^{-1}$. Because T and T' are θ-stable, $g^{-1}\theta(g)$ normalizes T. But $g^{-1}\theta(g)$ is in U; it follows that $g^{-1}\theta(g) = 1$, that is, $g \in U^\theta$. ■

Lemma 3 *The following conditions are equivalent:*

(i) B *is* θ*-stable.*
(ii) $B^{\theta,0}$ *is a Borel subgroup of* G^θ*.*

Proof By Lemma 2, we can choose a θ-stable maximal torus T of B.

(i)⇒(ii) Because B is θ-stable, the same holds for U. Let B^- be the Borel subgroup of G such that $B^- \cap B = T$; then B^- and its unipotent part U^- are θ-stable as well. Because $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{t} \oplus \mathfrak{u}^-$, we have

$$\mathfrak{g}^\theta = \mathfrak{u}^\theta \oplus \mathfrak{t}^\theta \oplus (\mathfrak{u}^-)^\theta.$$

It follows that $\mathfrak{b}^\theta$ and $(\mathfrak{b}^-)^\theta$ are opposite Borel subalgebras of $\mathfrak{g}^\theta$.

(ii)⇒(i) Observe that θ acts on Φ; if moreover B is not θ-stable, then we can find $\alpha \in \theta(\Phi^+) \cap \Phi^-$. Now $X_\alpha + \theta(X_\alpha)$ and $X_{-\alpha} + \theta(X_{-\alpha})$ are eigenvectors of T^θ in $\mathfrak{g}^\theta$ of opposite weights. Because $\mathfrak{b}^\theta$ is a Borel subalgebra of the Lie algebra of the reductive group G^θ, it follows that one of these vectors is in $\mathfrak{b}^\theta$, in particular in $\mathfrak{b}$. This contradicts the assumption that $\alpha \in \Phi^-$ and $\theta(\alpha) \in \Phi^+$. ■

Lemma 4 *For a* θ*-stable maximal torus* T *of* G*, the following conditions are equivalent:*

(i) T *is contained in a* θ*-stable Borel subgroup of* G*.*
(ii) $T^{\theta,0}$ *is a regular subtorus of* G*.*

All θ*-stable maximal tori* T *satisfying (i) or (ii) are conjugate under* $G^{\theta,0}$*. If moreover* G^θ *is connected, then* T^θ *is connected as well.*

Proof (i)⇒(ii) We may assume that B is θ-stable. If there exists $\alpha \in \Phi^+$ which vanishes identically on $T^{\theta,0}$, then, for all $t \in T$, we have $\alpha\bigl(t\theta(t)\bigr) = 1$, because $t\theta(t) \in T^{\theta,0}$. Thus, $\alpha + \theta(\alpha) = 0$, which contradicts the fact that $\theta(\alpha) \in \Phi^+$.

(ii)⇒(i) Observe that $T^{\theta,0}$ is a maximal subtorus of G^θ. Let Γ be a Borel subgroup of G^θ containing $T^{\theta,0}$, and let B be a Borel subgroup of G containing Γ. Then $\Gamma = B^{\theta,0}$, whence B is θ-stable by Lemma 3. Furthermore, B contains T, because B contains the regular subtorus $T^{\theta,0}$.

If moreover G^θ is connected, then B^θ is connected (because it is contained in the normalizer in G^θ of the Borel subgroup Γ). Because $B^\theta = U^\theta T^\theta$, it follows that T^θ is connected.

Let T' be another θ-stable maximal torus of G satisfying (ii). Then $T^{\theta,0}$ and $T'^{\theta,0}$ are maximal subtori of $G^{\theta,0}$, so that they are conjugate in this group. Taking centralizers in G, we see that T and T' are conjugate in $G^{\theta,0}$, too. ■

1.2 Parametrization of Orbits

Let $\mathcal{B}(G)$ be the flag variety of G. Recall that the set of G^θ-orbits in $\mathcal{B}(G)$ is in bijection with the set of G^θ-conjugacy classes of pairs (B, T) where $B \subseteq G$ is a Borel subgroup, and $T \subseteq B$ is a θ-stable maximal torus; the inverse bijection maps the G^θ-conjugacy class of (B, T) to that of B. As a consequence, $\mathcal{B}(G)$ contains only finitely many G^θ-orbits (see [14, 1.2 and 1.3] for simple proofs of these results).

We begin by generalizing this to the variety $\mathcal{P}(G)$ of all parabolic subgroups of G.

Proposition 1 *There is a bijection from the set of G^θ-orbits in $\mathcal{P}(G)$ onto the set of G^θ-conjugacy classes of triples (P, B, T) where*

(i) P is a parabolic subgroup of G,
(ii) B is a Borel subgroup of P such that the product $P^\theta B$ is open in P, and
(iii) T is a θ-stable maximal torus of B.

The inverse bijection maps the G^θ-conjugacy class of (P, B, T) to that of P.

Proof Let P be a parabolic subgroup of G. For a Borel subgroup B of P, the product $G^\theta P$ is a union of finitely many (G^θ, B)-double cosets. Because the quotient $G^\theta \setminus G^\theta P$ is a P-orbit, it is irreducible; thus, $G^\theta P$ contains a unique open (G^θ, B)-double coset. Replacing B by a conjugate in P, we may assume that $G^\theta B$ is open in $G^\theta P$. It follows that $P^\theta B = (G^\theta B) \cap P$ is open in P. Furthermore, B contains a θ-stable maximal torus by Lemma 2. Thus, there exists a pair (B, T) satisfying (ii) and (iii).

To complete the proof, it suffices to check that all such pairs are conjugate under P^θ, the G^θ-isotropy group of the point P of $\mathcal{P}(G)$. Let (B', T') be another such pair. We can write $B' = pBp^{-1}$ for some $p \in P$. Then $P^\theta B$ and $P^\theta pB$ are open (P^θ, B)-double cosets in the irreducible variety P. Thus, they are equal, and p is in $P^\theta B$: we may assume that $p \in P^\theta$. Now T and $p^{-1}T'p$ are θ-stable maximal subtori of B: by Lemma 2 again, there exists $b \in B^\theta$ such that $p^{-1}T'p = bTb^{-1}$. Then $T' = pbT(pb)^{-1}$ and $B' = pbB(pb)^{-1}$ with $pb \in P^\theta$. ■

From now on we assume that T is a θ-stable maximal torus of G; then its normalizer N is θ-stable, too. Set

$$\mathcal{V} := \{g \in G \mid g^{-1}\theta(g) \in N\}.$$

Then $\mathcal{V}$ is the set of all $g \in G$ such that the maximal torus gTg^{-1} is θ-stable. Clearly, $\mathcal{V}$ is stable under left multiplication by G^θ and right multiplication by N. In fact, by [16] and [9], any (G^θ, B)-double coset in G meets $\mathcal{V}$, along a unique (G^θ, T)-double coset. As an easy consequence of this result, we shall obtain a similar parametrization of the (G^θ, P)-double cosets in G.

For $g \in G$, define an involution ψ_g of G by

$$\psi_g := \operatorname{Int}(g^{-1}) \circ \theta \circ \operatorname{Int}(g) = \operatorname{Int}\bigl(g^{-1}\theta(g)\bigr) \circ \theta.$$

Then $G^{\psi_g} = g^{-1}G^\theta g$. Observe also that

$$\mathcal{V} = \{g \in G \mid T \text{ is } \psi_g\text{-stable}\}.$$

Set finally

$$\mathcal{V}^P := \{g \in \mathcal{V} \mid G^\theta gB \text{ is open in } G^\theta gP\}.$$

Proposition 2 *Any (G^θ, P)-double coset in G meets $\mathcal{V}^P$, along a unique (G^θ, T)-double coset. Furthermore, $\mathcal{V}^P$ is the set of all $g \in \mathcal{V}$ such that $P^{\psi_g}B$ is open in P.*

Proof Let $\mathcal{O}$ be a (G^θ, P)-double coset in G. Then $\mathcal{O}$ contains a unique open (G^θ, B)-double coset $\mathcal{O}^B$. The latter meets $\mathcal{V}$ along a unique (G^θ, T)-double coset $\mathcal{O}^P$. Let $g \in \mathcal{O}^P$, then $G^\theta gB$ is open in $G^\theta gP$. This is equivalent to: $G^{\psi_g} B$ is open in $G^{\psi_g} P$, and also to: $P^{\psi_g} B$ is open in P. Indeed, the G^{ψ_g}-variety $G^{\psi_g} P$ is the quotient of $G^{\psi_g} \times P$ by the action of P^{ψ_g} defined as follows: $x \cdot (g, p) = (gx^{-1}, xp)$. Thus, a subset E of P is open if and only if $G^{\psi_g} E$ is open in $G^{\psi_g} P$. ■

1.3 θ-Stable Levi Subgroups

In this subsection, we assume that P contains a θ-stable Levi subgroup. Let G^P be the set of all $g \in G$ such that gPg^{-1} contains a θ-stable Levi subgroup. Clearly, G^P is a union of (G^θ, P)-double cosets, which we will parametrize.

Recall that L denotes the Levi subgroup of P which contains T. We begin with the easy

Lemma 5 *L is θ-stable, and any θ-stable Levi subgroup of P is conjugate to L in $R_u(P)^\theta$.*

Proof Let M be a θ-stable Levi subgroup of P. Then M is a Levi subgroup of $P \cap \theta(P)$. The latter contains $L \cap \theta(L)$ as its Levi subgroup containing T. Thus, M and $L \cap \theta(L)$ are conjugate; in particular, $\dim L = \dim M = \dim L \cap \theta(L)$. It follows that L is θ-stable. The proof of the other assertion is similar to that of Lemma 2. ■

Let $S = Z(L)^0$ denote the connected center of L, and $N_G(S)$ resp. $Z_G(S)$ the normalizer, resp. centralizer of S in G. Then $L = Z_G(S)$, $N_G(L) = N_G(S)$, and these groups are θ-stable. Let $\mathcal{V}^S = \{g \in G \mid g^{-1}\theta(g) \in N_G(S)\}$, a union of $\big(G^\theta, N_G(S)\big)$-double cosets contained in G^P. Finally, let $\mathcal{V}^{S,P} = \mathcal{V}^S \cap \mathcal{V}^P$.

Proposition 3 *Any (G^θ, P)-double coset in G^P meets $\mathcal{V}^S$ along a unique (G^θ, L)-double coset. The latter meets $\mathcal{V}^{S,P}$ along a unique (G^θ, T)-double coset.*

Proof Let $g \in G^P$, then gPg^{-1} contains a θ-stable Levi subgroup of the form $guLu^{-1}g^{-1}$ for some $u \in R_u(P)$. Then $gu \in \mathcal{V}^S$ so that $G^\theta gP$ meets $\mathcal{V}^S$. If g and gu are in $\mathcal{V}^S$ for u as above, then gLg^{-1} and $guLu^{-1}g^{-1}$ are θ-stable Levi subgroups of gPg^{-1}. By Lemma 5, $gug^{-1} \in G^\theta$. Thus, $gu \in G^\theta g$, which proves the first assertion.

Let $g \in \mathcal{V}^S$, then $G^\theta gP$ meets $\mathcal{V}^P$ along a unique (G^θ, T)-double coset. Moving g in its (G^θ, L)-double coset, we may assume that there exists $u \in R_u(P)$ such that $gu \in \mathcal{V}^P$. Then $gPg^{-1} = guPu^{-1}g^{-1}$ contains a θ-stable Levi subgroup, and contains the θ-stable maximal torus $guTu^{-1}g^{-1}$. By Lemma 5, it follows that $guLu^{-1}g^{-1}$ is θ-stable, that is, $gu \in \mathcal{V}^S$. By the first part of the proof, $gu \in G^\theta g$. ■

Set $V^S := G^\theta \setminus \mathcal{V}^S / L$; then we have $V^S = G^\theta \setminus G^P / P = G^\theta \setminus \mathcal{V}^{S,P} / T$. The action of $N_G(S)$ on $\mathcal{V}^S$ by right multiplication induces an action of the Weyl group $W(S) := N_G(S)/Z_G(S)$ on V^S. We interpret the orbit set $V^S/W(S)$ in terms of certain conjugacy classes of θ-stable tori, as follows.

Let $\mathcal{S}$ be the set of all conjugates of S by elements of G. This is an affine variety, isomorphic to $G/N_G(S)$, on which θ acts. Let $\mathcal{S}^\theta$ be the fixed point set of θ, then $\mathcal{S}^\theta$ is the set of conjugates of S by elements of $\mathcal{V}^S$. It is an affine variety, on which G^θ acts by conjugation.

The bijective map $\mathcal{V}^S/N_G(S) \to \mathcal{S}^\theta \colon gN_G(S) \mapsto gSg^{-1}$ is G^θ-equivariant; thus, the induced map $V^S/W(S) \to \mathcal{S}^\theta/G^\theta$ is bijective. In the case that $P = B$ this was observed in [13, Proposition 2.7].

For S a maximal k_0-split torus of G, where $k_0 \subseteq k$ is a subfield of k and G, θ are defined over k_0, the sets $\mathcal{V}^S$ and $\mathcal{S}^\theta/G^\theta$ are discussed in more detail in [8]. This includes a characterization of $\mathcal{S}^\theta/G^\theta$; the case where S is a maximal torus is treated in [7].

1.4 Fixed Points in Parabolic Subgroups

For a parabolic subgroup $P \supseteq B$, we describe the subgroup P^θ, and its image in the quotient of P by its unipotent radical $R_u(P)$. Recall that P is the semidirect product of $R_u(P)$ with its Levi subgroup $L \supseteq T$; we shall identify $P/R_u(P)$ with L.

Theorem 1 *With notation as above, $R_u(P)^\theta$ is a connected unipotent normal subgroup of P^θ. Furthermore, the quotient $P^\theta/R_u(P)^\theta$ (the image of P^θ in L) is the semidirect product of $L\cap\theta\big(R_u(P)\big)$ (the unipotent radical of $L\cap\theta(P)$, a parabolic subgroup of L) with L^θ (a reductive group).*

Proof Set $Q := \theta(P)$, a parabolic subgroup of G containing T, and set $M := \theta(L)$, the Levi subgroup of Q containing T. Then $P \cap Q$ is θ-stable and contains P^θ as its fixed point subgroup.

We claim that $P\cap Q$ is the semidirect product of its unipotent radical $R_u(P\cap Q)$ with the θ-stable connected reductive subgroup $L \cap M$. Furthermore, $R_u(P \cap Q)$ contains $R_u(P) \cap R_u(Q)$ as a θ-stable connected normal subgroup, and the quotient

$$R_u(P \cap Q)/R_u(P) \cap R_u(Q)$$

is the direct product of $L \cap R_u(Q)$ with $R_u(P) \cap M$, where θ acts by exchanging both factors (this analysis of $P \cap Q$ is implicit in [3, pp. 86–88].)

Indeed, both $R_u(P)\cap Q$ and $P\cap R_u(Q)$ are unipotent normal subgroups of $P\cap Q$; because they are normalized by T, they are connected. Furthermore, we have isomorphisms

$$(P \cap Q)/\big(R_u(P) \cap Q\big)\big(P \cap R_u(Q)\big) \cong (L \cap Q)/\big(L \cap R_u(Q)\big) \cong L \cap M$$

and the latter is a connected reductive group. Thus, the unipotent radical of $P \cap Q$ is

$$\big(R_u(P) \cap Q\big)\big(P \cap R_u(Q)\big) = \big(R_u(P) \cap R_u(Q)\big)\big(R_u(P) \cap M\big)\big(L \cap R_u(Q)\big),$$

a product of three subgroups with trivial pairwise intersections. And $R_u(P) \cap R_u(Q)$ is a normal subgroup of $R_u(P \cap Q)$, and contains all commutators $[g, h]$ where $g \in L \cap R_u(Q)$ and $h \in R_u(P) \cap M$. This proves the claim.

By that claim and Lemma 1(iv), $R_u(P)^\theta = \big(R_u(P) \cap R_u(Q)\big)^\theta$ is connected, and the quotient

$$P^\theta/R_u(P)^\theta = (P \cap Q)/\big(R_u(P) \cap R_u(Q)\big)^\theta$$

is the semidirect product of the group of all pairs $\big(g, \theta(g)\big)$ where $g \in L \cap R_u(Q)$, with $(L \cap M)^\theta = L^\theta$. It follows that the image of P^θ in L is the semidirect product of $L \cap R_u(Q)$ with L^θ. Furthermore, $L\cap Q$ is a parabolic subgroup of L, with unipotent radical $L\cap R_u(Q)$ and Levi subgroup $L \cap M$. ■

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1.5 Decomposition of Double Cosets

With notation as in 1.2, let $g \in \mathcal{V}$. We shall decompose $G^\theta gP$ into (G^θ, B)-double cosets.

Set $L_g := L \cap \psi_g(L)$, then L_g is a ψ_g-stable Levi subgroup of the parabolic subgroup $L \cap \psi_g(P)$ of L, and T is a ψ_g-stable maximal torus of L_g with normalizer $N \cap L_g$. Furthermore, $L^{\psi_g} = L_g^{\psi_g}$. Set

$$\mathcal{V}_g := \{x \in L_g \mid x^{-1}\psi_g(x) \in N \cap L_g\}.$$

By the results recalled in 1.2, the map $L^{\psi_g} \setminus \mathcal{V}_g/T \to L^{\psi_g} \setminus L_g/B \cap L_g$ is bijective.

Finally, denote by N_g the set of all $n \in N \cap L$ such that $B \cap L_g$ is contained in $n(B \cap L)n^{-1}$. Then, by the Bruhat decomposition, the map $N_g/T \to L \cap \psi_g(P) \setminus L/B \cap L$ is bijective.

Proposition 4 *With notation as above, we have*

$$G^\theta gP = \bigcup_{l \in \mathcal{V}_g, n \in N_g} G^\theta glnB.$$

Furthermore, $G^\theta glnB = G^\theta gl'n'B$ if and only if: $L^{\psi_g}lT = L^{\psi_g}l'T$ and $nT = n'T$. This defines a bijection

$$L^{\psi_g} \setminus \mathcal{V}_g/T \times N_g/T \to G^\theta \setminus G^\theta gP/B.$$

Proof Observe that

$$G^\theta \setminus G^\theta gP/B = g^{-1}G^\theta g \setminus g^{-1}G^\theta gP/B = G^{\psi_g} \setminus G^{\psi_g}P/B.$$

Now any (G^{ψ_g}, B)-double coset in $G^{\psi_g}P$ meets P, along a unique (P^{ψ_g}, B)-double coset. Thus, we have

$$G^{\psi_g} \setminus G^{\psi_g}P/B = P^{\psi_g} \setminus P/B = \mathrm{Im}(P^{\psi_g}) \setminus L/B \cap L$$

where $\mathrm{Im}(P^{\psi_g})$ is the image of P^{ψ_g} in L. But $\mathrm{Im}(P^{\psi_g}) = L \cap \psi_g\big(R_u(P)\big)L^{\psi_g}$ by Theorem 1. For simplicity, set $Q := \psi_g(P)$, $Q_L := Q \cap L$ (a parabolic subgroup of L, with Levi subgroup L_g) and $B_L := B \cap L$ (a Borel subgroup of L); then $L \cap \psi_g\big(R_u(P)\big) = R_u(Q_L)$. Each $\big(R_u(Q_L)L^{\psi_g}, B_L\big)$-double coset in L is contained in a unique (Q_L, B_L)-double coset. The latter meets N_g along a unique T-coset. This defines a surjective map

$$\mathrm{Im}(P^{\psi_g}) \setminus L/B \cap L = R_u(Q_L)L^{\psi_g} \setminus L/B_L \to Q_L \setminus L/B_L = N_g/T.$$

For $n \in N_g$, the fiber of this map over nT is

$$R_u(Q_L)L^{\psi_g} \setminus Q_L nB_L/B_L = R_u(Q_L)L^{\psi_g} \setminus Q_L/Q_L \cap nB_Ln^{-1} = L^{\psi_g} \setminus L_g/B \cap L_g.$$

Indeed, as nB_Ln^{-1} contains $B \cap L_g$, the image of $Q_L \cap nB_Ln^{-1}$ in $L_g = Q_L/R_u(Q_L)$ is $B \cap L_g$. Finally, each $(L^{\psi_g}, B \cap L_g)$-double coset in L_g meets $\mathcal{V}_g$ into a unique (L^{ψ_g}, T)-double coset. Tracing through all identifications completes the proof. ∎

2 Combinatorics and Geometry of Orbits

2.1 Parabolic Subgroups Associated with Double Cosets

Any double coset $G^\theta gB$ defines two parabolic subgroups containing B: its right stabilizer, that is, the set of all $x \in G$ such that $G^\theta gBx = G^\theta gB$, and the right stabilizer of its closure $\overline{G^\theta gB}$. We shall describe both parabolic subgroups in terms of the combinatorics of root systems and involutions, which we recall below; as an application, we shall characterize the set $\mathcal{V}^P$ introduced in 1.2.

For each $\alpha \in \Phi$, let $U_\alpha \subset G$ be the corresponding root subgroup. Each simple root $\alpha \in \Delta$ defines a parabolic subgroup P_α of semisimple rank one, generated by B and $U_{-\alpha}$. We denote by L_α the Levi subgroup of P_α which contains T, and by G_α the quotient of L_α by its center; then G_α is isomorphic to $\mathrm{PSL}(2)$. We shall identify U_α and $U_{-\alpha}$ with their images in G_α, and we denote by T_α the image of T; we set $B_{\pm\alpha} = U_{\pm\alpha}T_\alpha$.

Recall that any parabolic subgroup $P \supseteq B$ is generated by the P_α's that it contains. We write $P = P_\Pi$ where Π is the set of all $\alpha \in \Delta$ such that $P_\alpha \subseteq P$. We denote by Φ_Π the sub-root system of Φ generated by Π, and by W_Π its Weyl group; we also denote $\mathcal{V}^P$ by $\mathcal{V}^\Pi$.

Because T is θ-stable, θ acts on Φ by an involution, still denoted by θ. Recall from [16] that $\alpha \in \Phi$ is called *real* if $\theta(\alpha) = -\alpha$, *imaginary* if $\theta(\alpha) = \alpha$ and *complex* if $\theta(\alpha) \neq \pm\alpha$. For real or imaginary α, the group L_α is θ-stable, and θ acts on G_α; recall that α is *compact* if θ fixes G_α pointwise (then α is imaginary). Observe that α is compact (resp. non-compact imaginary) if and only if $\theta(X_\alpha) = X_\alpha$ (resp. $\theta(X_\alpha) = -X_\alpha$).

The following result is an easy consequence of [13, Section 4] or of Theorem 1.

Lemma 6 *The image of $P_\alpha^{\theta,0}$ in G_α is*

- *G_α if α is compact,*
- *T_α if α is non-compact imaginary,*
- *a copy of the multiplicative group, distinct from T_α, if α is real,*
- *B_α if α is complex and in $\theta(\Phi^+)$,*
- *$B_{-\alpha}$ if α is complex and in $\theta(\Phi^-)$.*

As a consequence, α is compact (resp. $\alpha \in \theta(\Phi^-)$; $\alpha \in \theta(\Phi^+)$) if and only if $P_\alpha^\theta B$ is equal to P_α (resp. is a proper open subset of P_α; is closed in P_α).

For $g \in \mathcal{V}$, the involution $\psi_g = \mathrm{Int}\big(g^{-1}\theta(g)\big) \circ \theta$ acts on Φ as well; if w_g denotes the image in W of $g^{-1}\theta(g) \in N$, then $\psi_g(\alpha) = w_g\theta(\alpha)$ for all $\alpha \in \Phi$. Thus, we can distinguish between ψ_g-real, imaginary, complex, ... roots. Let Δ_c be the set of all ψ_g-compact simple roots.

Proposition 5 *Let $g \in \mathcal{V}$.*

(i) The right stabilizer of $G^\theta gB$ is generated by the P_α where $\alpha \in \Delta_c$.
(ii) The right stabilizer of $\overline{G^\theta gB}$ is generated by the P_α where α is in Δ_c or in $\Delta \cap \psi_g(\Phi^-)$.
(iii) $G^\theta gB$ is open in $G^\theta gP$ (that is, $g \in \mathcal{V}^\Pi$) if and only if Π is contained in $\Delta_c \cup \psi_g(\Phi^-)$.
(iv) $G^\theta gB$ is closed in $G^\theta gP$ if and only if Π is contained in $\psi_g(\Phi^+)$.

Proof As in 1.5, we may reduce to the case where $g = 1$; then $\psi_g = \theta$.

(i) The right stabilizer of $G^\theta B$ is generated by the P_α ($\alpha \in \Delta$) such that $G^\theta B = G^\theta P_\alpha$. This amounts to: $P_\alpha^\theta B = P_\alpha$, that is, α is θ-compact by Lemma 6.

(ii) Similarly, the right stabilizer of $\overline{G^\theta B}$ is generated by the P_α ($\alpha \in \Delta$) such that $\overline{G^\theta B} = \overline{G^\theta B}P_\alpha = \overline{G^\theta P_\alpha}$, that is, $G^\theta B$ is open in $G^\theta P_\alpha$. This amounts to: $P_\alpha^\theta B$ is open in P_α, or to: α is either θ-compact or in $\theta(\Phi^-)$.

(iii) is a direct consequence of (ii).

(iv) Observe that $G^\theta B$ is closed in $G^\theta P$ if and only if $P^\theta B$ is closed in P. If this holds, then, intersecting with P_α for $\alpha \in \Pi$, we have that $P_\alpha^\theta B$ is closed in P_α. By the Lemma, we then have $\alpha \in \theta(\Phi^+)$.

Conversely, if $\Pi \subseteq \theta(\Phi^+)$, we claim that $B \cap \theta(B)$ is a Borel subgroup of $P \cap \theta(P)$. Indeed, the assumption implies that $B \cap \theta(B) = B \cap \theta(P) = P \cap \theta(B)$. Thus, $B \cap \theta(B)$ contains both $R_u(P) \cap \theta(P)$ and $P \cap \theta\big(R_u(P)\big)$. By the structure of $P \cap \theta(P)$ given in the proof of Theorem 1, it follows that $B \cap \theta(B)$ contains the unipotent radical of $P \cap \theta(P)$. Furthermore, $B \cap \theta(B)$ contains $B \cap L \cap \theta(L)$; the latter is a Borel subgroup of the Levi subgroup $L \cap \theta(L)$ of $P \cap \theta(P)$. This proves the claim.

This claim and Lemma 3 imply that $B^{\theta,0}$ is a Borel subgroup of P^θ. This implies in turn that P^θ/B^θ is complete, hence closed in P/B. It follows that $P^\theta B$ is closed in P. ■

In the case where $P = G$, we obtain the following result, which is also a consequence of [9, Proposition 9.2 and Lemma 1.7].

Corollary 1 *With notation as above, $G^\theta g B$ is open (resp. closed) in G if and only if each simple root is either ψ_g-compact or in $\psi_g(\Phi^-)$ (resp. each simple root is in $\psi_g(\Phi^+)$, that is, B is ψ_g-stable).*

2.2 Isotropy Groups

Let $g \in \mathcal{V}^\Pi$. The G^θ-isotropy group of the point gP of G/P is $G^\theta \cap gPg^{-1} = gP^{\psi_g}g^{-1}$. To describe this group, or, equivalently, P^{ψ_g}, we need more notation. Set

$$\Pi_g := \{\alpha \in \Pi \mid \psi_g(\alpha) \in \Phi_\Pi\}.$$

Then Π_g contains Π_c (the set of all ψ_g-compact roots of Π); we denote by Φ_{Π_g}, Φ_{Π_c} the corresponding sub-root systems of Φ. Let Φ_c (resp. Φ_C) be the set of all ψ_g-compact (resp. complex) roots.

Finally, recall that a parabolic subgroup Q of G is *split* with respect to an involution ψ if the parabolic subgroup $\psi(Q)$ is opposite to Q, that is, if $Q \cap \psi(Q)$ is a Levi subgroup of Q and of $\psi(Q)$.

Proposition 6

(i) The group $L_g := L \cap \psi_g(L)$ is equal to L_{Π_g}; in particular, Φ_{Π_g} is ψ_g-stable. Furthermore,

$$\psi_g(\Phi_{\Pi_g}^+ - \Phi_{\Pi_c}^+) = \Phi_{\Pi_g}^- - \Phi_{\Pi_c}^-.$$

Thus, Φ_{Π_c} is the set of all ψ_g-compact roots of Φ_{Π_g}, and $P_{\Pi_c} \cap L_g$ is a minimal ψ_g-split parabolic subgroup of L_g.

(ii) *The group P^{ψ_g} is the semi-direct product of a connected unipotent normal subgroup of dimension*

$$|\Phi_c^+ - \Phi_{\Pi_c}^+| + \frac{1}{2}|\Phi_C^+ \cap \psi_g(\Phi^+)| + |\Phi_\Pi^+ - \Phi_{\Pi_g}^+|$$

with the reductive subgroup $L_g^{\psi_g}$.

Proof (i) By Proposition 5(iii), we have $\Pi \subseteq \psi_g(\Phi^- \cup \Pi)$ whence

$$\Phi_\Pi^+ \subseteq \psi_g(\Phi^- \cup \Phi_\Pi).$$

It follows that $B \cap L$ is contained in $\psi_g(P^-) \cap L$. The latter is a parabolic subgroup of L, with $L \cap \psi_g(L)$ as its Levi subgroup containing T. Thus, there exists a subset $\Pi' \subseteq \Pi$ such that $L \cap \psi_g(L) = L_{\Pi'}$. Then we must have $\Pi' = \Pi_g$.

Let $\alpha \in \Pi_g - \Pi_c$. Then $\psi_g(\alpha) \in \Phi_{\Pi_g}^- - \Phi_{\Pi_c}$ by Proposition 5(iii) again. Thus, the coefficients of $\psi_g(\alpha)$ on all elements of $\Pi_g - \Pi_c$ are non-positive, one of them being negative. It follows that $\psi_g(\Phi_{\Pi_g}^+ - \Phi_{\Pi_c}^+)$ consists of negative roots.

(ii) By Theorem 1, the group $L^{\psi_g} = L_g^{\psi_g}$ is a maximal reductive subgroup of P^{ψ_g}, and $R_u(P^{\psi_g})$ is an extension of $L \cap \psi_g\big(R_u(P)\big)$ by $R_u(P)^{\psi_g}$. Furthermore, $L \cap \psi_g\big(R_u(P)\big)$ is the unipotent radical of $L \cap \psi_g(P)$, a parabolic subgroup of L with Levi subgroup L_g. Thus, we have

$$\dim L \cap \psi_g\big(R_u(P)\big) = |\Phi_\Pi^+ - \Phi_{\Pi_g}^+|.$$

To compute the dimension of $R_u(P)^{\psi_g}$, we use the notation of the proof of Lemma 3. The X_α ($\alpha \in \Phi^+ - \Phi_\Pi$) are a basis of the Lie algebra of $R_u(P)$. Thus, a basis of the Lie algebra of $R_u(P)^{\psi_g}$ consists of the X_α (where $\alpha \in \Phi_c^+ - \Phi_\Pi$) together with the $X_\alpha + \psi_g(X_\alpha)$ (where α is complex and both $\alpha, \psi_g(\alpha)$ are in $\Phi^+ - \Phi_\Pi$).

Observe that

$$\Phi_c^+ - \Phi_\Pi = \Phi_c^+ - \big(\Phi_\Pi \cap \psi_g(\Phi_\Pi)\big) = \Phi_c^+ - \Phi_{\Pi_g} = \Phi_c^+ - \Phi_{\Pi_c}.$$

Finally, we check that the set of all complex roots $\alpha \in \Phi^+ - \Phi_\Pi$ such that $\psi_g(\alpha) \in \Phi^+ - \Phi_\Pi$ is $\Phi_C^+ \cap \psi_g(\Phi^+)$. Indeed, there is no complex $\alpha \in \Phi_\Pi^+$ such that $\psi_g(\alpha) \in \Phi^+$ (otherwise, $\psi_g(\alpha) \in \Phi_\Pi^+$ by the proof of (i), whence $\alpha \in \Phi_{\Pi_g}$; but any complex root $\alpha \in \Phi_{\Pi_g}$ satisfies $\psi_g(\alpha) \in \Phi^-$, by (i)). And for $\alpha \in \Phi^+ - \Phi_\Pi$, the condition: $\psi_g(\alpha) \in \Phi^+ - \Phi_\Pi$ is equivalent to: $\psi_g(\alpha) \in \Phi^+$. ∎

As an application, we describe the isotropy groups for the G^θ-action on G/B; this sharpens [16, Proposition 4.8]. Let $g \in \mathcal{V}$, then the G^θ-isotropy group of gB/B is

$$(gBg^{-1})^\theta = gB^{\psi_g}g^{-1}.$$

By Proposition 5(i), the parabolic subgroup P_{Δ_c} is the right stabilizer of $G^\theta gB$, and moreover $g \in \mathcal{V}^{\Delta_c}$. Clearly, L_{Δ_c} is ψ_g-stable, and its derived subgroup consists of ψ_g-fixed points. It then follows from Theorem 1 that

$$P_{\Delta_c}^{\psi_g} = R_u(P_{\Delta_c})^{\psi_g} L_{\Delta_c}^{\psi_g}.$$

Intersecting with B, we obtain the following

Corollary 2 *With notation as above, B^{ψ_g} is the semi-direct product of the connected unipotent normal subgroup*

$$R_u(P_{\Delta_c})^{\psi_g}(U \cap L_{\Delta_c})$$

with the diagonalizable subgroup T^{ψ_g}, and we have

$$\dim R_u(P_{\Delta_c})^{\psi_g} = \frac{1}{2}|\Phi_C^+ \cap \psi_g(\Phi^+)|.$$

2.3 Affine Orbits

Let $g \in \mathcal{V}^P$. We give a criterion for the orbit $G^\theta gP/P \subseteq G/P$ to be affine. As G^θ is reductive and the isotropy group $G^\theta \cap gPg^{-1}$ is equal to $gP^{\psi_g}g^{-1}$, this is equivalent to: P^{ψ_g} is reductive.

This condition holds if P is ψ_g-split: then $P^{\psi_g} = \big(P \cap \psi_g(P)\big)^{\psi_g} = L^{\psi_g}$. Another example of an affine orbit occurs when the symmetric space G/G^θ is *Hermitian*, that is, there exists a parabolic subgroup $Q \subseteq G$ and a Levi subgroup $M \subseteq Q$ such that $G^{\theta,0} = M$. Then $Q^\theta = M$ is reductive; the corresponding orbit $G^\theta Q/Q = G^\theta/G^{\theta,0}$ is finite. In the general case, we shall see that affine orbits arise from a combination of both examples.

Let Δ_n be the set of all non-compact imaginary simple roots for ψ_g. Write $P = P_\Pi$ and consider the Dynkin diagram of $\Pi \cup \Delta_n$. Let $\overline{\Delta}_n$ be the union of all connected components of this diagram which meet $\Delta_n - \Pi$, and let Π^0 be the union of the other components. Then $\Phi_{\Pi\cup\Delta_n}$ is the disjoint union of Φ_{Π^0} and $\Phi_{\overline{\Delta}_n}$.

Proposition 7 *With notation as above, P^{ψ_g} is reductive if and only if g satisfies the following three conditions:*

a) *Φ_Π is ψ_g-stable and contains all ψ_g-compact roots of Φ.*
b) *$P_{\Pi\cup\Delta_n}$ is ψ_g-split.*
c) *$\overline{\Delta}_n$ is contained in $\Delta_n \cup \Pi_c$.*

Then $P^{\psi_g,0} = L_{\Pi\cup\Delta_n}^{\psi_g,0}$, both L_{Π^0} and $L_{\overline{\Delta}_n}$ are ψ_g-stable, and the symmetric space $L_{\overline{\Delta}_n}/L_{\overline{\Delta}_n}^{\psi_g}$ is Hermitian with Levi subgroup $L_{\Pi_c\cap\overline{\Delta}_n}$.

Proof We use the notation of 2.2. If P^{ψ_g} is reductive, then $|\Phi_\Pi^+ - \Phi_{\Pi_g}^+| = 0$ whence Φ_Π is ψ_g-stable. Furthermore, $|\Phi_c^+ - \Phi_{\Pi_c}^+| = 0$ whence Φ_Π contains all ψ_g-compact roots, and a) holds. Finally, $|\Phi_C^+ \cap \psi_g(\Phi^+)| = 0$ whence

$$\psi_g(\Phi^+ - \Phi_i) = \Phi^- - \Phi_i$$

where $\Phi_i \subseteq \Phi$ denotes the subset of ψ_g-imaginary roots. It follows that $\Phi_i = \Phi_{\Delta_i}$ where $\Delta_i = \Delta \cap \Phi_i$. Indeed, let $\beta \in \Phi_i^+$. Write $\beta = \sum_{\alpha\in\Delta} n_\alpha\alpha$, then $\beta - \sum_{\alpha\in\Delta_i} n_\alpha\alpha$ is fixed by ψ_g and belongs to the convex cone generated by $\Phi^+ - \Phi_i$. Thus, it also belongs to the convex cone generated by $\Phi^- - \Phi_i$. It follows that $\beta - \sum_{\alpha\in\Delta_i} n_\alpha\alpha = 0$.

Because Φ_Π contains all ψ_g-compact roots, we have $\Pi \cup \Delta_i = \Pi \cup \Delta_n$. Furthermore, $\Phi_{\Pi\cup\Delta_n}$ is ψ_g-stable and

$$\psi_g(\Phi^+ - \Phi_{\Pi\cup\Delta_n}) = \Phi^- - \Phi_{\Pi\cup\Delta_n}$$

whence b) holds.

Let I be a connected component of the Dynkin diagram of $\Pi \cup \Delta_n$, which meets Π and $\Delta_n - \Pi$. Let J be a connected component of $I \cap \Pi$, and let α be the sum of all simple roots of J. Then $\alpha \in \Phi_\Pi^+$ and we can find $\beta \in (\Delta_n - \Pi) \cap I$ which is connected to α. Thus, $\alpha + \beta \in \Phi^+$. It follows that $\psi_g(\alpha + \beta) = \psi_g(\alpha) + \beta \in \Phi^+$, whence $\alpha + \beta \in \Phi_i$ and α is imaginary. Because $\alpha \in \Phi_\Pi = \Phi_{\Pi_g}$, Proposition 6 implies that $\alpha \in \Phi_{\Pi_c}$. Thus, $I \cap \Pi \subseteq \Pi_c$. This implies c).

Conversely, assume that a), b) and c) hold. By b), we have $P^{\psi_g} \subseteq L_{\Pi \cup \Delta_n}$, and the latter is ψ_g-stable. Thus, we may assume that $\Delta = \Pi \cup \Delta_n$. Let $G_{\overline{\Delta}_n}$ be the connected adjoint semisimple group with root system $\Phi_{\overline{\Delta}_n}$; then ψ_g induces an involution of $G_{\overline{\Delta}_n}$, and we have a ψ_g-equivariant quotient map $q\colon G \to G_{\overline{\Delta}_n}$. Because ψ_g fixes $\overline{\Delta}_n$ pointwise, it acts on $G_{\overline{\Delta}_n}$ by conjugation by an element of $q(T)$. Thus, $G_{\overline{\Delta}_n}^{\psi_g}$ contains $q(T)$, and its roots are the ψ_g-compact roots of $\Phi_{\overline{\Delta}_n}$. By a) and c), this set of roots is $\Phi_{\Pi_c \cap \overline{\Delta}_n}$. In other words,

$$G_{\overline{\Delta}_n}^{\psi_g,0} = q(L_{\Pi_c \cap \overline{\Delta}_n}).$$

Because $q^{-1}q(L_{\Pi_c \cap \overline{\Delta}_n}) = L_\Pi$, it follows that $G^{\psi_g,0} \subseteq L_\Pi$, that is, $P^{\psi_g,0} = G^{\psi_g,0}$. ■

Corollary 3 *The parabolic subgroup P is θ-split if and only if the orbit $G^\theta P/P$ is an open affine subset of G/P. Then this orbit consists of all θ-split G-conjugates of P.*

Proof Choose $B \subseteq P$ such that $G^\theta B$ is open in $G^\theta P$. Then, by Proposition 5(iii), each $\alpha \in \Pi$ is either θ-compact or in $\theta(\Phi^-)$.

If P is θ-split, then $\theta(\Phi^+ - \Phi_\Pi) = \Phi^- - \Phi_\Pi$. Thus, each $\alpha \in \Delta - \Pi$ is in $\theta(\Phi^-)$. Now Corollary 1 implies that $G^\theta B$ is open in G. Then $G^\theta P/P \simeq G^\theta/P^\theta = G^\theta/L^\theta$ is an open affine subset of G/P.

Conversely, if $G^\theta P/P$ is an open affine subset of G/P, then $G^\theta B$ is open in G. It follows that all imaginary roots are compact, *e.g.* by Proposition 6(i). Applying Proposition 7 with $\Delta_n = \varnothing$, we see that P is θ-split. Let now Q be a θ-split conjugate of P. Write $Q = gPg^{-1}$, then $G^\theta gP$ is open in G, whence $G^\theta gP = G^\theta P$ and $g \in G^\theta P$. Thus, Q is conjugate to P in G^θ. ■

2.4 Examples

1) (see [13, 10.1]) Let $\mathbf{G}$ be a connected reductive group, $\mathbf{B} \subseteq \mathbf{G}$ a Borel subgroup, and $\mathbf{T} \subset \mathbf{B}$ a maximal torus. Consider $G = \mathbf{G} \times \mathbf{G}$ with involution θ defined by $\theta(g_1, g_2) = (g_2, g_1)$. Then G^θ is the diagonal $\operatorname{diag}(\mathbf{G})$. The maximal torus $T = \mathbf{T} \times \mathbf{T}$ and the Borel subgroup $B = \mathbf{B} \times \mathbf{B}$ are θ-stable.

The map $(g_1, g_2) \mapsto g_1^{-1} g_2$ induces a bijection $G^\theta \setminus G/B \to \mathbf{B} \setminus \mathbf{G}/\mathbf{B}$. More generally, let P be a parabolic subgroup of G containing B; then $P = \mathbf{P}_1 \times \mathbf{P}_2$ where $\mathbf{P}_1$ and $\mathbf{P}_2$ are parabolic subgroups of $\mathbf{G}$ containing $\mathbf{B}$, and we have a bijection $G^\theta \setminus G/P \to \mathbf{P}_1 \setminus \mathbf{G}/\mathbf{P}_2$ which is compatible with the partial orderings given by inclusion of closures. Thus, our results in this case can be derived more directly from the Bruhat decomposition.

The root system of (G, T) is the disjoint union of two copies of the root system Φ of $(\mathbf{G}, \mathbf{T})$; we shall denote these copies by $\Phi \times 0$ and $0 \times \Phi$. Let $\mathbf{N}$ be the normalizer of $\mathbf{T}$ in $\mathbf{G}$; then

$$\mathcal{V} = \{(g_1, g_2) \mid g_1^{-1} g_2 \in \mathbf{N}\} = \mathrm{diag}(\mathbf{G})(1 \times \mathbf{N}).$$

For $g = (g_1, g_2) \in \mathcal{V}$, let w be the image of $g_1^{-1} g_2$ in $\mathbf{W} = \mathbf{N}/\mathbf{T}$. Then ψ_g acts on G by $\psi_g(x_1, x_2) = (n x_2 n^{-1}, n^{-1} x_1 n)$, and on roots by $\psi_g(\alpha, 0) = \big(0, w^{-1}(\alpha)\big)$, $\psi_g(0, \alpha) = \big(w(\alpha), 0\big)$. In particular, there are no ψ_g-imaginary roots.

Let $\Pi = (\Pi_1 \times 0) \cup (0 \times \Pi_2)$ be a subset of the set of simple roots, and let $g \in \mathcal{V}$. By Proposition 5, $g \in \mathcal{V}^{\Pi}$ if and only if $w(\Pi_1)$ and $w^{-1}(\Pi_2)$ are contained in Φ^-. This amounts to: w is the element of maximal length in its $(\mathbf{W}_{\Pi_1}, \mathbf{W}_{\Pi_2})$-double coset. Furthermore, we have $P_\Pi = \mathbf{P}_1 \times \mathbf{P}_2$ and

$$P_\Pi^{\psi_g} = \{(x_1, x_2) \in \mathbf{P}_1 \times \mathbf{P}_2 \mid x_1 = n x_2 n^{-1}\} \simeq \mathbf{P}_1 \cap w \mathbf{P}_2 w^{-1}.$$

And P_Π is ψ_g-split if and only if the parabolic subgroups $\mathbf{P}_1$, $w(\mathbf{P}_2)$ are opposite. This is also equivalent to: $P_\Pi^{\psi_g}$ is reductive (this can be seen directly, or deduced from Proposition 7 together with non-existence of imaginary roots.)

2) (see [13, 10.2]) Let $G = \mathrm{GL}_n$ with involution θ defined by $\theta(g) = (g^{-1})^t$; then G^θ is the orthogonal group O_n. Let B be the Borel subgroup of G consisting of upper triangular matrices, and let T be the maximal torus of diagonal matrices. Then T is θ-stable, and B is θ-split; we have $\theta(\alpha) = -\alpha$ for all $\alpha \in \Phi$.

For $g \in \mathcal{V}$, we have $w_g^2 = 1$, and the map $g \mapsto w_g$ induces a bijection from $G^\theta \setminus G/B = G^\theta \setminus \mathcal{V}/T$ onto the set of elements of W of order ≤ 2, see [13, 10.2]. We identify W with the symmetric group S_n, and Φ with the set of pairs (i, j) of distinct integers between 1 and n; then Δ consists of the pairs $\alpha_i = (i, i{+}1)$, $1 \leq i \leq n{-}1$. We have $\psi_g(i, j) = \big(w_g(j), w_g(i)\big)$; as a consequence, the ψ_g-imaginary roots are the pairs $\big(i, w_g(i)\big)$.

We claim that there are no ψ_g-compact roots. To see this, let Γ be the copy of GL_2 in G associated with the the pair $\big(i, w_g(i)\big)$. Then ψ_g stabilizes Γ, and acts there by inverse transpose followed with conjugation by a symmetric monomial matrix. A matrix computation shows that $\psi_g(E_{i, w_g(i)}) = -E_{i, w_g(i)}$ where $E_{i,j}$ denotes the elementary $n \times n$ matrix; this proves the claim. As a consequence, the imaginary simple roots are the pairs $(i, i + 1)$ such that $w_g(i) = i + 1$; because $w_g^2 = 1$, these simple roots are pairwise orthogonal.

Let Π be a subset of Δ and let $g \in \mathcal{V}$. By the claim and Proposition 5(iii), $g \in \mathcal{V}^\Pi$ if and only if $w_g(i) < w_g(i + 1)$ for any $(i, i + 1) \in \Pi$. If $g \in \mathcal{V}^\Pi$, then it follows easily that Π_g consists of those pairs in Π that are fixed by w_g. In particular, Φ_Π is ψ_g-stable if and only if w_g fixes Π pointwise.

For any subset Π' of Δ, the parabolic subgroup $P_{\Pi'}$ is ψ_g-split if and only if w_g stabilizes $\Phi^+ \cup \Phi_{\Pi'}$ (because ψ_g acts on roots by $-w_g$). This amounts to: $w_g \in W_{\Pi'}$. Using these remarks, Proposition 7 simplifies as follows: for $\Pi \subset \Delta$ and $g \in \mathcal{V}^\Pi$, the group $P_\Pi^{\psi_g}$ is reductive if and only if w_g fixes Π and is a product of simple transpositions with disjoint supports.

3) (see [13, 10.5]) Let $G = \mathrm{GL}_n$ with involution θ such that $\theta(g) = zgz^{-1}$ where $z = \mathrm{diag}(1, \dots, 1, -1)$; then $G^\theta = \mathrm{GL}_{n-1} \times k^*$. Let B and T be as in the previous example;

then T is θ-fixed, and B is θ-stable. One checks that a system of representatives of $G^\theta \setminus \mathcal{V}/T$ consists of the

$$g_{i,j}\colon (e_1,\dots,e_n) \mapsto (e_1,\dots,e_{i-1},e_i+e_n,e_{i+1},\dots,e_{j-1},e_i-e_n,e_j,\dots,e_{n-1}) \qquad (1 \le i < j \le n)$$

together with the

$$g_{i,i}\colon (e_1,\dots,e_n) \mapsto (e_1,\dots,e_{i-1},e_n,e_i,e_{i+1},\dots,e_{n-1}) \quad (1 \le i \le n).$$

Furthermore, for $i < j$, the corresponding involution $\psi_{g_{i,j}}$ is conjugation by the permutation matrix associated with the transposition $(i\,j)$; and $\psi_{g_{i,i}}$ is conjugation by $\operatorname{diag}(1,\dots,1,-1,1,\dots,1)$ where -1 occurs at the i-th place. As a consequence, for a subset Π of Δ, we have: $g_{i,j} \in \mathcal{V}^\Pi$ if and only if α_{i-1} and α_j are not in Π.

We sketch a geometric interpretation of this result. Consider G/B as the variety of complete flags

$$\underline{V} = (V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = k^n)$$

where each V_i is a linear subspace of dimension i. Observe that G^θ is the isotropy group in G of the pair (ℓ, H) where ℓ is the line spanned by e_n, and H is the hyperplane spanned by $e_1,\dots,e_{n-1}$. For $1 \le i \le j \le n$, set

$$X_{i,j} := \{\underline{V} \in G/B \mid \ell \subset V_j \text{ and } V_{i-1} \subset H\}.$$

Then one checks that the $X_{i,j}$ are the G^θ-orbit closures in G/B. More precisely, denoting by $\mathcal{O}_{i,j}$ the G^θ-orbit of $g_{i,j}B$ in G/B, we have

$$X_{i,j} = \overline{\mathcal{O}_{i,j}} = \mathcal{O}_{i,j} \cup X_{i+1,j} \cup X_{i,j-1}$$

where $X_{a,b}$ is empty if $a > b$. In particular, the closed orbits are the $X_{i,i} = \mathcal{O}_{i,i}$ $(1 \le i \le n)$.

The right stabilizer of $\overline{G^\theta g_{i,j}B}$ is the largest parabolic subgroup $P^{i,j} = P \supseteq B$ such that $X_{i,j}$ is the pull-back of a subvariety of G/P under the projection $G/B \to G/P$. As a consequence, we see that $P^{i,j}$ is generated by the P_α's with $\alpha \notin \{\alpha_{i-1}, \alpha_j\}$.

3 Closed Orbits

3.1 Parametrization of Closed Orbits

For simplicity, we assume from now on that G^θ is connected; by [17], this holds if G is semisimple and simply connected. In order to describe closed G^θ-orbits in G/P, it will be convenient to choose a *standard pair* (B,T), that is, $B \subseteq G$ is a θ-stable Borel subgroup, and $T \subseteq B$ is a θ-stable maximal torus (such pairs exist by [17, Theorem 7.5]). Then T^θ is a regular subtorus of G by Lemma 4, and hence a maximal subtorus of G^θ. Furthermore, B^θ is a Borel subgroup of G^θ by Lemma 3.

With notation as in 2.1, the θ-action on Φ stabilizes Φ^+ and hence Δ. Let $P = P_\Pi$ be a parabolic subgroup of G containing B; then $\theta(P) = P_{\theta(\Pi)}$. Finally, for $g \in \mathcal{V}$, recall that w_g denotes the image in W of $g^{-1}\theta(g)$.

Proposition 8 *For $g \in \mathcal{V}$, the following conditions are equivalent:*

(i) $G^\theta gP$ *is closed in* G.
(ii) $P \cap \psi_g(P)$ *is a parabolic subgroup of* G.
(iii) $w_g \in W_\Pi W_{\theta(\Pi)}$.

In particular, $G^\theta gB$ is closed in G if and only if $w_g = 1$, that is, $g^{-1}\theta(g) \in T$ (this follows also from Corollary 1).

Proof (i)⇒(ii) Observe that $G^{\psi_g}P$ is closed in G, whence G^{ψ_g}/P^{ψ_g} is closed in G/P. Thus, P^{ψ_g} contains a Borel subgroup B' of G^{ψ_g}. In turn, B' is contained in a Borel subgroup B'' of P. Then B'' is ψ_g-stable by Lemma 3. Thus, $P \cap \psi_g(P) \supseteq B''$ is a parabolic subgroup of G.

(ii)⇒(iii) Because $P \cap \psi_g(P)$ contains T, it contains a Borel subgroup xBx^{-1} for some $x \in W$. Then $x \in W_\Pi$ (because $xBx^{-1} \subseteq P$) and $x(\Phi^+) \subseteq \psi_g(\Phi^+ \cup \Phi_\Pi)$ (because $xBx^{-1} \subseteq \psi_g(P)$). But $\psi_g = w_g\theta$ and Φ^+ is θ-stable. Thus,

$$\theta w_g^{-1} x\theta(\Phi^+) \subseteq \Phi^+ \cup \Phi_\Pi.$$

Because $\theta w_g^{-1} x\theta \in W$, we must have $\theta w_g^{-1} x\theta \in W_\Pi$, that is, $w_g^{-1}x \in W_{\theta(\Pi)}$. We conclude that $w_g \in W_\Pi W_{\theta(\Pi)}$.

(iii)⇒(i) is checked by reversing the previous arguments. ■

The statement (i)⇔(ii) also follows from [9, Lemma 1.7].

To parametrize the closed double cosets, we need more notation. Let

$$q \colon N \to N/T = W$$

be the quotient map; then $q(N^\theta)$ is a subgroup of W^θ. Because T^θ is a regular subtorus of T, we have

$$N_{G^\theta}(T^\theta) = N_{G^\theta}(T) = N^\theta.$$

It follows that $q(N^\theta)$ is isomorphic to the Weyl group $W(G^\theta, T^\theta)$.

Finally, let

$$Q = P \cap \theta(P) = P_{\Pi \cap \theta(\Pi)}$$

be the largest θ-stable parabolic subgroup contained in P. Then θ acts on G/Q.

Proposition 9

(i) *Any closed (G^θ, P)-double coset in G meets $q^{-1}(W^\theta)$, along a unique $\big(N^\theta, q^{-1}(W_\Pi^\theta)\big)$-double coset. This defines a bijection from the set of closed G^θ-orbits in G/P, onto $q(N^\theta) \backslash W^\theta / W_\Pi^\theta$.*
(ii) *The union of all closed G^θ-orbits in G/Q is the subset of all θ-fixed points; under the projection $G/Q \to G/P$, this subset is mapped isomorphically to the union of all closed G^θ-orbits in G/P.*

Proof Let $G^\theta gP \subseteq G$ be a closed double coset. As it contains a closed (G^θ, B)-double coset, we may assume that $G^\theta gB$ is closed in G, too. Then the G^θ-orbit $G^\theta gB/B$ is closed in G/B; thus, it contains a fixed point of B^θ. So we may assume further that $B^\theta \subseteq gBg^{-1}$. Then gBg^{-1} is θ-stable by Lemma 3. Furthermore, gBg^{-1} contains the regular torus T^θ, whence it contains T. It follows that $g \in NB$; we may assume further that $g \in N$. Now, because gBg^{-1} is θ-stable, we have $\theta(g) \in gB$. Thus, $g \in q^{-1}(W^\theta)$. Conversely, if $g \in q^{-1}(W^\theta)$ then $G^\theta gP$ is closed in G by Proposition 8.

Let now $g' \in G^\theta gP \cap q^{-1}(W^\theta)$. Then g' normalizes T^θ and hence $g'P/P$ is a T^θ-fixed point in $G^\theta gP/P$. The latter is a complete homogeneous space under G^θ. Thus, $g' \in N_{G^\theta}(T^\theta)gP = N^\theta gP$. Because g and g' are in $q^{-1}(W^\theta)$, it follows that g' is in $N^\theta g\big(P \cap q^{-1}(W^\theta)\big) = N^\theta g q^{-1}(W^\theta_\Pi)$. This proves (i).

For the first assertion of (ii), let $G^\theta gQ$ be a closed double class. We may assume that $g \in q^{-1}(W^\theta)$ by (i). Then $g^{-1}\theta(g) \in T$ whence $\theta(gQ) = gQ$: any closed G^θ-orbit in G/Q consists of θ-fixed points. Conversely, let $g \in G$ such that $gQ \in G/Q$ is θ-fixed; we may assume that $g \in \mathcal{V}$. Then gQg^{-1} is θ-stable, whence $g^{-1}\theta(g) \in Q$. But $g^{-1}\theta(g) \in N$ so that $g^{-1}\theta(g) \in N \cap L \cap \theta(L)$, and $w_g \in W_{\Pi\cap\theta(\Pi)}$. By Proposition 8, $G^\theta gQ$ is closed in G.

For the second assertion of (ii), observe that

$$W^\theta_\Pi = \big(W_\Pi \cap \theta(W_\Pi)\big)^\theta = W^\theta_{\Pi\cap\theta(\Pi)}.$$

Thus, the map $G/Q \to G/P$ induces a bijection on the subsets of closed orbits. Furthermore, for $g \in q^{-1}(W^\theta)$, we have:

$$\begin{aligned} G^\theta gQ/Q \simeq G^\theta/(gQg^{-1})^\theta &= G^\theta/\big(gPg^{-1} \cap \theta(gPg^{-1})\big)^\theta \\ &= G^\theta/(gPg^{-1})^\theta \simeq G^\theta gP/P \end{aligned}$$

because $\theta(gPg^{-1}) = g\theta(P)g^{-1}$. So the map $G^\theta gQ/Q \to G^\theta gP/P$ is an isomorphism. ∎

3.2 Standard Representatives

We begin by constructing a set of representatives for closed (G^θ, P)-double cosets in G or, equivalently, for $\big(q(N^\theta), W^\theta_\Pi\big)$-double cosets in W^θ. An element $w \in W^\theta$ will be called *standard* if $(wBw^{-1})^\theta = B^\theta$.

Proposition 10 *For any $w \in W^\theta$, the double coset $q(N^\theta)wW^\theta_\Pi$ contains a unique standard $u \in W^\theta$ such that $u(\Pi) \subseteq \Phi^+$.*

Proof By Proposition 8, $G^\theta wB$ is closed in G. Thus, $G^\theta wB/B$ is a closed G^θ-orbit in G/B, with wB/B as a T^θ-fixed point. It follows that there exists $x \in N^\theta$ such that xwB/B is fixed by B^θ. In other words, $B^\theta = (xwBw^{-1}x^{-1})^\theta$. Replacing w by $q(x)w$, we may assume that w is standard. Then there exist unique u, v in W such that: $u(\Pi) \subseteq \Phi^+$, $v \in W_\Pi$ and $w = uv$. Because θ stabilizes Π and Φ^+, it follows that u and v are in W^θ.

We claim that $(wUw^{-1})^\theta = (uUu^{-1})^\theta$; then u will be a standard representative of w. For this, denote by L_Π the Levi subgroup of P_Π containing T, and set $U_\Pi = U \cap L_\Pi$. Observe

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that $(wU_\Pi w^{-1})^\theta \subseteq U$. But $wU_\Pi w^{-1} \subseteq uL_\Pi u^{-1}$, and $uL_\Pi u^{-1} \cap U = uU_\Pi u^{-1}$ because $u(\Pi) \subseteq \Phi^+$. Thus,

$$(wU_\Pi w^{-1})^\theta \subseteq (uU_\Pi u^{-1})^\theta.$$

Furthermore,

$$wR_u(P_\Pi)w^{-1} = uR_u(P_\Pi)u^{-1}$$

because $v \in W_\Pi$. As wUw^{-1} is the semi-direct product of the θ-stable normal subgroup $wR_u(P_\Pi)w^{-1}$ with the θ-stable subgroup $wU_\Pi w^{-1}$, it follows that

$$(wUw^{-1})^\theta \subseteq (uUu^{-1})^\theta.$$

But $(wUw^{-1})^\theta = U^\theta$ is a maximal unipotent subgroup of G^θ, which implies our claim.

Let u' be another standard representative of w such that $u'(\Pi) \subseteq \Phi^+$. Then $u'B/B$ is a B^θ-fixed point in $G^\theta uP_\Pi/B$. Under the map $G/B \to G/P_\Pi$, the latter is mapped to $G^\theta uP_\Pi/P_\Pi$, a complete G^θ-orbit with a unique B^θ-fixed point uP_Π/P_Π. Thus, $u'B/B$ is in the fiber uP_Π/B, that is, $u' \in uP_\Pi$. Because u and u' are in W, we have $u' \in uW_\Pi$. It follows that $u' = u$, as both $u(\Pi)$ and $u'(\Pi)$ are contained in Φ^+. ■

We now give two characterizations of standard elements. As in 2.2, denote by Φ_c (resp. Φ_C) the set of all compact (resp. complex) roots for θ; there are no real roots because Φ^+ is θ-stable. Let $\Delta_i \subseteq \Delta$ be the subset of all imaginary simple roots; then θ acts trivially on Φ_{Δ_i}.

Proposition 11 *For $w \in W^\theta$, the following conditions are equivalent:*

(i) *w is standard.*
(ii) *$\Phi_c^+ \cup \Phi_C^+ \subseteq w(\Phi^+)$.*
(iii) *$w \in W_{\Delta_i}$ and $\Phi_{\Delta_i}^+ \cap \Phi_c \subseteq w(\Phi_{\Delta_i}^+)$.*

Proof (i)⇔(ii) As in the proof of Proposition 8, observe that w is standard if and only if $U^\theta \subseteq wUw^{-1}$, that is, $\mathfrak{u}^\theta \subseteq w\mathfrak{u}w^{-1}$. Furthermore, a basis of $\mathfrak{u}^\theta$ consists of the X_α ($\alpha \in \Phi_c^+$) together with the $X_\alpha + \theta(X_\alpha)$ ($\alpha \in \Phi_C^+$). This basis is contained in $w\mathfrak{u}w^{-1}$ if and only if $\Phi_c^+ \cup \Phi_C^+ \subseteq w(\Phi^+)$, because θ stabilizes Φ_C^+ and commutes with w.

(ii)⇒(iii) We argue by induction on the length $l(w)$. The case where $w = 1$ is trivial. Otherwise, we can find $\alpha \in \Delta$ and $\tau \in W$ such that $w = s_\alpha\tau$ and $l(w) = l(\tau) + 1$ where l is the length function on W. Then $w^{-1}(\alpha) \in \Phi^-$; thus, $\alpha \notin \Phi_c^+ \cup \Phi_C^+$, that is, α is non-compact imaginary. In particular, $\alpha \in \Delta_i$; as a consequence, $\tau \in W^\theta$. Furthermore,

$$\Phi^+ \cap w(\Phi^+) = \big(\Phi^+ \cap \tau(\Phi^+)\big) - \{\alpha\}.$$

Thus, $\Phi_c^+ \cup \Phi_C^+$ is contained in $\tau(\Phi^+)$. By the induction hypothesis, $\tau \in W_{\Delta_i}$ whence $w \in W_{\Delta_i}$ as well. It follows that

$$\Phi_{\Delta_i,c}^+ \subseteq w(\Phi^+) \cap \Phi_{\Delta_i} = w(\Phi_{\Delta_i}^+).$$

(iii)⇒(ii) If $w \in W_{\Delta_i}$ then w stabilizes $\Phi^+ - \Phi_{\Delta_i}$. The latter contains all positive complex roots. ■

Examples We determine the standard elements in the cases considered in Section 2.4.

1) The pair (B, T) is standard. As there are no imaginary roots, the identity is the unique standard element. This agrees with the fact that the unique closed orbit of $\operatorname{diag}(\mathbf{G})$ in $\mathbf{G}/\mathbf{P}_1 \times \mathbf{G}/\mathbf{P}_2$ is the orbit of the base point, isomorphic to $\mathbf{G}/\mathbf{P}_1 \cap \mathbf{P}_2$.

2)′ We modify slightly Example 2, because the pair (B, T) is not standard there, and G^θ is not always connected. As in [13, 10.3], consider $G = \mathrm{SL}_n$ with involution θ given by $\theta(g) = \operatorname{Int}(d_0)(g^{-1})^t$, where $d_0 \in \mathrm{GL}_n$ maps each e_i to e_{n+1-i}. Then G^θ is the special orthogonal group for the quadratic form $q(x_1, \dots, x_n) = \sum_{i=1}^n x_i x_{n+1-i}$. The pair (B, T) is standard, and θ acts on roots by $\theta(\alpha_i) = \alpha_{n-i}$. If n is odd, then the set Δ_i is empty, and the unique standard element is the identity. If $n = 2n'$ is even, then Δ_i consists of the non-compact root $\alpha_{n'}$; thus, the standard elements are 1 and the transposition $(n', n'+1)$. This agrees with the fact that $\mathrm{SO}_{2n'}$ has two closed orbits in the Grassmanian of n'-dimensional subspaces of $k^{2n'}$, associated with two types of null subspaces.

3) The pair (B, T) is standard, and all roots are imaginary; the compact roots are the pairs (i, j) with $1 \le i, j \le n-1$. Thus, $w \in S_n$ is standard if and only if $w^{-1}(1) < w^{-1}(2) < \cdots < w^{-1}(n-1)$, that is, w is the image in S_n of $g_{i,i}$ for some i, $1 \le i \le n$; denote this image by w_i.

If Π is the complement of $\{\alpha_{i-1}, \alpha_j\}$ in Δ, then the standard elements w such that $w(\Pi) \subset \Phi^+$ are 1, w_{i-1} and w_j. They represent the three closed G^θ-orbits in $G/P_\Pi = G/P^{i,j}$, consisting of all pairs $(V_{i-1} \subset V_j)$ such that $V_j \subset H$ (resp. $\ell \subset V_{i-1}$; $V_{i-1} \subset H$ and $\ell \subset V_j$.)

Let $\pi_{i,j} \colon G/B \to G/P^{i,j}$ be the projection. Geometrically, $\pi_{i,j}$ maps each complete flag $\underline{V}$ to $(V_{i-1} \subset V_j)$. Thus, the orbit closure $X_{i,j}$ is the pull-back via $\pi_{i,j}$ of the closed orbit $G^\theta w_j P^{i,j}/P^{i,j}$. The latter identifies, via the map $(V_{i-1} \subset V_j) \mapsto (V_{i-1} \subset V_j \cap H)$, to the variety of partial flags of dimensions $i-1$, $j-1$ in H. And each fiber of

$$\pi_{i,j} \colon X_{i,j} \to G^\theta w_j P^{i,j}/P^{i,j}$$

is isomorphic to the complete flag variety for $\mathrm{GL}_{i-1} \times \mathrm{GL}_{j-i+1} \times \mathrm{GL}_{n-j}$, a Levi subgroup of $P^{i,j}$.

Thus, each orbit closure of GL_{n-1} in GL_n/B is an "induced flag variety".

3.3 θ-Stable Parabolic Subgroups

As an application of the results in 3.1 and 3.2, we describe the G^θ-conjugacy classes of θ-stable parabolic subgroups, and their relation to parabolic subgroups of G^θ.

Theorem 2 *Let $Q \subseteq G$ be a θ-stable parabolic subgroup; let Π be the subset of Δ such that Q is G-conjugate to P_Π. Then Π is θ-stable, and Q is G^θ-conjugate to $wP_\Pi w^{-1}$ for a unique standard $w \in W^\theta$ such that $w(\Pi) \subseteq \Phi^+$.*

As a consequence, $Q^\theta \subseteq G^\theta$ is a parabolic subgroup, G^θ-conjugate to $(wP_\Pi w^{-1})^\theta$. Conversely, any parabolic subgroup of G^θ is G^θ-conjugate to $(wP_\Pi w^{-1})^\theta$ for some Π and w as above.

Proof Let $g \in G$ such that $Q = gP_\Pi g^{-1}$. Moving g in its (G^θ, B)-double coset, we may assume that $g \in \mathcal{V}$. As Q is θ-stable, we have $(w_g\theta)(P_\Pi) = P_\Pi$. In terms of roots, this means that $(w_g\theta)(\Phi^+ \cup \Phi_\Pi) = \Phi^+ \cup \Phi_\Pi$. Thus,

$$\theta w_g \theta(\Phi^+) \subseteq \Phi^+ \cup \Phi_{\theta(\Pi)}.$$

Because $\theta w_g \theta \in W$, it follows that $\theta w_g \theta \in W_{\theta(\Pi)}$ and that $w_g \in W_\Pi$, whence

$$\theta(P_\Pi) = w_g^{-1}(P_\Pi) = P_\Pi.$$

Thus, Π is θ-stable.

Now the θ-stable G-conjugates of P_Π are the θ-fixed points in G/P_Π, that is, the points with closed G^θ-orbit by Proposition 8. By Propositions 9 and 10, there exists $h \in G^\theta$ and a unique standard $w \in W^\theta$ such that $w(\Pi) \subseteq \Phi^+$ and that $Q = hwP_\Pi w^{-1}h^{-1}$. Then

$$Q^\theta = h(wP_\Pi w^{-1})^\theta h^{-1} \supseteq hB^\theta h^{-1}$$

so that Q^θ is a parabolic subgroup of G^θ (this follows also from Lemma 3).

Conversely, let $\Gamma \subseteq G^\theta$ be a parabolic subgroup. For a multiplicative one-parameter subgroup $\lambda\colon \mathbf{G}_m \to G$, set

$$G(\lambda) := \{g \in G \mid \lim_{t\to 0} \lambda(t) g \lambda(t^{-1}) \text{ exists}\}.$$

Then $G(\lambda)$ is a parabolic subgroup of G; moreover, all parabolic subgroups of G are obtained in this way. Applying this to the connected reductive group G^θ, we obtain $\lambda\colon \mathbf{G}_m \to G^\theta$ such that $\Gamma = G^\theta(\lambda)$. Then $Q := G(\lambda)$ is a θ-stable parabolic subgroup of G, and $Q^\theta = \Gamma$. ∎

Remark Given a parabolic subgroup Γ of G^θ containing B^θ, there may exist several θ-stable parabolic subgroups Q such that $Q^\theta = \Gamma$ (*e.g.* if $\Gamma = B^\theta$ and there are several standard elements). And there may exist no parabolic subgroup P of G *containing* B such that $P^\theta = \Gamma$.

Consider for example $G = \mathrm{SP}_4$, the group which preserves the symplectic form $(\ ,\ )$ on k^4 such that $(e_1, e_4) = (e_2, e_3) = 1$ and $(e_i, e_j) = 0$ if $i + j \neq 5$. Let B (resp. T) be the standard Borel subgroup (resp. maximal torus) of G. Let θ be the conjugation by $\mathrm{diag}(1, -1, -1, 1)$, then $G^\theta = \mathrm{SL}_2 \times \mathrm{SL}_2$ contains T, and the pair (B, T) is standard. Let α, β be the simple roots of (G, T) where α is short; then the roots of (G^θ, T) are $\pm\beta, \pm(2\alpha+\beta)$. Let Γ be the parabolic subgroup of G^θ containing T, with roots β and $\pm(2\alpha + \beta)$; then Γ contains B^θ but is not contained in a proper parabolic subgroup $P \supseteq B$.

4 Orbit Closures and Restriction of Representations

4.1 Induced Flag Varieties

From now on, we assume that the characteristic of the ground field k is zero. As in Section 3, we also assume that G^θ is connected, and we choose a standard pair (B, T). Let P be a θ-stable parabolic subgroup containing B; let

$$\pi\colon G/B \to G/P$$

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be the projection. The pull-back under π of a closed G^θ-orbit will be called an *induced flag variety*.

Recall that any closed G^θ-orbit in G/P can be written as $G^\theta wP/P$ for a unique standard $w \in W^\theta$ such that $w(\Pi) \subseteq \Phi^+$. Because $w \in W^\theta$, the group

$$Q := wPw^{-1}$$

is a θ-stable parabolic subgroup of G, with

$$M := wLw^{-1}$$

as a θ-stable Levi subgroup containing T. Furthermore, Q^θ contains B^θ (because w is standard), and

$$B \cap M = w(B \cap L)w^{-1}$$

(because $w(\Pi) \subseteq \Phi^+$). It follows that $(B \cap M)^\theta$ is a Borel subgroup of M^θ. The latter is a Levi subgroup of Q^θ.

Set

$$X := \pi^{-1}(G^\theta wP/P) = G^\theta wP/B.$$

Then the image of X under π is the homogeneous space $G^\theta wP/P \simeq G^\theta/Q^\theta$, and the fiber $\pi^{-1}(wP/P)$ is isomorphic to $wP/B = wL/B\cap L$. This isomorphism is Q-equivariant, where Q acts on $wL/B \cap L$ through the quotient map $Q \to Q/R_u(Q) \simeq M$. It follows that

$$X \simeq G^\theta \times_{Q^\theta} (wL/B \cap L) \simeq G^\theta \times_{Q^\theta} (M/B \cap M)$$

where Q^θ acts on the flag variety $M/B \cap M$ through M^θ. This explains the terminology of "induced flag variety".

Let λ be a character of T; then it extends uniquely to a character of B, also denoted by λ. Let $\mathcal{L}_\lambda$ be the associated line bundle on G/B. Then

$$H^0(G/B, \mathcal{L}_\lambda) = \operatorname{Ind}_B^G(-\lambda)$$

(the induced module from B to G of the one-dimensional B-module with weight $-\lambda$). This is a simple G-module with lowest weight $-\lambda$, if λ is dominant (against roots of B); otherwise, $H^0(G/B, \mathcal{L}_\lambda) = 0$.

Theorem 3 *Let X be as above and let λ be a dominant character of T.*

(i) The restriction map

$$\operatorname{res}_X \colon H^0(G/B, \mathcal{L}_\lambda) \to H^0(X, \mathcal{L}_\lambda)$$

is surjective, and $H^i(X, \mathcal{L}_\lambda) = 0$ for all $i \geq 1$.

(ii) We have an isomorphism of G^θ-modules

$$H^0(X, \mathcal{L}_\lambda) \cong \operatorname{Ind}_{Q^\theta}^{G^\theta} H^0(M/B \cap M, \mathcal{L}_{w(\lambda)})$$

where Q^θ acts on $H^0(M/B \cap M, \mathcal{L}_{w(\lambda)})$ via the quotient map $Q^\theta \to M^\theta$.

(iii) *The M^θ-module $H^0(M/B \cap M, \mathcal{L}_{w(\lambda)})$ is a direct sum of simple modules with G^θ-antidominant lowest weights.*

(iv) *The kernel of res_X is a direct sum of simple G^θ-modules with lowest weights of the form $\mu + \nu$ where μ is the lowest weight of a simple M^θ-submodule of $H^0(M/B \cap M, \mathcal{L}_{w(\lambda)})$, and ν is the restriction to T^θ of a non-trivial sum of non-compact roots in $w(\Phi^+ - \Phi_\Pi)$.*

Proof Under the isomorphism $X \simeq G^\theta \times_{Q^\theta} (M/B \cap M)$, the restriction of $\mathcal{L}_\lambda$ to X identifies with $G^\theta \times_{Q^\theta} \mathcal{L}_{w(\lambda)}$, the G^θ-linearized line bundle whose restriction to $M/B \cap M$ is $\mathcal{L}_{w(\lambda)}$. This implies (ii).

Composing res_X with the restriction map

$$r' \colon H^0(X, \mathcal{L}_\lambda) \to H^0(wP/B, \mathcal{L}_\lambda) \simeq H^0(M/B \cap M, \mathcal{L}_{w(\lambda)}),$$

we obtain the restriction map

$$r'' \colon H^0(G/B, \mathcal{L}_\lambda) \to H^0(M/B \cap M, \mathcal{L}_{w(\lambda)}).$$

Observe that $H^0(M/B \cap M, \mathcal{L}_{w(\lambda)})$ is a simple M-module with lowest weight $-w(\lambda)$. Furthermore, r'' is non-zero (because $\mathcal{L}_\lambda$ is generated by its global sections) whence r'' is surjective. Thus, the same holds for r'. Decompose the M^θ-module $H^0(M/B \cap M, \mathcal{L}_{w(\lambda)})$ into a direct sum of simple submodules; each of them is of the form

$$\mathrm{Ind}_{(B \cap M)^\theta}^{M^\theta}(-\omega) = \mathrm{Ind}_{B^\theta}^{Q^\theta}(-\omega).$$

By (ii), the G^θ-module $H^0(X, \mathcal{L}_\lambda)$ decomposes into the direct sum of the corresponding induced modules

$$\mathrm{Ind}_{Q^\theta}^{G^\theta} \mathrm{Ind}_{B^\theta}^{Q^\theta}(-\omega) = \mathrm{Ind}_{B^\theta}^{G^\theta}(-\omega).$$

Because r' is surjective, all these induced modules are non-zero. Thus, their lowest weight vectors $\mu = -\omega$ are G^θ-antidominant, which proves (iii). Furthermore, by surjectivity of r'', the image of res_X meets all these induced modules. Because the latter are simple, res_X is surjective.

To prove vanishing of $H^i(X, \mathcal{L}_\lambda)$ for $i \geq 1$, observe that $R^j \pi_* \mathcal{L}_\lambda = 0$ for all $j \geq 1$, because λ is dominant. Thus, we obtain isomorphisms

$$H^i(X, \mathcal{L}_\lambda) \simeq H^i(G^\theta wP/P, \pi_* \mathcal{L}_\lambda) = H^i(G^\theta/Q^\theta, \pi_* \mathcal{L}_\lambda).$$

The restriction of $\pi_* \mathcal{L}_\lambda$ to the G^θ-orbit G^θ/Q^θ is the homogeneous vector bundle associated with the Q^θ-module $H^0(M/B \cap M, \mathcal{L}_{w(\lambda)})$. By (iii), this module is semisimple and its lowest weights are G^θ-antidominant. So $H^i(G^\theta/Q^\theta, \pi_* \mathcal{L}_\lambda) = 0$ for $i \geq 1$, by Bott's theorem.

Let $\mathcal{I} \subset \mathcal{O}_{G/B}$ be the ideal sheaf of X in G/B, then the kernel of res_X is $H^0(G/B, \mathcal{I} \otimes \mathcal{L}_\lambda)$. To study the lowest weight vectors of this G^θ-module, we embed it into a larger module, as follows. Let P^- be the parabolic subgroup of G such that $P^- \cap P = L$; set $Q^- := wP^- w^{-1}$. Then G/B contains

$$Q^- wP/B = wP^- P/B$$

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as an open affine subset, stable under Q^-. Thus, the restriction map

$$H^0(G/B, \mathcal{I} \otimes \mathcal{L}_\lambda) \to H^0(Q^- wP/B, \mathcal{I} \otimes \mathcal{L}_\lambda)$$

is injective, and equivariant for the action of $(Q^-)^\theta$. The latter is a parabolic subgroup of G^θ, with unipotent radical $R_u(Q^-)^\theta$ and Levi subgroup M^θ (because Q^- is a θ-stable parabolic subgroup of G). Furthermore, $(Q^-)^\theta$ meets Q^θ along M^θ, their common Levi subgroup containing T^θ. Thus, Q^θ and $(Q^-)^\theta$ are opposite parabolic subgroups of G^θ.

Let B^- be the Borel subgroup of G such that $B^- \cap B = T$. Then B^- is θ-stable, and $(B^-)^\theta$ is the Borel subgroup of G^θ such that $(B^-)^\theta \cap B^\theta = T^\theta$. Because B^θ is contained in Q^θ, it follows that $(B^-)^\theta$ is contained in $(Q^-)^\theta$. Thus, $(B^-)^\theta$ is the semidirect product of $R_u(Q^-)^\theta$ with

$$(B^- \cap M)^\theta = (B^- \cap wLw^{-1})^\theta = \big(w(B^- \cap L)w^{-1}\big)^\theta$$

(indeed, $B^- \cap wLw^{-1} = w(B^- \cap L)w^{-1}$ because $w(\Pi) \subseteq \Phi^+$).

By the Bruhat decomposition, the product map

$$R_u(Q^-) \times wP/B \to Q^- wP/B$$

is an isomorphism. Combining this with Lemma 1(i), we obtain a $(Q^-)^\theta$-equivariant isomorphism

$$R_u(Q^-)^\theta \times \tau\big(R_u(Q^-)\big) \times wL/B \cap L \simeq Q^- wP/B$$

which restricts to an equivariant isomorphism

$$R_u(Q^-)^\theta \times \{1\} \times wL/B \cap L \simeq (G^\theta wP \cap Q^- wP)/B.$$

Let $p_2 \colon Q^- wP/B \to \tau\big(R_u(Q^-)\big)$ and $p_3 \colon Q^- wP/B \to wL/B \cap L$ be the corresponding projection maps. Let I be the ideal of $k[R_u(Q^-)]$ (the algebra of regular functions on $R_u(Q^-)$) consisting of functions that vanish at 1. Then the isomorphism above identifies $\mathcal{I}|_{Q^- wP/B}$ with $p_2^* I$, and $\mathcal{L}_\lambda|_{Q^- wP/B}$ with $p_3^* \mathcal{L}_\lambda$. Thus, we obtain a $(Q^-)^\theta$-equivariant isomorphism

$$H^0(Q^- wP/B, \mathcal{I} \otimes \mathcal{L}_\lambda) \simeq k[R_u(Q^-)^\theta] \otimes I \otimes H^0(wL/B \cap L, \mathcal{L}_\lambda).$$

It identifies the subset of $(B^-)^\theta$-eigenvectors in the left hand side (that is, the subset of lowest weight vectors), with the subset of $(B^- \cap M)^\theta$-eigenvectors in

$$I \otimes H^0(wL/B \cap L, \mathcal{L}_\lambda) = I \otimes H^0(M/B \cap M, \mathcal{L}_{w(\lambda)}).$$

The latter being the tensor product of two M^θ-modules, each of its lowest weights is the sum of a weight of T^θ in $\mathcal{I}$ with a lowest weight of $H^0(M/B \cap M, \mathcal{L}_{w(\lambda)})$.

To complete the proof, we check that the weights of T^θ in $\mathcal{I}$ are non-trivial sums of non-compact roots in $w(\Phi^+ - \Phi_\Pi)$. Indeed, the T-variety $R_u(Q^-)$ is isomorphic to a module with set of weights $w(\Phi^- - \Phi_\Pi)$. Thus, the T^θ-variety $\tau\big(R_u(Q^-)\big)$ is isomorphic to a module with weights $\alpha|_{T^\theta}$ where α is a non-compact element of $w(\Phi^- - \Phi_\Pi)$. Furthermore, the weights of T^θ in I are non-trivial sums of opposites of weights in $\tau\big(R_u(Q^-)\big)$. ∎

For λ as above, let V_λ be the dual of the G-module $H^0(G/B,\mathcal{L}_\lambda)$ and let $\mathcal{C}_\lambda \subseteq V_\lambda$ be the G-orbit closure of a highest weight vector. If λ is regular, then $\mathcal{C}_\lambda$ is the affine cone over G/B for its projective embedding associated with $\mathcal{L}_\lambda$; this cone is smooth outside the origin.

Recall that $\mathcal{C}_\lambda$ is normal, with a rational singularity at the origin (see [12] for a proof in arbitrary characteristics). We shall see that the same holds for the affine cone $\tilde{X}_\lambda \subseteq \mathcal{C}_\lambda$ over $X \subseteq G/B$; because X is smooth, $\tilde{X}_\lambda$ is smooth outside the origin.

Corollary 4 *Let X be as above and let λ be a regular dominant weight. Then $\tilde{X}_\lambda$ is normal, with a rational singularity at the origin.*

Proof Let

$$R = \bigoplus_{n=0}^{\infty} H^0(X, \mathcal{L}_\lambda^{\otimes n}) = \bigoplus_{n=0}^{\infty} H^0(X, \mathcal{L}_{n\lambda}).$$

Because X is smooth, the algebra R is normal. The algebra S of regular functions over $\tilde{X}_\lambda$ is the subalgebra of R generated by $H^0(X,\mathcal{L}_\lambda)$. But

$$\mathrm{res}_X \colon H^0(G/B,\mathcal{L}_{n\lambda}) \to H^0(X,\mathcal{L}_{n\lambda})$$

is surjective, and the graded algebra

$$\bigoplus_{n=0}^{\infty} H^0(G/B,\mathcal{L}_{n\lambda})$$

is generated by its elements of degree 1. It follows that $S = R$, that is, $\tilde{X}_\lambda$ is normal.

Let $p\colon Z \to \tilde{X}_\lambda$ be the blow-up of the origin. Then Z is the total space of the line bundle over X, dual of the restriction of $\mathcal{L}_\lambda$. It follows that Z is smooth, and that

$$H^i(Z,\mathcal{O}_Z) = \bigoplus_{n=0}^{\infty} H^i(X,\mathcal{L}_{n\lambda})$$

for all $i \geq 0$. By Theorem 3, we thus have $H^i(Z,\mathcal{O}_Z) = 0$ for $i \geq 1$. This means that $\tilde{X}_\lambda$ has rational singularities. ∎

4.2 Restriction of Representations

We begin by applying Theorem 3 to the decomposition of simple G-modules into G^θ-modules.

The map $T \to T^\theta\colon t \mapsto t\theta(t)$ is surjective, and its restriction to T^θ is the map $t \mapsto t^2$. Using this map, we shall identify the character group of T^θ with the set of all $\chi+\theta(\chi)$ where χ is a character of T.

Corollary 5 *Let ω be a G^θ-dominant character of T^θ and let λ be a dominant character of T. Then we have for multiplicities:*

$$[\mathrm{Ind}_B^G(-\lambda) : \mathrm{Ind}_{B^\theta}^{G^\theta}(-\omega)] \geq \big[\mathrm{Ind}_{B\cap M}^M\big(-w(\lambda)\big) : \mathrm{Ind}_{(B\cap M)^\theta}^{M^\theta}(-\omega)\big]$$

with equality if $\lambda+\theta(\lambda)-2w^{-1}(\omega)$ *is a sum of positive roots in* Φ_Π. *Furthermore, if* $\operatorname{Ind}_{B^\theta}^{G^\theta}(-\omega)$ *occurs in the* G^θ*-module* $\operatorname{Ind}_B^G(-\lambda)$*, then* $\lambda+\theta(\lambda)-2w^{-1}(\omega)$ *is a sum of positive roots.*

Proof The inequality follows from surjectivity of res_X and the structure of $H^0(X,\mathcal{L}_\lambda)$ (Theorem 3(i) and (ii).)

Assume moreover that $\lambda+\theta(\lambda)-2w^{-1}(\omega)$ is a sum of positive roots in Φ_Π. To prove equality, it is enough to check that $\operatorname{Ind}_{B^\theta}^{G^\theta}(-\omega)$ does not occur in the kernel of res_X. Otherwise, we can write $\omega=-\mu-\nu$ where $\operatorname{Ind}_{B^\theta}^{G^\theta}(\mu)$ occurs in $H^0(M/B\cap M,\mathcal{L}_{w(\lambda)})$, and ν is a sum of roots in $w(\Phi^+-\Phi_\Pi)$ (Theorem 3(iv).) In particular, μ is a weight of T^θ in $H^0(M/B\cap M,\mathcal{L}_{w(\lambda)})$. But each weight of T in that module can be written as $-w(\lambda)+\chi$ where χ is a sum of elements of $w(\Phi_\Pi^+)$. It follows that

$$w(\lambda)+\theta\big(w(\lambda)\big)+2\mu=w\big(\lambda+\theta(\lambda)\big)+2\mu$$

is a sum of elements of $w(\Phi_\Pi^+)$. Thus,

$$\lambda+\theta(\lambda)-2w^{-1}(\omega)=\lambda+\theta(\lambda)+2w^{-1}(\mu)+2w^{-1}(\nu)$$

is a sum of positive roots, not all in Φ_Π, a contradiction.

The proof of the latter assertion is similar. ■

Define a polytope $\mathcal{C}(G,\theta,\lambda)$ as the convex hull of the set of all G^θ-dominant weights ω such that $\operatorname{Ind}_{B^\theta}^{G^\theta}(\omega)$ occurs in the G^θ-module $\operatorname{Ind}_B^G(\lambda)$. Applying Corollary 5 with $\Pi=\varnothing$, we see that $w(\lambda)$ is a vertex of $\mathcal{C}(G,\theta,\lambda)$ and that the corresponding multiplicity is 1. More generally, for a subset $\Pi\subseteq\Delta$ such that $w(\Pi)\subset\Phi^+$, we see that $\mathcal{C}\big(wL_\Pi w^{-1},\theta,w(\lambda)\big)$ is a face of $\mathcal{C}(G,\theta,\lambda)$ and that the multiplicity functions agree on that face. This will be developed elsewhere, in relation to "moment polytopes" [4].

For a reductive subgroup K of G, the pair (G,K) is *multiplicity-free* if the multiplicity of any simple K-module in any simple G-module is at most 1. Equivalently, a Borel subgroup $B_K\subseteq K$ has a dense orbit in G/B.

By [10] or [5], any multiplicity-free pair with G semisimple and simply connected is a product of (the simply connected cover of) one of the following indecomposable pairs:

$$(\mathrm{SL}_n,\mathrm{GL}_{n-1}),\quad(\mathrm{SO}_n,\mathrm{SO}_{n-1}),\quad(\mathrm{SO}_8,\mathrm{Spin}_7).$$

In particular, multiplicity-free pairs are symmetric; their associated polytopes are described in [15]. We check that the corresponding orbit closures in flag varieties have a very nice structure.

Proposition 12 *If* (G,G^θ) *is multiplicity-free, then any* G^θ*-orbit closure* $X\subseteq G/B$ *is an induced flag variety; writing* $X=G^\theta\times_{Q^\theta}(M/B\cap M)$*, the pair* (M,M^θ) *is multiplicity-free as well. In particular, all* G^θ*-orbit closures in* G/B *are smooth.*

Proof We may assume that the pair (G, G^θ) is indecomposable. In the case of $(\mathrm{SL}_n, \mathrm{GL}_{n-1})$, our assertion has been checked in Example 3 in 3.2. Consider the case of $(\mathrm{SO}_n, \mathrm{SO}_{n-1})$ where $n = 2n'$ is even. Then G/B is the set of all flags

$$\underline{V} = (V_0 \subset V_1 \subset \cdots \subset V_{n'-1})$$

of null subspaces of $k^{2n'}$ of dimensions $0, 1, \ldots, n' - 1$. Let $H \subset k^{2n'}$ be the unique hyperplane stabilized by $\mathrm{SO}_{2n'-1}$. One checks that the $\mathrm{SO}_{2n'-1}$-orbit closures of $\mathrm{SO}_{2n'-1}$ in $\mathrm{SO}_{2n'}/B$ are the

$$X_i := \{\underline{V} \mid V_{i-1} \subset H\}$$

for $1 \le i \le n'$. In particular, $X_{n'}$ is the closed orbit, isomorphic to the flag variety of $\mathrm{SO}_{2n'-1}$. More generally, one checks that the map

$$\pi_i \colon \underline{V} \mapsto (V_0 \subset V_1 \subset \cdots \subset V_{i-1})$$

makes X_i an induced flag variety with $M/M^\theta = \mathrm{SO}_{2n'-2i} / \mathrm{SO}_{2n'-2i-1}$.

The case of $(\mathrm{SO}_n, \mathrm{SO}_{n-1})$ where $n = 2n' + 1$ is odd, is similar: the variety G/B is now the set of all flags

$$\underline{V} = (V_0 \subset V_1 \subset \cdots \subset V_{n'})$$

of null subspaces of dimensions $0, 1, \ldots, n'$. The orbit closures of $\mathrm{SO}_{2n'}$ in $\mathrm{SO}_{2n'+1}/B$ are the varieties $X_1, \ldots, X_{n'-1}$ defined as above, plus two varieties $X^1_{n'}$, $X^2_{n'}$ defined by: $V_{n'} \subset H$ (the unique hyperplane of $k^{2n'+1}$ stabilized by $\mathrm{SO}_{2n'}$), and $V_{n'}$ belongs to a fixed orbit under $\mathrm{SO}_{2n'}$ of n'-dimensional null subspaces of $k^{2n'}$ (there are two such orbits). Then $X^1_{n'}$ and $X^2_{n'}$ are the closed orbits, isomorphic to the flag variety of $\mathrm{SO}_{2n'}$; the other X_i's are induced flag varieties as above.

Finally, the analysis of $(\mathrm{SO}_8, \mathrm{Spin}_7)$ follows from that of $(\mathrm{SO}_8, \mathrm{SO}_7)$ by applying a triality automorphism. ∎

4.3 An Example Where res_X is Not Surjective

As in Example 2 in 3.2, consider $G = \mathrm{SL}_n$ with involution θ defined by $\theta(g) = (g^{-1})^t$. The standard Borel subgroup B of G is the isotropy group of the flag

$$k^1 \subset k^2 \subset \cdots \subset k^n$$

where each k^i is the span of the i first basis vectors of k^n. And G/B is the variety of complete flags

$$\underline{V} = (V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = k^n)$$

where each V_i is a linear subspace of dimension i.

For $1 \le i \le n - 1$, let $X_i \subset G/B$ be the subset of flags $\underline{V}$ such that restriction of q to V_i is degenerate (where q denotes the standard quadratic form on k^n). Then the pull-back of X_i in G is the subset of all g such that restriction of $g^{-1}q$ to k^i is degenerate, that is, the discriminant of $g^{-1}q|_{k^i}$ is zero. This discriminant is invariant for the action of SO_n by left multiplication, and is an eigenvector of weight $2\pi_i$ for the action of B by right multiplication; here π_i denotes the highest weight of the simple GL_n-module $\wedge^i k^n$. Thus, X_i is the divisor of a SO_n-invariant section of $\mathcal{L}_{2\pi_i}$. Observe that each X_i is irreducible if $n \geq 3$ (which we will assume from now on.)

Let λ be a weight, then we have an exact sequence of sheaves on G/B:

$$0 \to \mathcal{L}_{\lambda-2\pi_i} \to \mathcal{L}_\lambda \to \mathcal{L}_\lambda \otimes_{\mathcal{O}_{G/B}} \mathcal{O}_{X_i} \to 0.$$

If moreover λ is dominant, then $H^1(G/B, \mathcal{L}_\lambda) = 0$ and we obtain an exact sequence

$$H^0(G/B, \mathcal{L}_\lambda) \to H^0(X_i, \mathcal{L}_\lambda) \to H^1(G/B, \mathcal{L}_{\lambda-2\pi_i}) \to 0.$$

Now choose

$$\lambda = \sum_{j \neq i} x_j \pi_j$$

where the x_j are integers such that $x_j \geq 0$ if $|j - i| \geq 2$, and $x_j \geq 1$ if $|j - i| = 1$. Let $\alpha_1, \dots, \alpha_{n-1}$ be the simple roots and $s_1, \dots, s_{n-1}$ the corresponding simple reflections; let ρ be the half sum of positive roots. Then

$$s_i(\lambda - 2\pi_i + \rho) - \rho = \lambda - 2\pi_i + \alpha_i = \lambda - \sum_{j,|j-i|=1} \pi_j$$

is dominant, and hence $H^1(G/B, \mathcal{L}_{\lambda-2\pi_i})$ is non-zero by Bott's theorem. In other words, the restriction map

$$\mathrm{res}_{X_i} \colon H^0(G/B, \mathcal{L}_\lambda) \to H^0(X_i, \mathcal{L}_\lambda)$$

is not surjective.

Let $P \subset G$ be the stabilizer of the line k^1. Then G/P is the projective space of lines in k^n; it contains a unique closed SO_n-orbit $\mathcal{Q}$, the quadric ($q = 0$). Let $\pi \colon G/B \to G/P$ be the projection, then $X_1 = \pi^{-1}(\mathcal{Q})$; in particular, X_1 is smooth. Thus, Theorem 3 does not extend to all parabolic subgroups (here P is not conjugate to a θ-stable parabolic subgroup!).

Observe finally that res_{X_i} is surjective for all X_i as above, and all *regular* dominant weights λ. In fact, we do not know any example of a symmetric subgroup $G^\theta \subset G$, a G^θ-orbit closure $X \subset G/B$ and a regular dominant weight λ such that $\mathrm{res}_X \colon H^0(G/B, \mathcal{L}_\lambda) \to H^0(X, \mathcal{L}_\lambda)$ fails to be surjective.

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