PRIMITIVE SEQUENCES WHOSE ELEMENTS HAVE NO LARGE PRIME FACTORS

by IAN ANDERSON

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A sequence $a_1 < a_2 < \ldots$ of positive integers is said to be *primitive* if no element of the sequence divides any other. The study of primitive sequences arose naturally out of investigations into the subject of abundant numbers, where sequences each of whose elements is of the form $p_1^{\alpha_1} \ldots p_r^{\alpha_r}$, the p_i being fixed primes, are of particular importance. Such a sequence is said to be *built up* from the primes $p_1 \ldots p_r$. Thus Dickson [**1**], in an early paper on abundant numbers, proved that a primitive sequence built up from a fixed set of primes is necessarily finite.

Instead of limiting the size of the prime factors of the elements, as we do when we consider sequences built up from primes $p \leqq x$, we can limit the size of the elements themselves. Erdös, Sárkösy, and Szemerédi [**2**] have obtained the best possible upper bound for

$$\sum_{a_i \leqq x} \frac{1}{a_i}$$

for primitive sequences, their result being that, given $\varepsilon > 0$,

$$\sum_{a_i \leqq x} \frac{1}{a_i} \leqq \left(\frac{1}{\sqrt{(2\pi)}} + \varepsilon \right) \frac{\log x}{\sqrt{\log \log x}}$$

provided that x is sufficiently large. In this paper† we shall return to the first type of primitive sequence, where it is the prime factors which are bounded by x, and we shall obtain the corresponding best possible result. First we consider the particular sequence for which the sum is essentially greatest. We prove

THEOREM 1. *Let $\{a_i'\}$ denote the sequence of integers built up from the primes $p \leqq x$ and containing exactly $K = [\log \log x]$ prime factors counted according to multiplicity. Then, as $x \to \infty$,*

$$\sum_i \frac{1}{a_i'} \sim \frac{e^c}{\sqrt{(2\pi)}} \frac{\log x}{\sqrt{\log \log x}},$$

where c denotes Euler's constant.

We remark that this sequence is clearly primitive. Further, the a_i' are of degree K, where the *degree* of an integer m is defined to be the number of prime factors of m counted according to multiplicity. We then apply this result to deduce

THEOREM 2. *Let $A = \{a_i\}$ be any primitive sequence built up from the primes $p \leqq x$. Then, provided that x is sufficiently large,*

$$\sum_i \frac{1}{a_i} \leqq \left(\frac{e^c}{\sqrt{(2\pi)}} + \varepsilon \right) \frac{\log x}{\sqrt{\log \log x}}.$$

In view of Theorem 1, this result is clearly best possible.

† The contents of this paper formed part of the author's Ph.D.(Nottingham) thesis.

Proof of Theorem 1. We require the fact that

$$\sum_{p \leqq x} \frac{1}{p} = \log \log x + a + O\left(\frac{1}{\log x}\right), \tag{1}$$

where

$$1 > a = c + \sum_{p} \left\{ \log\left(1 - \frac{1}{p}\right) + \frac{1}{p} \right\} > 0.$$

It follows that, for sufficiently large x,

$$S = \sum_{p \leqq x} \frac{1}{p} > \log \log x. \tag{2}$$

We shall write

$$U = \sum_{\substack{p \leqq x \\ m \geqq 2}} \frac{\cos m\theta}{mp^m}, \qquad W = \sum_{\substack{p \leqq x \\ m \geqq 2}} \frac{\sin m\theta}{mp^m},$$

$$Y(z) = \sum_{\substack{p \leqq x \\ m \geqq 2}} \frac{z^m}{mp^m},$$

so that

$$Y(1) = c - a.$$

Let $A_K = \sum_i \frac{1}{a_i'}$. Since A_K is the coefficient of z^K in

$$\prod_{p \leqq x} \left(1 - \frac{z}{p}\right)^{-1},$$

we have by Cauchy's theorem that

$$\begin{aligned} A_K &= \frac{1}{2\pi i} \int_{|z|=1} z^{-K-1} \exp\left\{ -\sum_{p \leqq x} \log\left(1 - \frac{z}{p}\right) \right\} dz \\ &= \frac{1}{2\pi i} \int_{|z|=1} z^{-K-1} \exp\{Sz + Y(z)\} \, dz \\ &= \frac{1}{2\pi i} \int_{-\pi}^{\pi} \exp\{S \cos\theta + i(S \sin\theta - K\theta) + U + iW\} \, d\theta \\ &= \frac{1}{\pi i} \int_{0}^{\pi} \exp(S\cos\theta + U) \,.\, \cos(S \sin\theta - K\theta + W) \, d\theta, \end{aligned} \tag{3}$$

since A_K is real.

In evaluating this integral, we show that the contribution from $[0, \delta]$, where $\delta = S^{-5/12}$, is the dominating part. First we require two lemmas.

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LEMMA 1. *$W = O(\delta)$ whenever $0 \leqq \theta \leqq \delta$.*

Proof. Let $N = [\delta^{-1}]$. Since $\left|\dfrac{\sin x}{x}\right| \leqq 1$,

$$\left|\sum_{p\leqq x}\sum_{2\leqq m<N}\frac{\sin m\theta}{mp^m}\right| \leqq \delta \sum_{\substack{p\leqq x\\ m\geqq 2}}\frac{1}{p^m} \leqq \delta\sum_p \frac{1}{p(p-1)} < \delta.$$

Further,

$$\left|\sum_{p\leqq x}\sum_{m\geqq N}\frac{\sin m\theta}{mp^m}\right| \leqq \frac{1}{N}\sum_{\substack{p\leqq x\\ m\geqq 2}}\frac{1}{p^m} = O(\delta),$$

and so the result follows.

Similarly, we can prove

LEMMA 2. *If $0 \leqq \theta \leqq \delta$, then $e^U = \{1+O(\delta)\}\, e^{Y(1)}$.*

We now return to (3). In $[0, \delta]$ the integrand is, by Lemma 2,

$$\{1+O(\delta)\}\exp\{Y(1)+S\cos\theta\}\cos(S\sin\theta-K\theta+W). \tag{4}$$

Now, by Lemma 1,

$$\begin{aligned}|S\sin\theta-K\theta+W| &\leqq S|\sin\theta-\theta|+|K-S|.|\theta|+O(\delta)\\ &= O(S\delta^3)+O(\delta) = O(\delta^{\frac{1}{2}}),\end{aligned}$$

so that

$$\cos(S\sin\theta-K\theta+W) = 1+O(\delta). \tag{5}$$

Since

$$\exp(S\cos\theta) = \exp\{S-\tfrac{1}{2}S\theta^2(1+O(S^{-5/6}))\},$$

it follows, by (4) and (5), that the integrand in $[0, \delta]$ is

$$(1+O(\delta))\exp\{S+Y(1)-\tfrac{1}{2}S\theta^2(1+O(S^{-5/6}))\}.$$

Thus, on letting $T = \delta\sqrt{[S(1+O(S^{-5/6}))]}$, the contribution to A_K is

$$\frac{1}{\pi}(1+O(\delta))\,e^{S+Y(1)}\,S^{-\frac{1}{2}}\int_0^T e^{-\frac{1}{2}t^2}\,dt \sim \frac{1}{\pi\sqrt{S}}\,e^{S+Y(1)}\int_0^\infty e^{-\frac{1}{2}t^2}\,dt \sim \frac{1}{(\sqrt{2\pi})}\,e^S S^{-\frac{1}{2}} e^{c-a}$$

as $S\to\infty$.

We must now consider the contribution to A_K from the remainder of the interval of integration. Here the integrand is

$$\ll \exp(S\cos\delta+Y(1)) \ll \exp(S-\tfrac{1}{4}S\delta^2),$$

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which is of smaller order than $e^S S^{-\frac{1}{2}}$. Thus finally, by (1),

$$A_K \sim \frac{1}{\sqrt{(2\pi S)}} \exp(S+c-a) \sim \frac{e^c}{\sqrt{(2\pi \log\log x)}} \exp\left\{\log\log x + O\left(\frac{1}{\log x}\right)\right\}$$
$$\sim \frac{e^c \log x}{\sqrt{(2\pi \log\log x)}}$$

as required. This completes the proof of Theorem 1.

Proof of Theorem 2. The proof is based on the method used by the authors of [**2**]. First we show that nothing is essentially lost by ignoring some of the elements of A. We introduce notation as follows: let $\Omega(a)$ denote the degree of a, and let $\omega(a)$ denote the number of distinct prime factors of a.

LEMMA 3. *Let $\{b_i\}$ be a sequence of positive integers built up from the primes $p \leqq x$, and suppose that $\Omega(b_i) - \omega(b_i) > 2\log K$ for each b_i, where K is as before. Then*

$$\sum_i \frac{1}{b_i} = o\left(\frac{\log x}{\sqrt{\log\log x}}\right).$$

Proof. Any such b can be written in the form $b = m^2 u$, where u is squarefree and $\Omega(m) > \log K$. Thus

$$\sum_i \frac{1}{b_i} < \left\{\prod_{p \leqq x}\left(1+\frac{1}{p}\right)\right\} \sum_{\Omega(m) > \log K} \frac{1}{m^2}$$
$$\ll \left\{\prod_{p \leqq x}\left(1-\frac{1}{p}\right)^{-1}\right\} \sum_{m > 2^{\log K}} \frac{1}{m^2}$$
$$\ll 2^{-\log K} \log x$$
$$= o\left(\frac{\log x}{\sqrt{K}}\right).$$

Thus we can now assume that $\Omega(a_i) - \omega(a_i) \leqq 2\log K$ for all a_i in A.

We next show that we can replace all a_i of degree less than K by integers of degree K without destroying primitiveness and without decreasing the sum.

Let $\{a_i^{(h)}\}$ be the subsequence of all elements of A of smallest degree $h < K$, and denote by $\{b_i^{(h+1)}\}$ all the distinct numbers of degree $h+1$ obtained by multiplying each $a_i^{(h)}$ by each $p \leqq x$. None of the $b_i^{(h+1)}$ can be in A, and the replacement of the $a_i^{(h)}$ by the $b_i^{(h+1)}$ preserves primitiveness. Further, since each $b_i^{(h+1)}$ arises from at most $\omega(b_i^{(h+1)}) \leqq h+1$ elements $a_i^{(h)}$, we have

$$\left(\sum_i \frac{1}{a_i^{(h)}}\right) S < (h+1)\left(\sum_i \frac{1}{b_i^{(h+1)}}\right).$$

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Thus, by (2), for sufficiently large x,

$$\sum_i \frac{1}{b_i^{(h+1)}} > \frac{K}{h+1} \sum_i \frac{1}{a_i^{(h)}} \geqq \sum_i \frac{1}{a_i^{(h)}}.$$

Repeating this process for the newly formed primitive sequence $K-h-1$ times, we eventually obtain a primitive sequence all of whose elements are of degree at least K and for which the sum is not less than that for A.

Finally, we have to consider those a_i which are of degree greater than K. By the result of Dickson to which we referred earlier, we can define H, the maximum degree occurring among the elements of A. For each k $(K \leqq k \leqq H)$ denote by $\{a_i^{(k)}\}$ the sequence of elements of A of degree k. We define integers $c_i^{(k)}$ as follows.

Let $\{c_i^{(H-1)}\}$ denote the set all divisors of the $a_i^{(H)}$ of degree $H-1$; they are distinct from the $a_i^{(H-1)}$. For $K < k < H$, let $\{c_i^{(k-1)}\}$ denote the sequence of all divisors of degree $k-1$ of the $a_i^{(k)}$ and the $c_i^{(k)}$. Again, these are all distinct from the $a_i^{(k-1)}$. In view of our assumption, stated after Lemma 3, for each $k > K$,

$$\left(\sum_i \frac{1}{c_i^{(k-1)}}\right)\left(\sum_{p \leqq x} \frac{1}{p}\right) \geqq (k-2\log K)\left\{\sum_i \frac{1}{c_i^{(k)}} + \sum_i \frac{1}{a_i^{(k)}}\right\},$$

so that, for sufficiently large x,

$$\sum_i \frac{1}{c_i^{(k-1)}} \geqq \frac{k-2\log K}{K+2}\left(\sum_i \frac{1}{c_i^{(k)}} + \sum_i \frac{1}{a_i^{(k)}}\right).$$

But, for sufficiently large x,

$$\frac{k-2\log K}{K+2} > \begin{cases} 1 & \text{if} \quad k \geqq K+3\log K, \\ 1-\dfrac{3\log K}{K} & \text{if} \quad K < k < K+3\log K. \end{cases}$$

Consequently, by repeated application of the above process, we obtain

$$\sum_i \frac{1}{c_i^{(K)}} > \left(1-\frac{3\log K}{K}\right)^{3\log K+1} \sum_{k>K} \frac{1}{a_i^{(k)}} = (1+o(1)) \sum_{k>K} \frac{1}{a_i^{(k)}}.$$

Thus finally

$$\sum_i \frac{1}{a_i} \leqq (1+o(1)) \sum_i \frac{1}{a_i''},$$

where $\{a_i''\}$ is a subsequence of the sequence $\{a_i'\}$ defined in Theorem 1. The result now follows from Theorem 1.

We end by posing the following problem. We have shown that $A_k < A_{k+1}$ for $k < K$, and that the maximum value of A_k must occur for some k with $K < k < K+3\log K$. We ask if this value of k is in fact $K+O(1)$, and if there exists a value K_1 such that, if

$$k_1 < k_2 < K_1 < k_3 < k_4,$$

then

$$A_{k_1} \leqq A_{k_2} \leqq A_{K_1}, \quad A_{K_1} \geqq A_{k_3} \geqq A_{k_4}.$$

REFERENCES

1. L. E. Dickson, Finiteness of the odd perfect and primitive numbers with n distinct prime factors, *Amer. J. Math.* **35** (1913), 413–422.

2. P. Erdös, A. Sárkösy and E. Szemerédi, On an extremal problem concerning primitive sequences, *J. London Math. Soc.* **42** (1967), 484–488.

University of Glasgow