A collisional and self-gravitational model to simulate numerically the dynamics of planetary disks

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Abstract. The dynamical evolution of the planetary rings is simulated by means of a numerical model in which particles interact through mutual attraction and inelastic collisions. We use a mixed simulation: a deterministic integration of the N - body problem for large distances ("particle-mesh" method with an expansion of density and potential in spherical harmonics) and a Monte Carlo treatment for the close encounters. The implementation is done in the Connection Machine in order to be able to make a detailed simulation using a greater number of particles (of the order of 10^5). The deterministic calculation of the action of a shepherding satellite on the particles will allow us to study the effect of resonances on the formation and the evolution of the sharp edges of the rings.

Key words: planetary rings / "particle-mesh" method / n-body simulation / Monte Carlo method / connection machine

1. Introduction

The observations made by the Voyager have revealed that the planetary rings are a dynamical system of a great complexity, not well understood yet. In particular, the confinement, the gaps, the sharp edges observed at many places in real rings.

We will present a collisional and self-gravitational model to simulate numerically the dynamics of the rings.

In our model we consider particle sizes of the order of the cm - m, so we neglect the force of Pointing - Robertson, the radiation pression, the solar winds, that are some important effects when we study the dust. We consider a great number of particles, of the order of 10^5 (in the previous models, this number was of the order of a hundred of particles, [1], [2]). We take into account the self - gravitation – resolution of Poisson equation by the method of a mean gravitational field (particle-mesh) – and we treat partially elastic collisions and gravitational encounters by a Monte Carlo method. Concerning the encounters, we improve the model of Petit, Hénon [3], [4], [5], [6], because they kept only the semi-major axis of each orbit as parameter in their simulation and we are also interested in the effects on the eccentricity.

2. Self - gravitation

Concerning the self-gravitation, we apply the method of particle-mesh, initially developped to study the dynamics of galaxies.

The Poisson equation

$$\nabla^2\psi = 4\pi G\rho \qquad (1)$$

S. Ferraz-Mello (ed.), Chaos, Resonance and Collective Dynamical Phenomena in the Solar System, 109–114.

(G : constant of gravitation)

is solved by an expansion of density ρ and potential ψ in spherical harmonics that are, in particular, the proper vectors of Poisson operator.

$$\rho(r,\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} P_n^m(\cos\theta)\,(A_{nm}(r)\cos(m\phi) + B_{nm}(r)sin(m\phi)) \qquad (2)$$

$$\psi(r,\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} P_n^m(\cos\theta)\,(C_{nm}(r)\cos(m\phi) + D_{nm}(r)sin(m\phi)) \qquad (3)$$

[The P_n^m's are associated Legendre functions of degree n and order m , [7].]

After the determination of the coefficients $A, B, C, D,$ we are able to evaluate the acceleration.

In order to solve this problem numerically we consider a spherical mesh which has, in particular, the advantage of allowing arbitrary spacings in the radial direction and so a better adaptation to the density distribution.

We assume that in each cell (i,j,k) the density is constant.
The expansions in Legendre functions are taken up to degree $n = 32$.
In the ϕ - direction, $4n$ cells oversample the expansion and in the θ - direction, we consider $2n$ cells.

After knowing the value of the acceleration at all mesh points, we know it, in particular, at the four coins of each cell and so, by a linear interpolation, we are able to determine the acceleration for each particle in each cell.

The time integration of the equation of motion is done with a leap-frog scheme.

But we have to adapt this formalism, which was developped for 3 - dimensional systems, [8], to planetary disks, which are essentially 2 - dimensional systems. So we consider

$$\begin{cases} \rho(r,\theta,\phi) = \frac{\sigma(r,\phi)}{2\,\delta\theta}, & \text{if } \theta \in J = [\frac{\Pi}{2} - \delta\theta, \frac{\Pi}{2} + \delta\theta] \ (\delta\theta \to 0) \\ \rho(r,\theta,\phi) = 0, & \text{if } \theta \notin J \end{cases}$$

where $\sigma(r,\phi)$ is the surface density.

We treat the density as constant over the domain of each cell.

$$\begin{cases} \rho(r,\theta,\phi) = \frac{M(r,\theta,\phi)}{\int_{\frac{\pi}{2}-\delta\theta}^{\frac{\pi}{2}+\delta\theta} d\theta \sin\theta \int_{\phi_k}^{\phi_{k+1}} d\phi \int_{r_i}^{r_{i+1}} r^2 dr}, & \text{if } \theta \in J \\ \rho(r,\theta,\phi) = 0, & \text{if } \theta \notin J \end{cases}$$

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The function A_{nm} is determined from the density distribution [9]:

$$A_{nm}(i)_{\delta\theta} = C \int_{\frac{\pi}{2}-\delta\theta}^{\frac{\pi}{2}+\delta\theta} d\theta \sin\theta \, P_n^m(\cos\theta) \sum_{k=1}^{4n} \int_{\phi_k}^{\phi_{k+1}} d\theta \frac{\cos(m\phi)\, M(i,j,k)}{\Delta\phi \int_{\frac{\pi}{2}-\delta\theta}^{\frac{\pi}{2}+\delta\theta} d\theta \sin\theta \int_{r_i}^{r_{i+1}} r^2 dr} \quad (4)$$

where

$$C = \frac{(2n+1)\epsilon_m}{4\pi} \frac{(n-m)!}{(n+m)!} \quad (5)$$

with $\epsilon_0 = 1$, $\epsilon_m = 2$ $(m = 1, 2, \ldots$.

Now

$$A_{nm}(i) = \lim_{\delta\theta \to 0} A_{nm}(i)_{\delta\theta} \quad (6)$$

[$B_{nm}(i)$ is found by replacing $\cos(m\phi)$ by $\sin(m\phi)$.]

$$A_{nm}(i) = C \, P_n^m(0) \, \frac{2\sin(\frac{m\Delta\phi}{2})}{m\Delta\phi} \sum_{k=1}^{4n} \frac{M(i,k)}{\int_{r_i}^{r_{i+1}} r^2 dr} \cos[\frac{m(\phi_k + \phi_{k+1})}{2}] \quad (7)$$

If we put

$$W_{nm} = C P_n^m(0) \frac{2}{m} \sin(\frac{m\Delta\phi}{2}) \quad (8)$$

we have

$$\bar{A}_{nm}(i) = W_{nm} \sum_{k=1}^{4n} M(i,k) \cos[\frac{m(\phi_k + \phi_{k+1})}{2}] \quad (9)$$

where $\bar{A}_{nm}(i)$ are the coefficients for the mass.

Similarly, and if we note

$$C_{nm}(r) = C_{1,n,m}(r) + C_{2,n,m}(r) \quad (10)$$

we obtain

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$$C_{1,n,m}(i) = -\frac{4\Pi G}{2n+1} r_i^{-n-1} \sum_{l=0}^{i-1} A_{nm}(l) \int_{r_i}^{r_{i+1}} s^{n+2} ds \tag{11}$$

$$C_{2,n,m}(i) = -\frac{4\Pi G}{2n+1} r_i^{n} \sum_{l=i}^{63} A_{nm}(l) \int_{r_i}^{r_{i+1}} s^{1-n} ds \tag{12}$$

[D_{nm} is found by replacing A_{nm} by B_{nm}.]

At the mesh corner (r_i, ϕ_k) the potential becomes

$$\psi(i,k) = \sum_{n=0}^{32} \sum_{m=0}^{n} P_n^m(0)[C_{n,m}(i)\cos(m\phi_k) + D_{n,m}(i)\sin(m\phi_k)] \tag{13}$$

and the components of the acceleration are then

$$a_r(i,k) = -\sum_{n=0}^{32} \sum_{m=0}^{n} P_n^m(0)\,[E_{n,m}(i)\cos(m\phi_k) + F_{n,m}(i)\sin(m\phi_k)] \tag{14}$$

$$a_\phi(i,k) = -\sum_{n=0}^{32} \sum_{m=0}^{n} m\, P_n^m(0)\,[H_{n,m}(i)\cos(m\phi_k) - G_{n,m}(i)\sin(m\phi_k)] \tag{15}$$

where

$$E_{nm}(i) = -(n+1)\frac{C_{1,n,m}(i)}{r} + n\frac{C_{2,n,m}(i)}{r} \tag{16}$$

$$G_{nm}(i) = \frac{C_{nm}(i)}{r} \tag{17}$$

[F_{nm} and H_{nm} are found by replacing C_{nm} by D_{nm}.]

The 3 - dimensional formalism to solve the Poisson equation with spherical harmonics is so adapted to the problem of a very thin disk.

Now it may happen that when we make the system evolve in time, there is a collision between two particles in the same cell. So the next problem to be discussed is how to treat the collisions.

3. Collisions

The previous method cannot account for close encounters, i.e. gravitational interactions when the particles are close together and physical inelastic collisions (we consider collisions that do not lead to acretion or fragmentation).

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We consider binary encounters and we make the assumption that we can apply an impulse approximation.

There is an encounter when one particle enters the sphere of influence of the other. The "sphere of influence" is defined such that inside it the gravitational force between the two particles is greater or equal to the differential attraction of the planet.

Since we have a dominant central mass, Saturn, we can apply Hill's problem.

So we consider the relative motion of the particles in a frame that rotates with the velocity of one of the particles. In this reference system, the motion of the other particle is an epicyclic one. We consider the following parameters (Hill parameters) :

h : impact parameter
– it is determined by the two semi - major axis of the two particles

k : reduced eccentricity
– it is determined by the two eccentricities of the two particles

ϕ : phase

And we replace the rigourous treatment of the encounters – which is too much time consuming – by a Monte Carlo method. To do this, we need to know the distribution function of the effect of the encounters, when averaging over the phase. It is determined by a separated detailed study. We know the parameters h and k before the encounter and this function will give us immediatly these parameters after the encounter. The phase ϕ gives us the exact configuration of the particles after the encounter. We take it as a parameter and we assume that it is uniformly distributed. It is selected at random.

Many numerical experiments are needed to derive the analytical function or several functions for different regions in the h, k space. From our previous calculations [10], which we are actually refining, we need five distribution functions.

4. Conclusion

Since we have 10^5 particles we need a supercomputer. We treat our problem in a connection machine.

The effects we want to simulate are essentially the following ones:

– the ring spreadening due to collisions among particles
– the ring confinement due to shepherding satellites
– the sharp edges

and finally

– some finer structures by the inclusion of large boddies imbedded in the ring.

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Discussion

P.Goldreich **– I am not sure that I understand what you mean by phase. However, if I do, then I have one comment. In regions perturbed by shepherd satellites or waves, collisions do not occur at random phase. This could be taken into account by drawing phase from an appropriate distribution**

Pereira Gama **– Thank you for your hint.**

H.Kinoshita **– Is the dynamical model of the encounter in your theory conservative or dissipative?**

Pereira Gama **– Dissipative.**

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