Canad. Math. Bull. Vol. **29** (1), 1986

FINITE EXTENSIONS OF VALUED FIELDS

BY
SETH WARNER

ABSTRACT. A corollary of the main result is that if L is a finite-dimensional Galois extension of a field K and if w is a valuation of L extending a valuation v of K, then K is closed in L if and only if all valuations of L extending v are dependent. A further consequence is a generalization of Ostrowski's criterion for a real-valued valuation to be henselian.

Let v be a valuation of a field K. If v is henselian, then K is closed in any separable algebraic extension L, furnished with the topology defined by the unique extension of v to L. Indeed, by Krasner's Lemma [2] (first proved by Ostrowski [5, Hilfssatz, p. 197]) for each $x \in L$, V_x, defined by $V_x = \{y \in L: K(x) \subseteq K(y)\}$, is a neighborhood of x; if $x \in L \backslash K$, then $V_x \subseteq L \backslash K$, so K is closed.

Here we shall seek, conversely, to determine what henselian-like properties follow from the assertion that K is closed in any finite-dimensional separable extension, furnished with a valuation extending v (Theorem 2). More generally, we shall relate the assertion that a finite-dimensional extension L admits only one ring topology inducing on K the topology defined by v with closure properties of various subspaces of L (Theorem 1).

Normality plays a simplifying role, as the following example shows: the 5-adic number field $\boldsymbol{Q}_5$ contains a cube root $\sqrt[3]{2}$ of 2 but does not contain a primitive cube root ω of unity since $x^3 \equiv 2 \pmod 5$ has an integral solution but $x^2 \equiv -3 \pmod 5$ does not. Therefore if u is the unique extension of the 5-adic valuation to an algebraic closure of $\boldsymbol{Q}_5$, its restrictions to the conjugate fields $\boldsymbol{Q}(\sqrt[3]{2})$ and $\boldsymbol{Q}(\omega\sqrt[3]{2})$ yield valued fields in one of which $\boldsymbol{Q}$ is dense, in the other, closed. Thus the 5-adic valuation on $\boldsymbol{Q}$ has two extensions to $\boldsymbol{Q}(\sqrt[3]{2})$, for one of which $\boldsymbol{Q}$ is dense, for the other, closed.

We shall use the well-known fact that if $\{e_1, \ldots, e_n\}$ is a base of the K-vector space L and if v is a proper valuation [absolute value] on K, w an extension of v to L, then $\{e_1, \ldots, e_n\}$ is a set of generators of the $K^\wedge$-vector space $L^\wedge$ (where $K^\wedge$ and $L^\wedge$ are the completions of K and L respectively); in particular, $[L^\wedge : K^\wedge] \leq [L:K]$. Indeed, $K^\wedge e_1 + \ldots + K^\wedge e_n$ is a closed subspace of $L^\wedge$ containing L [1, Cor., p. 121] and hence is $L^\wedge$. Also, for any $a \in L$, $K^\wedge(a) = K(a)^\wedge$ since $K^\wedge(a)$ is a finite-dimensional

Received by the editors August 13, 1984.
AMS Subject Classification (1980): 12J10.

$K^\wedge$-vector space and hence is a complete vector space in which $K(a)$ is dense. We shall also call upon the following Lemma, a simple consequence of the product formula for successive extensions and the inequality just mentioned:

LEMMA. *If v is a proper valuation [absolute value] of a field K and if w is an extension of v to L such that $[L^\wedge : K^\wedge] = [L : K]$, then for any subfields E and F such that $K \subseteq E \subseteq F \subseteq L$, $[F^\wedge : E^\wedge] = [F : E]$.*

THEOREM 1. *Let v be a proper valuation [absolute value] of a field K, let Ω_0 be an algebraic closure of the completion $K^\wedge$ of K for v, let w_0 be an extension of $v^\wedge$ to Ω_0, and let w be the restriction of w_0 to the algebraic closure Ω of K in Ω_0. Let $\boldsymbol{T}_w$ and $\boldsymbol{T}_v$ be the topologies on Ω and K determined by w and v respectively. Let L be a finite-dimensional extension of K contained in Ω. Let (S) be the statement: there exist $c_0, c_1, \ldots, c_n \in L$ such that $K(c_0)$ is the separable closure L_s of K in L, $L = K(c_0, c_1, \ldots, c_n)$, and the subfields $K(c_0, \ldots, c_i)$, where $i \in [0, n]$, are closed in L. The following statements are equivalent:*

(1) $[L^\wedge : K^\wedge] = [L : K]$.

(2a) [2b] *For some [every] base $\{e_1, \ldots, e_n\}$ of L over K, $(\lambda_1, \ldots, \lambda_n) \mapsto \sum_{i=1}^n \lambda_i e_i$ is a topological K-isomorphism from the topological K-vector space K^n to the topological K-vector space L.*

(3) *Every subspace of the K-vector space L is closed.*

(4) *$\boldsymbol{T}_w$ induces on L the only ring topology that induces $\boldsymbol{T}_v$ on K.*

(5) *All valuations [absolute values] on L extending v are dependent [equivalent], and (S) holds.*

(6a) [6b] *Every K-monomorphism σ from L to Ω is continuous [a topological isomorphism from L to $\sigma(L)$], and (S) holds.*

(7) *The minimal polynomial of each element of L_s is irreducible over $K^\wedge$, and (S) holds.*

These statements imply (8) *and are equivalent to* (8) *if L is a normal extension of K:*

(8) *K is closed in L and (S) holds.*

PROOF. (1) implies (2b): Let $\{e_1, \ldots, e_n\}$ be a base of the K-vector space L. Since $\{e_1, \ldots, e_n\}$ generates the $K^\wedge$-vector space $L^\wedge$, by (1) $\{e_1, \ldots, e_n\}$ is a base of $L^\wedge$. Therefore $(\lambda_1, \ldots, \lambda_n) \rightarrow \sum_{i=1}^n \lambda_i e_i$ is a topological $K^\wedge$-isomorphism from $K^{\wedge n}$ to $L^\wedge$ [1, Prop. 4, p. 120]. The restriction to K^n of that topological isomorphism is therefore a topological K-isomorphism from K^n to L. (2a) implies (1): the given topological isomorphism extends by continuity to a topological isomorphism from $K^{\wedge n}$ to $L^\wedge$, which necessarily is the mapping $(\lambda_1, \ldots, \lambda_n) \mapsto \sum_{i=1}^n \lambda_i e_i$ as that mapping is continuous from $K^{\wedge n}$ to $L^\wedge$. Thus $\{e_1, \ldots, e_n\}$ is a base of the $K^\wedge$-vector space $L^\wedge$, so $[L^\wedge : K^\wedge] = [L : K]$. Clearly (2b) implies (3), and by [3, Theorems 2 and 6], (3) implies (2b). Thus (1) $-$ (3) are all equivalent.

(1) implies (4): Let $b_1, \ldots, b_n \in L$ be such that $L = K(b_1, \ldots, b_n)$, and let $K_0 = K$,

$K_i = K(b_1, \ldots, b_i)$ for all $i \in [1, n]$. For each $i \in [0, n]$ let (P_i) be the statement: $\boldsymbol{T}_w$ induces on K_i the only ring topology inducing $\boldsymbol{T}_v$ on K. It suffices to show that if $i \in [0, n-1]$, (P_{i+1}) follows from (P_i). By the Lemma, $[K_{i+1}^{\wedge} : K_i^{\wedge}] = [K_{i+1} : K_i]$, so by [7, Theorem 3] $\boldsymbol{T}_w$ induces on K_{i+1} the strongest ring topology inducing on K_i the same topology as $\boldsymbol{T}_w$. But the topology $\boldsymbol{T}_w$ induces on K_{i+1}, defined by the restriction of w to K_{i+1}, is minimal among all ring topologies inducing on K_i the same topology as $\boldsymbol{T}_w$ by [7, Theorem 1]. Therefore $\boldsymbol{T}_w$ induces on K_{i+1} the only ring topology inducing on K_i the same topology as $\boldsymbol{T}_w$. Now let $\boldsymbol{T}$ be any ring topology on K_{i+1} inducing $\boldsymbol{T}_v$ on K. By (P_i), $\boldsymbol{T}$ induces on K_i the same topology as $\boldsymbol{T}_w$. By what we have just proved, therefore, $\boldsymbol{T}$ is the topology on K_{i+1} induced by $\boldsymbol{T}_w$, that is, (P_{i+1}) holds.

(4) implies (5): We shall actually prove that every subfield of L containing L_s is closed. Suppose that F is a nonclosed subfield of L containing L_s, and let $c \in \bar{F} \backslash F$. As L is a purely inseparable extension of F, the minimal polynomial of c over F is $X^{p^n} - a$ for some $a \in F$, where the prime p is the characteristic of K and $n \geq 1$. Now $X^{p^n} - a = (X - c)^{p^n}$ in $F^{\wedge}[X]$, so by [7, Theorem 1], for each $j \in [1, p^n]$ there is a ring topology $\boldsymbol{T}_j$ on $F(c)$ inducing on F the same topology as $\boldsymbol{T}_w$ such that the completion of $F(c)$ for $\boldsymbol{T}_j$ is $F^{\wedge}(c_j)$ where $(X - c)^j$ is the minimal polynomial of c_j. Let $j \in [2, p^n]$. Then $c - c_j$ is a nonzero nilpotent element of $F^{\wedge}(c_j)$. Since any algebra topology on a finite-dimensional algebra over $F^{\wedge}$ is defined by a norm, it is locally bounded. Therefore $\boldsymbol{T}_j$ is a normed and hence locally bounded topology on $F(c)$ and hence, by [8, Satz 1.6], is the restriction to $F(c)$ of a locally bounded (and hence ring) topology $\boldsymbol{T}$ on L. The completion of L for $\boldsymbol{T}$ then contains nonzero nilpotent elements, in contradiction to (4).

(5) implies (6b): If σ is a K-monomorphism from L to Ω and if $\boldsymbol{T}_{w,\sigma}$ is the topology induced on $\sigma(L)$ by $\boldsymbol{T}_w$, then $\sigma^{-1}(\boldsymbol{T}_{w,\sigma})$ is the topology on L defined by the valuation $w \circ \sigma$. By (5), $\sigma^{-1}(\boldsymbol{T}_{w,\sigma})$ is also the topology induced on L by $\boldsymbol{T}_w$, that is, σ is a topological isomorphism from L to $\sigma(L)$. Also (6a) implies (6b), for if σ is continuous, $w \circ \sigma$ defines on L a topology weaker than the topology induced by $\boldsymbol{T}_w$; these topologies are then identical, since a valuation topology on L inducing $\boldsymbol{T}_v$ on K is minimal among all ring topologies on L inducing $\boldsymbol{T}_v$ on K by, for example, [7, Theorem 1].

Each of (6b) and (7) implies (1): With the notation of (S), $L_s = K(c_0)$. We shall first show that $[L_s^{\wedge} : K^{\wedge}] = [L_s : K]$. Assume (6b). If σ is any K-monomorphism from L into Ω, then by (6b) σ has a continuous extension to a $K^{\wedge}$-isomorphism from $L^{\wedge}$ to the subfield $\sigma^{\wedge}(L^{\wedge})$ of Ω_0. Thus every K-conjugate of c_0 is also a $K^{\wedge}$-conjugate, so as c_0 is separable over K, the minimal polynomials of c_0 over K and over $K^{\wedge}$ are identical. Consequently, each of (6b) and (7) implies that $[L_s^{\wedge} : K^{\wedge}] = [L_s : K]$. With the notation of (S), let $L_j = K(c_0, \ldots, c_j)$ for each $j \in [0, n]$. By the product formula for successive extensions, it suffices to show that $[L_j^{\wedge} : L_{j-1}^{\wedge}] = [L_j : L_{j-1}]$ for each $j \in [1, n]$. As L_j is a finite-dimensional, purely inseparable extension of L_{j-1}, there exists $m \geq 0$ such that $L_j \subseteq \tau^{-1}(L_{j-1})$ where τ is the automorphism $x \mapsto x^{p^m}$ of Ω_0. As τ is continuous, $L_j^{\wedge} \subseteq \tau^{-1}(L_{j-1}^{\wedge})$, and hence $L_j^{\wedge}$ is a purely inseparable extension of $L_{j-1}^{\wedge}$. Thus the minimal polynomial of c_j over $L_{j-1}^{\wedge}$ is $X^{p^r} - b$ where $r \geq 0$ and $b \in L_{j-1}^{\wedge}$. As $b =$

$c_j^{p^r} \in L_j$, $b \in L_j \cap L_{j-1}^{\wedge} \subseteq L \cap L_{j-1}^{\wedge} = L_{j-1}$, since L_{j-1} is closed in L. Thus $X^{p^r} - b \in L_{j-1}[X]$ and hence is the minimal polynomial of c_j over L_{j-1}. Therefore $[L_j^{\wedge} : L_{j-1}^{\wedge}] = [L_{j-1}^{\wedge}(c_j) : L_{j-1}^{\wedge}] = [L_{j-1}(c_j) : L_{j-1}] = [L_j : L_{j-1}]$.

(1) implies (7): If $a \in L$, $[K^{\wedge}(a) : K^{\wedge}] = [K(a) : K]$ by the Lemma, so the minimal polynomials of a over K and $K^{\wedge}$ coincide. Also as (1) implies (3), (S) holds.

Clearly (3) and (7) imply (8). Conversely, assume (8), and suppose that L is a normal extension of K. To prove (7), let $b \in L_s$ and let f be the minimal polynomial of b over K. Then there exist $b_1, \dots, b_r \in L$ such that $f = \Pi_{i=1}^{r}(X - b_i)$, so the minimal polynomial g of b over $K^{\wedge}$ is $\Pi_{i \in J}(X - b_i)$ for some nonempty subset J of $[1, r]$. Each coefficient of g is an elementary symmetric polynomial of the b_i's, where $i \in J$, and hence belongs to L. Thus $g \in (L \cap K^{\wedge})[X] = K[X]$, as K is closed in L. Therefore $g = f$. Thus (7) holds.

The example given earlier shows that (8) may hold but (6a) fail if L is not a normal extension of K.

Nagata [4, p. 56] (see also [1, Exercise 14 c), p. 193]) has given an example of a valuation v of a field K of prime characteristic p such that $K^{\wedge}$ is a simple, purely inseparable extension of K of degree p. Consequently, K and $L = K^{\wedge}$ satisfy the first halves of (5)−(7) but not (1)−(4).

In the statement of Theorem 1, we may, of course, replace "(S) holds" with "L_s is closed" in (5)−(8) if L is a simple extension of K.

COROLLARY 1. *Let v be a proper valuation [absolute value] of a field K, and let w be an extension of v to a finite-dimensional Galois extension L. The following statements are equivalent:*

(1) $[L^{\wedge} : K^{\wedge}] = [L : K]$.

(2) *For some [every] base $\{e_1, \dots, e_n\}$ of L over K, $(\lambda_1, \dots, \lambda_n) \mapsto \Sigma_{i=1}^{n} \lambda_i e_i$ is a topological isomorphism from K^n to L.*

(3) *Every subspace of the K-vector space L is closed.*

(4) *The topology defined by w is the only ring topology on L inducing on K the topology defined by v.*

(5) *All valuations [absolute values] on L extending v are dependent [equivalent].*

(6) *Every K-automorphism of L is continuous.*

(7) *The minimal polynomial over K of each element of L is irreducible over $K^{\wedge}$.*

(8) *K is closed in L.*

COROLLARY 2. *Let v be a proper valuation [absolute value] of a field K, and let $\{w_1, \dots, w_r\}$ be a complete set of independent valuations [inequivalent absolute values] extending v on a finite-dimensional Galois extension L of K. For each $i \in [1, r]$* let *$\overline{K}_i$ be the closure of K in L for the topology $\boldsymbol{T}_i$ defined by w_i.* (1) *For each $i \in [1, r]$, $\overline{K}_i$ is the smallest of the subfields E containing K such that all extensions to L of the*

https://doi.org/10.4153/CMB-1986-012-x Published online by Cambridge University Press

restriction of w_i *to* E *are dependent* [*equivalent*]. (2) $[\bar{K}_i : K] = [\bar{K}_j : K]$ *for all* i, $j \in [1, r]$.

PROOF. (1) If E satisfies the description of (1), then E is closed in L for $\boldsymbol{T}_i$ by Corollary 1, so $E \supseteq \bar{K}_i$. But again by Corollary 1, all extensions to L of the restriction of w_i to $\bar{K}_i$ are dependent [equivalent]. Thus $\bar{K}_i$ is the smallest such field. (2) By Krull's theorem [1, Ch. 6, Cor. 1, p. 152] there is a K-automorphism σ of L such that $w_j = w_i \circ \sigma$. Hence σ is a topological K-automorphism from L, furnished with $\boldsymbol{T}_j$, to L, furnished with $\boldsymbol{T}_i$. Consequently, $[\bar{K}_i : K] = [\sigma(\bar{K}_j) : \sigma(K)] = [\bar{K}_j : K]$.

THEOREM 2. *Let* v *be a proper valuation* [*absolute value*] *of a field* K. *The following statements are equivalent:*

(1) *K is closed in every finite-dimensional separable extension field L, furnished with a valuation* [*absolute value*] *extending* v.

(2) *For every finite-dimensional extension L of K, all valuations* [*absolute values*] *on L extending v are dependent* [*equivalent*].

(3) *Each finite-dimensional separable extension L of K admits only one ring topology inducing on K the topology defined by v (namely, the topology defined by a valuation* [*absolute value*] *extending* v).

(4) *K is separably algebraically closed in* $K^{\wedge}$.

PROOF. Since every finite-dimensional separable extension of K is contained in a finite-dimensional Galois extension, the equivalence of (1)−(3) follows from Theorem 1. (1) implies (4), for if $a \in K^{\wedge} \backslash K$ were separable algebraic over K, then K would be a proper dense subfield of $K(a)$ for the topology induced by $v^{\wedge}$, in contradiction to (1). Conversely, assume (4), and let L satisfy the hypotheses of (1). If $a \in \bar{K} \backslash K$, then a would be an element of $K^{\wedge} \backslash K$ that is separable over K, in contradiction to (4).

The equivalence of (2) and (5) of the following Corollary is a classical theorem of Ostrowski [6, §25], rediscovered by Nagata [4, Cor., p. 51].

COROLLARY. *Let v be a proper real-valued valuation* [*absolute value*] *of a field K. The following statements are equivalent:*

(1) *K is closed in every finite-dimensional separable extension field of K, furnished with a valuation* [*absolute value*] *extending* v.

(2) *v is henselian.*

(3) *K is closed in the separable algebraic closure Ω_s of K, furnished with a valuation* [*absolute value*] *extending* v.

(4) *Each finite-dimensional separable extension field of K admits only one ring topology inducing on K the topology defined by* v.

(5) *K is separably algebraically closed in* $K^{\wedge}$.

PROOF. Since v is real-valued, (2) of Theorem 2 implies that each finite-dimensional separable extension L of K admits a unique extension of v (with values in the divisible group generated by the value group of v), and hence that v is henselian. As mentioned earlier, Krasner's Lemma establishes that (2) implies (3).

Although Nagata's example shows that, in general, some requirement is needed concerning the closure of subfields of L containing the separable closure L_s of K for the validity of (1)−(4) of Theorem 1, no requirement is needed for a special class of fields:

THEOREM 3. *Let L be an algebraic function field in one variable over a subfield k of prime characteristic, let K be a finite-codimensional subfield of L containing k, let w be a proper valuation of L inducing the improper valuation on k, and let v be its restriction to K. The first seven statements of Corollary 1 of Theorem 1 are equivalent, and if L is a normal extension of K, the eight statements of that Corollary are equivalent.*

PROOF. By Theorem 1, it suffices to show that if F is a finite-codimensional subfield of L containing k such that L is a purely inseparable extension of F, then F is closed. Suppose not. The proof that (4) implies (5) in Theorem 1 shows that L admits a locally bounded topology $\boldsymbol{T}$, inducing on K the topology defined by v, such that the completion $L^{\wedge}$ of L for $\boldsymbol{T}$ contains nonzero nilpotent elements. In particular, k is bounded for $\boldsymbol{T}$. By a theorem of Weber [8, Satz 4.4], $\boldsymbol{T}$ is "special" (as defined in [8, p. 167]). The completion of L for a special topology is the local direct sum of a family of complete, discretely valued fields relative to their valuation subrings. In particular, $L^{\wedge}$ is algebraically isomorphic to a subring of a cartesian product of fields, and hence has no nonzero nilpotents, a contradiction.

REFERENCES

1. N. Bourbaki, *Algèbre Commutative*, Ch. 5−6, Hermann, Paris, 1964.

2. M. Krasner, *Théorie non abélienne des corps de classes pour les extensions finies et séparables des corps valués complets; principes fondamentaux; espaces de polynomes et transformation T; lois d'unicité, d'ordination et d'existence*, C.R. Acad. Sci. Paris **222** (1946), pp. 626−628.

3. L. Nachbin, *On strictly minimal topological division rings*, Bull. Amer. Math. Soc. **55** (1949), pp. 1128−1136.

4. M. Nagata, *On the theory of Henselian rings*, Nagoya Math. J. **5** (1953), pp. 45−57.

5. A. Ostrowski, *Über sogenannte perfekte Körper*, J. Reine Angew. Math. **147** (1917), pp. 191−204.

6. A. Ostrowski, *Untersuchungen zur arithmetischen Theorie der Körper*, Math. Z. **39** (1935), pp. 269−404.

7. T. Rigo and S. Warner, *Topologies extending valuations*, Canad. J. Math. **30** (1978), pp. 164−169.

8. H. Weber, *Topologische Charakterisierung globaler Körper und algebraischer Funktionenkörper in einer Variablen*, Math. Z. **169** (1979), pp. 167−177.

DUKE UNIVERSITY
DURHAM, NC 27706