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CONGRUENCE PERMUTABLE EXTENSIONS OF DISTRIBUTIVE DOUBLE *p*-ALGEBRAS*

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Every distributive double p-algebra L is shown to have a congruence permutable extension K such that every congruence of L has a unique extension to K.

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1. Introduction

It is well-known that any distributive lattice L can be embedded into a distributive relatively complemented lattice K in such a way that every congruence of L has exactly one extension to K (see [4]). Furthermore, Katriňák [5] has shown that any distributive p-algebra L can be embedded into a distributive p-algebra K whose dense filter is relatively complemented in such a way that every congruence of L has exactly one extension to K. However, a distributive lattice is congruence permitable if and only if it is relatively complemented and Berman [2] has shown that a distributive p-algebra is congruence permutable if and only if its dense filter is relatively complemented. Consequently, every distributive lattice L and every distributive p-algebra L has a congruence permutable extension K such that every congruence of L has exactly one extension to K. In this note we obtain an analogous result for distributive double p-algebras, using Priestley duality.

2. Preliminaries

Although we assume some acquaintance with Priestley duality for distributive (0, 1)-lattices, we begin by reviewing some notation and terminology and the restriction of Priestley duality to distributive double p-algebras.

Let $(X, \tau, \leqq)$ be an ordered topological space and let $Y \subseteq X$. The set Y is said to be *decreasing* (*increasing*) if $(Y]_X = Y([Y)_X = Y)$, where

$$(Y]_X = \{x \in X : x \leqq y \text{ for some } y \in Y\}$$

and $[Y)_X$ is defined dually.

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We will write $\mathrm{Max}_X(Y)$ for $[Y)_X \cap \mathrm{Max}(X)$ and $\mathrm{Min}_X(Y)$ for $(Y]_X \cap \mathrm{Min}(X)$, where $\mathrm{Max}(X)$ and $\mathrm{Min}(X)$ denote the set of maximal elements and minimal elements of X, respectively. We also define $\mathrm{Ext}_X(Y) = \mathrm{Max}_X(Y) \cup \mathrm{Min}_X(Y)$, $\mathrm{Ext}(X) = \mathrm{Ext}_X(X)$, $\mathrm{Mid}(X) = X \backslash \mathrm{Ext}(X)$ and, when $Y = \{x\}$, we write $\mathrm{Max}_X(x)$ for $\mathrm{Max}_X(Y)$ and $\mathrm{Min}_X(x)$ for $\mathrm{Min}_X(Y)$.

$(X, \tau, \leqq)$ is called a *Priestley space* if it is compact and *totally order disconnected*; in the sense that, for every $x, y \in X$ with $x \not\leqq y$, there exists a clopen decreasing set $Y \subseteq X$ such that $y \in Y$ and $x \notin Y$. In such spaces, the sets $\mathrm{Max}_X(x)$ and $\mathrm{Min}_X(x)$ are non-empty, for any $x \in X$, and the following separation property holds:

(s) For any closed $Y, Z \subseteq X$ with $Y \cap (Z]_X = \phi$ there exists a clopen decreasing set D such that $(Z]_X \subseteq D$ and $Y \cap D = \phi$.

If $\mathscr{P}$ is the category of all Priestley spaces and continuous order preserving mappings and $\mathscr{D}$ is the category of all distributive $(0,1)$-lattices and $(0,1)$-lattice homomorphisms then Priestley ([**6**, **7**]) has shown that there exist contravariant functors D and P from $\mathscr{P}$ into $\mathscr{D}$ and $\mathscr{D}$ into $\mathscr{P}$, respectively, such that the composite functors $P \circ D$ and $D \circ P$ are naturally equivalent to the identity functors on their domains. Furthermore, a morphism f in $\mathscr{P}$ is subjective if and only if $D(f)$ is an embedding.

Recall now that a *distributive double p-algebra* is an algebra $(L; \vee, \wedge, {}^*, {}^+, 0, 1)$ in which $(L; \vee, \wedge, 0, 1)$ is a distributive $(0,1)$-lattice and, for $a \in L$, a^* is characterized by $x \leqq a^* \Leftrightarrow a \wedge x = 0$ and a^+ is characterized in a dual fashion. Priestley [**8**] has described the duals of distributive double p-algebras as follows:

(1) For an object $\mathbf{X} = (X, \tau, \leqq)$ in $\mathscr{P}$, $D(\mathbf{X})$ is a double p-algebra if and only if $[Y)_X$ is clopen for every clopen decreasing set $Y \subseteq X$ and $(Y]_X$ is clopen for every clopen increasing set $Y \subseteq X$.

(2) For a morphism $f : (X, \tau, \leqq) \to (X', \tau', \leqq')$ in $\mathscr{P}$, $D(f)$ is a double p-algebra homomorphism if and only if $f(\mathrm{Max}_X(x)) = \mathrm{Max}_{X'}(f(x))$ and $f(\mathrm{Min}_X(x)) = \mathrm{Min}_{X'}(f(x))$, for every $x \in X$.

If $D(\mathbf{X})$ is a double p-algebra then $\mathbf{X}$ is called a *dp-space* and in such spaces $\mathrm{Max}(X)$ and $\mathrm{Min}(X)$ are closed. If $D(f)$ is a double p-algebra homomorphism than f is called a *dp-map*.

Finally, we recall from [**3**], the following facts. If $\mathbf{X} = (X, \tau, \leqq)$ is a dp-space, $L = D(\mathbf{X})$ and Y is a *closed c-set*, i.e. Y is a closed subset of X satisfying $\mathrm{Ext}_X(Y) \subseteq Y$, then the binary relation $\Theta_L(Y)$ defined on L by

$$U \equiv V(\Theta_L(Y)) \Leftrightarrow U \cap Y = V \cap Y$$

is a congruence and the map $Y \mapsto \Theta_L(Y)$ is a 1-1 correspondence between the lattice of closed c-sets of $\mathbf{X}$ and the congruence lattice of the distributive double p-algebra L.

3. The construction

Our decision to employ Priestley duality to achieve our goal was motivated partly by the following result.

A distributive double p-algrbra has permutable congruences if and only if there is no 4-element chain in its dp-space.

This and other characterizations of congruence permutable distributive double p-algebras may be found in [**1**].

Theorem. *Every distributive double p-algebra L has a congruence permutable extension K such that every congruence of L has exactly one extension to K.*

Proof. Let $P=(X,\tau,\leqq)$ be the Priestley dual of the distributive double p-algebra L and let $Q=(X,\tau,\preceq)$ where $\preceq$ is the binary relation defined on X by

$$u\preceq v \Leftrightarrow \{u,v\}\cap \mathrm{Ext}(P)\neq\phi \quad \text{and} \quad u\leqq v.$$

Clearly $\preceq$ is a partial ordering of X, $\mathrm{Max}(Q)=\mathrm{Max}(P)$ and $\mathrm{Min}(Q)=\mathrm{Min}(P)$, so that $\mathrm{Ext}(Q)=\mathrm{Ext}(P)$. Furthermore, $\mathrm{Mid}(Q)$ is an unordered copy of $\mathrm{Mid}(P)$, $\mathrm{Max}_Q(x)=\mathrm{Max}_P(x)$ and $\mathrm{Min}_Q(x)=\mathrm{Min}_P(x)$, for every $x\in X$. Observe that there is no 4-element chain in Q and that the identity mapping f: $Q\to P$ is continuous, preserves order and, subject only to our showing that Q is a dp-space, has the properties necessary for it to qualify as a dp-map. We proceed by showing that Q is, indeed, a dp-space. Clearly, Q is compact. With the intention of proving that Q is totally order disconnected, suppose that $x,y\in X$ and $x\npreceq y$.

Let us assume that $x\nleqq y$. Then there exists a clopen $\leqq$-decreasing set $C\subseteq X$ such that $y\in C$ and $x\notin C$. However, the set $C=f^{-1}(C)$ is clopen and $\preceq$-decreasing because f is continuous and order preserving.

In the event that $x\leqq y$ and $x\npreceq y$, we have $x<y$ and it follows from the definition of $\preceq$ that $x,y\in\mathrm{Mid}(Q)=\mathrm{Mid}(P)$. Since $y\notin\mathrm{Max}(P)$, we have $(y]_P\cap\mathrm{Max}(P)=\phi$ and so that separation property (s) guarantees the existence of a clopen $\leqq$-decreasing set A_0 such that $y\in A_0$ and $A_0\cap\mathrm{Max}(P)=\phi$. Also, since $y\notin\mathrm{Min}(P)$, the closed set $\{x\}\cup\mathrm{Min}(P)$ has empty intersection with $[y)_P$ and the dual of the separation property (s) guarantees the existence of a clopen $\leqq$-increasing set A_1 such that $y\in A_1$ and $A_1\cap(\{x\}\cup\mathrm{Min}(P))=\phi$. Then set $A=A_0\cap A_1$ is, therefore, a clopen convex subset of $\mathrm{Mid}(P)$ having the property that $y\in A$ and $x\notin A$. The set A also has these properties with respect to Q, since f preserves order and is continuous. Furthermore, since $x\notin\mathrm{Min}(P)$, we can use the separation property (s) to obtain a clopen $\leqq$-decreasing set B such that $x\notin B$ and $\mathrm{Min}(P)\subseteq B$. Again, B is clopen and $\preceq$-decreasing. We claim that $C=A\cup B$ fulfills our needs. Obviously, C is clopen, $y\in C$ and $x\notin C$. To show that C is $\preceq$-decreasing, suppose that $z\preceq c\in C$ and $z\neq c$. If $c\in B$ then $z\in B\subseteq C$, since B is $\preceq$-decreasing, whereas if $c\in A$ then $c\in\mathrm{Mid}(Q)$ and so $z\in\mathrm{Min}(Q)\subseteq B\subseteq C$. Thus, Q is a Priestly space.

Next, we show that $[D)_Q$ is clopen, for any clopen $\preceq$-decreasing set $D\subseteq X$. Since D is $\preceq$-decreasing, we have $\mathrm{Min}_Q(D)=D\cap\mathrm{Min}(Q)$. Moreover, because $m\preceq x$ is equivalent to $m\leqq x$, for any $m\in\mathrm{Min}(Q)=\mathrm{Min}(P)$, we have $\mathrm{Min}_P(D)=\mathrm{Min}_Q(D)=D\cap\mathrm{Min}(P)=M$. This relation between the two orders also implies that $[M)_P=[M)_Q$. However, $[D)_Q=[M)_Q$, since D is $\preceq$-decreasing. Therefore $[M)_P=[D)_Q$. Now, $M\subseteq D$ implies that $\{x\}\cap M=\phi$, for any $x\notin D$. Since M is closed and $\leqq$-decreasing, the separation property (s) ensures the existence of a clopen $\leqq$-decreasing set E_x such that $M\subseteq E_x$ *and* $x\notin E_x$.

Now, $\cup\{X\backslash E_x : x \in X\backslash D\} \supseteq X\backslash D$ and $X\backslash D$ is compact, since D is open. Therefore there is a finite $F \subseteq X\backslash D$ such that $\cup\{X\backslash E_x : x \in F\} \supseteq X\backslash D$. The set $E = \cap\{E_x : x \in F\}$ is clopen, $\leqq$-decreasing and satisfies $M \subseteq E \subseteq D$. Clearly, $M \subseteq \mathrm{Min}_P(E) \subseteq \mathrm{Min}_P(D) = M$ so that $\mathrm{Min}_P(E) = M$ and therefore $[M)_P = [E)_P$, since E is $\leqq$-decreasing. It follows now that $[D)_Q = [E)_P$ which is clopen, since E is clopen and P is a dp-space. This, together with a dual argument, completes the proof of the fact that Q is a dp-space. Summarizing, thus far, $K = D(Q)$ is a congruence permutable extension of L.

Finally, observe that a subset of X is a closed c-set in P if and only if it is a closed c-set in Q, since ordered pairs involving extremal elements are the same in either dp-space, and so $\Theta_K(C)$ is the unique extension of $\Theta_L(C)$ to K, for any clopen c-set C.

Corollary. *The congruence lattice of any distributive double p-algebra is isomorphic to the congruence lattice of some congruence permutable distributive double p-algebra.*

Concluding remarks. Recall that an algebra is *congruence regular* if each of its congruences is uniquely determined by any one of its classes. Varlet [9] has shown that the congruence regular distributive double p-algebras are precisely those having no 3-element chain in their dp-spaces. It is an open question as to whether or not the congruence lattice of an arbitrary distributive double p-algebra is isomorphic to the congruence lattice of some congruence regular distributive double p-algebra.

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