Proceedings of the Edinburgh Mathematical Society (1986) 29, 97–100 ©

NUCLEAR SUBALGEBRAS OF UHF C*-ALGEBRAS

by R. J. ARCHBOLD and ALEXANDER KUMJIAN

(Received 6th February 1985)

1. Introduction

A C*-algebra A is said to be *approximately finite dimensional* (AF) if it is the inductive limit of a sequence of finite dimensional C*-algebras (see [2], [5]). It is said to be *nuclear* if, for each C*-algebra B, there is a unique C*-norm on the *-algebraic tensor product $A \otimes B$ [11]. Since finite dimensional C*-algebras are nuclear, and inductive limits of nuclear C*-algebras are nuclear [16], every AF C*-algebra is nuclear. The family of nuclear C*-algebras is a large and well-behaved class (see [12]). The AF C*-algebras form a particularly tractable sub-class which has been completely classified in terms of the invariant K_0 [7], [5].

Choi [4] showed that a C*-subalgebra of a nuclear C*-algebra can fail to be nuclear. Since the larger algebra in Choi's example is not AF, there remained the question as to whether a C*-subalgebra of an AF C*-algebra is ncessarily nuclear (see [6, Problem 10] and [12, Problem 3]). Blackadar [1] has recently answered this question negatively by showing that every non-type I C*-algebra [13] contains a non-nuclear C*-subalgebra. Nevertheless, in special circumstances it is possible to give a positive answer. In Theorem 3 we show that if D is a C*-subalgebra of a uniformly hyperfinite C*-algebra A (see below) and if D contains a suitable maximal abelian self-adjoint subalgebra (masa) of A then D is not only nuclear but even AF. In Remark 4 we indicate how this result may be generalized to the case where A is a unital AF C*-algebra.

2. Preliminaries

A C*-algebra A is said to be *uniformly hyperfinite* (UHF) if it is the norm-closure of the union of an ascending sequence of subalgebras each of which is *-isomorphic to a full complex matrix algebra and contains the identity of A [9]. Thus any UHF C*-algebra is automatically AF. If A is a UHF C*-algebra then it is *-isomorphic to an infinite tensor product $\bigotimes_{i \geqq 1} A_i$ where $A_i = M_{n(i)}$ (the C*-algebra of all $n(i) \times n(i)$ complex matrices) for some positive integer $n(i)$ [14, Section 4]. A masa B of a UHF C*-algebra A is said to be a standard *diagonal* if there exists a *-isomorphism θ of A onto $\bigotimes_{i \geqq 1} A_i$, where each $A_i = M_{n(i)}$ for some $n(i)$, such that $\theta(B) = \bigotimes_{i \geqq 1} B_i$ where each B_i is a masa of A_i (see [10] for a more general notion). It is well-known that $\bigotimes_{i \geqq 1} B_i$ is indeed a masa of $\bigotimes_{i \geqq 1} A_i$ (see [17, Corollary 11] for a more general result).

Lemma 1. *Let* $A = \bigotimes_{i \geqq 1} A_i$, *where each* $A_i = M_{n(i)}$ *for some* $n(i)$, *and let* $B = \bigotimes_{i \geqq 1} B_i$

E

https://doi.org/10.1017/S0013091500017454 Published online by Cambridge University Press

where each B_i is a masa of A_i, so that $B_i = \mathbb{C}e_1^{(i)} + \cdots + \mathbb{C}e_{n(i)}^{(i)}$ for some maximal orthogonal family $\{e_1^{(i)}, \ldots, e_{n(i)}^{(i)}\}$ of minimal nonzero projections in A_i.

For each i, let R_i be the projection of norm one from A_i onto B_i given by

$$R_i(a) = \sum_{j=1}^{n(i)} e_j^{(i)} a e_j^{(i)} \qquad (a \in A_i).$$

Let $C_i = A_1 \otimes \cdots \otimes A_i \otimes (\bigotimes_{j>i} B_j)$. For $n > i$ let $P_i^{(n)} = \mathrm{id}_1 \otimes \cdots \otimes \cdots \otimes \mathrm{id}_i \otimes R_{i+1} \otimes \cdots \otimes R_n \otimes \mathrm{id}$. Then there exists a projection of norm one $P_i = \mathrm{id}_1 \otimes \cdots \otimes \mathrm{id}_i \otimes (\bigotimes_{j>i} R_j)$ of A onto C_i such that

(i) $P_i(x) = \lim P_i^{(n)}(x) \qquad (x \in A)$,

(ii) $P_i = P_i P_{i+1}$ *for each i,*

(iii) *for any* C*-*algebra C satisfying $B \subseteq C \subseteq A$, $P_i(C) \subseteq C$,*

(iv) $x = \lim P_i(x) \qquad (x \in A)$.

Proof. The construction of P_i is routine so we merely sketch the argument. Let $a \in A$ and let $a = \lim a_n$ where $a_n \in A_1 \otimes \cdots \otimes A_n \otimes \mathbb{C}1$. For $m > n > i$,

$$\|P_i^{(n)}(a_n) - P_i^{(m)}(a_m)\| = \|P_i^{(m)}(a_n) - P_i^{(m)}(a_m)\| \leqq \|a_n - a_m\|.$$

So $(P_i^{(n)}(a_n))_{n \geqq i}$ is a Cauchy sequence in C_i. The limit is independent of (a_n) and is denoted by $P_i(a)$. It is straightforward to check that P_i is a projection of norm one from A onto C_i.

Since $P_i^{(n)}(a) \to P_i(a)$ whenever $a \in A_1 \otimes \cdots \otimes A_k \otimes \mathbb{C}1$ for some k, (i) follows from the fact that $\|P_i^{(n)}\| = 1$ for all $n > i$.

Clearly $P_i = P_i P_{i+1}$ on each algebra $A_1 \otimes \cdots \otimes A_k \otimes \mathbb{C}1$, and so (ii) holds by continuity.

Let $c \in C$. Since $B \subseteq C$, it follows from the definition of the R_k's that $P_i^{(n)}(c) \in C$. Hence (iii) follows from (i).

Since $P_i(a) \to a$ whenever $a \in A_1 \otimes \cdots \otimes A_k \otimes \mathbb{C}1$ for some k, (iv) follows from the fact that $\|P_i\| = 1$ for all i.

Proposition 2. *Let $A = M_n \otimes B$ where B is a unital abelian AF C*-algebra. Let $\{e_1, \ldots, e_n\}$ be a maximal orthogonal family of minimal nonzero projections in M_n. Suppose that D is a C*-subalgebra of A such that $D \supseteq e_i \otimes B\,(1 \leqq i \leqq n)$. Then D is an AF C*-algebra.*

Proof. For $1 \leqq i \leqq n$ *let* J_i be the norm-closure of $D(e_i \otimes 1)D$, so that J_i is a norm-closed two sided ideal of D. Let

$$D_i = (e_i \otimes 1)J_i(e_i \otimes 1) = \mathbb{C}e_i \otimes B.$$

Since $J_i(e_i \otimes 1)J_i \supseteq D(e_i \otimes 1)D$, D_i is a full hereditary C*-subalgebra of J_i (see [3]). Since J_i is separable, $J_i \otimes \mathcal{K} \cong D_i \otimes \mathcal{K}$ [3, Theorem 2.8] (where $\mathcal{K}$ denotes the algebra of compact linear operators on a Hilbert space of countably infinite dimension). Thus

https://doi.org/10.1017/S0013091500017454 Published online by Cambridge University Press

$J_i \otimes \mathcal{K}$ is AF. If e is a minimal nonzero projection in $\mathcal{K}$ then $J_i \cong (1 \otimes e)(J_i \otimes \mathcal{K})(1 \otimes e)$ which is AF since it is a hereditary subalgebra of the AF algebra $J_i \otimes \mathcal{K}$ (see [8, Theorem 3.1], where a more general notion of "AF" is considered, and [5, Lemma 9.4]).

Since $(J_1+J_2)/J_1 \cong J_2/(J_1 \cap J_2)$, $(J_1+J_2)/J_1$ is AF. It follows from the work of Brown and Elliott [5, Theorem 9.9] that J_1+J_2 is AF. Proceeding in the obvious way, one obtains that $J_1+J_2+\cdots+J_n=D$ is AF.

3. The main result

Theorem 3. *Let A be a UHF algebra and let D be a C*-subalgebra of A which contains a standard diagonal B of A. Then D is an AF C*-algebra (and hence is nuclear).*

Proof. We may assume $B=\bigotimes_{i\geq 1} B_i \subseteq D \subseteq A=\bigotimes_{i\geq 1} A_i$ where B_i is a masa of $A_i=M_{n(i)}$. Let $P_i (i \geq 1)$ be as in Lemma 1. Let $D_i=P_i(D)$. By (iii),

$$D \cap C_i \subseteq P_i(D) \subseteq D \cap C_i.$$

Thus $D_i=D \cap C_i$, a (necessarily liminary) C*-subalgebra of the liminary C*-algebra C_i. Since $B \subseteq D \cap C_{i+1}=D_{i+1}$, it follows from (ii) and (iii) that $D_i=P_i(D_{i+1}) \subseteq D_{i+1}$. It follows from (iv) that $D=$norm-closure $(\bigcup_{i\geq 1} D_i)$. Since D_i is liminary and hence nuclear [16], the nuclearity of D is now apparent. However we shall show that D is actually AF.

By Proposition 2, D_i is AF $(i \geq 1)$. Since D is separable, it follows from [2, Theorem 2.2] that D is AF.

Remark 4. We indicate how the above result may be extended to the case where A is a unital AF C*-algebra. Let $\{A_i\}$ be an increasing sequence of unital finite dimensional subalgebras. To construct a standard diagonal in A, one chooses a sequence of masas $B_i \subseteq A_i$ inductively as in [15]; given B_i, let B_{i+1} be generated by B_i and a masa in the relative commutant of A_i in A_{i+1}. Let B denote the closure of $\cup B_i$ and note that B is a hyperfinite diagonal in A (see [10, Section 6.16]).

As in Lemma 1 above, there is for each i a conditional expectational, $P_i: A \to C_i$ (where C_i is the subalgebra generated by B and A_i), satisfying properties (ii), (iii), (iv), (see [10, Proposition 6.15]). Hence, if D is a subalgebra of A which contains B, then it is generated by the family of subalgebras $\{D \cap C_i\}$. Since $D \cap C_i$ is an AF C*-algebra (C_i may be expressed as a finite direct sum of C*-algebras of the form considered in Proposition 2), D is an AF C*-algebra, as well.

Acknowledgements. The first-named author is grateful to the University of California for supporting a visit to the Mathematical Sciences Research Institute at Berkeley during which this work was completed. The second author wishes to acknowledge the support of the Australian Research Grants Scheme and the hospitality of the M.S.R.I.

https://doi.org/10.1017/S0013091500017454 Published online by Cambridge University Press

REFERENCES

1. B. Blackadar, Nonnuclear subalgebras of C*-algebras, *J. Operator Theory*. (to appear).

2. O. Bratteli, Inductive limits of finite dimensional C*-algebras, *Trans. Amer. Math. Soc.* **171** (1972), 195–234.

3. L. Brown, Stable isomorphism of hereditary subalgebras of C*-algebras, *Pacific J. Math.* **71** (1977), 335–348.

4. M-D. Choi, A simple C*-algebra generated by two finite-order unitaries, *Canad. J. Math.* **31** (1979), 867–880.

5. E. G. Effros, *Dimensions and C*-algebras* (CBMS Regional Conference Series No. 46, Amer. Math. Soc., Providence, R.I., 1981).

6. E. G. Effros, On the structure of C*-algebras: Some old and new problems, *Proc. Symp. Pure Math.* **38** (1982), 19–34.

7. G. A. Elliott, On the classification of inductive limits of sequences of semi-simple finite dimensional algebras, *J. Algebra* **38** (1976), 29–44.

8. G. A. Elliott, Automorphisms determined by multipliers on ideals of a C*-algebra, *J. Functional Analysis* **23** (1976), 1–10.

9. J. Glimm, On a certain class of operator algebras, *Trans. Amer. Math. Soc.* **95** (1960), 318–340.

10. A. Kumjian, On *C**-diagonals, preprint.

11. E. C. Lance, On nuclear C*-algebras, *J. Functional Analysis* **12** (1973), 157–176.

12. E. C. Lance, Tensor products and nuclear C*-algebras, *Proc. Symp. Pure Math.* **38** (1982), 379–399.

13. G. K. Pedersen, *C*-algebras and their automorphism groups* (Academic Press, London, 1979).

14. R. T. Powers, Representations of uniformly hyperfinite algebras and their associated Von Neumann rings, *Annals of Math.* **86** (1967), 138–171.

15. S. Strătilă and D. Voiculescu, *Representations of AF algebras and of the group* $U(\infty)$ (Lecture notes in Math. 486, Springer-Verlag, Berlin, 1975).

16. M. Takesaki, On the cross-norm of the direct product of C*-algebras, *Tôhoku Math. J.* **16** (1964), 111–122.

17. A. S. Wassermann, The slice map problem for C*-algebras, *Proc. London Math. Soc.* (3) **32** (1976), 537–559.

Department of Mathematics
University of California
Santa Barbara
California, 93106
USA

Permanent Address:
Department of Mathematics
The Edward Wright Building
Dunbar Street
Aberdeen AB9 2TY
UK

School of Mathematics
University of New South Wales
Kensington
N.S.W. 2033
Australia

https://doi.org/10.1017/S0013091500017454 Published online by Cambridge University Press