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POSITIVE ALMOST PERIODIC SOLUTIONS FOR THE HEMATOPOIESIS MODEL VIA THE HILBERT PROJECTIVE METRIC

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Abstract

The aim of this work is to prove the existence of a positive almost periodic solution to a multifinite time delayed nonlinear differential equation that describes the so-called hematopoiesis model. The approach uses the Hilbert projective metric in a cone. With some additional assumptions, we construct a fixed point theorem to prove the desired existence and uniqueness of the solution.

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1. Introduction

To describe some physiological control systems in the classic study of population dynamics, Mackey and Glass [8] proposed the following autonomous nonlinear delay differential equation

$$\rho'(t) = -\gamma\rho(t) + \frac{\beta\rho^m(t-\tau)}{1+\rho^n(t-\tau)}, \tag{1.1}$$

where $\gamma, \beta, n \in (0, +\infty)$, $\tau, m \in [0, +\infty)$, $\rho(t)$ denotes the density of mature blood cells circulating in the bloodstream and γ is the rate of loss of blood cells from the circulation. Here, $f(\rho(t-\tau)) = \beta\rho^m(t-\tau)/(1+\rho^n(t-\tau))$ is the flux of blood cells into the bloodstream from the stem cell compartment and depends on the delayed density, $\rho(t-\tau)$, of mature cells in circulation, where τ is the time delay between the production of the immature cells in the bone marrow and the release of the mature cells into the bloodstream.

In [9, 13], the model (1.1) is extended to the following nonautonomous nonlinear delay differential equation with time-varying coefficients and delays, which takes more account of real phenomena, such as the important role played by variations in the environment: namely,

$$x'(t) = -a(t)x(t) + \sum_{i=1}^{k} \frac{b_i(t)x^m(t-\tau_i(t))}{1 + x^n(t-\tau_i(t))}, \tag{1.2}$$

where $0 \le m < n$, the functions $a, b_i, \tau_i : \mathbb{R} \longrightarrow (0, +\infty)$ are continuous for $i = 1, 2, \ldots, k$, $x(t)$ is the density of mature blood cells circulating in the bloodstream and $a(t)$ is the rate of loss of blood cells from the circulation at time t. Now, $f(x(t-\tau_i(t))) = b_i(t)x^m(t-\tau_i(t))/(1 + x^n(t-\tau_i(t)))$ is the flux of blood cells into the bloodstream from the ith stem cell compartment for $i = 1, 2, \ldots, k$ and $\tau_i(t)$ is the corresponding time delay between the production of immature cells in the bone morrow in the ith compartment and the release of the mature cells into the bloodstream. The model (1.1) and its extension (1.2) are referred to as a model of hematopoiesis (cell production).

The existence of periodic and almost periodic solutions to the hematopoiesis model has been investigated extensively (see, for example, [4, 7, 12, 14, 15] and references therein). Some interesting results concerning the existence of almost periodic solutions to the hematopoiesis model (1.2) were obtained by Zhang *et al.* using the contraction mapping principle in the case $m = 0$ [14] and by using some additional conditions in the cases $m = 0$ or $m = 1$ [12, 14, 15]. For $0 \le m \le 1$ and assuming an *a priori* estimate that controls the balance between the loss rate and the flux, Liu [7] proved the existence of a positive almost periodic solution to (1.2). Recently, Diagana *et al.* [4] established the existence of a positive almost periodic solution to (1.2) via a fixed-point theorem in a cone.

Our purpose in this work is also to give criteria for the existence and uniqueness of a positive almost periodic solution to (1.2). By contrast with Diagana *et al.* [4], we use here the Hilbert projective metric in a cone, which allows the interior to accommodate a complete metric space structure. We follow the development presented in [5].

This paper is organised as follows. In the next section, we present some preliminaries that will be used to prove the main result. Section 3 deals with the main result on the existence and uniqueness of a positive almost periodic solution to (1.2) and the last section gives a concrete illustration of our result.

2. Preliminaries

We recall some definitions, notation and lemmas, which will be used later.

Definition 2.1 [2, 10]. Let X be a Banach space. A continuous function $f : \mathbb{R} \to X$ is called almost periodic if, for each $\epsilon > 0$, there exists $l(\epsilon) > 0$ such that every interval of length $l(\epsilon)$ contains a number τ with the property that $\|f(t+\tau) - f(t)\| \le \epsilon$ for each $t \in \mathbb{R}$. The number τ is called an ϵ-translation number of $f(t)$. Denote the set of such functions by $AP(\mathbb{R})$.

Lemma 2.2. *The intersection of two relatively dense subsets of* $\mathbb{R}$ *is relatively dense.*

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PROOF. Let $\mathcal{T}$ and $\mathcal{T}'$ be two relatively dense subsets of $\mathbb{R}$ and put $\tilde{\mathcal{T}} = \mathcal{T} \cap \mathcal{T}'$. For all $\epsilon > 0$, there exist $L(\mathcal{T}, \epsilon)$ and $L(\mathcal{T}', \epsilon)$ such that $\mathcal{T} \cap [t, t + L(\mathcal{T}, \epsilon)] \neq \emptyset$ and $\mathcal{T}' \cap [t, t + L(\mathcal{T}', \epsilon)] \neq \emptyset$ for all $t \in \mathbb{R}$. If $L(\tilde{\mathcal{T}}, \epsilon) = \max(L(\mathcal{T}, \epsilon), L(\mathcal{T}', \epsilon))$, then $\tilde{\mathcal{T}} \cap [t, t + L(\tilde{\mathcal{T}}, \epsilon)] \neq \emptyset$, which achieves the proof. □

2.1. Exponential dichotomy.

DEFINITION 2.3 [6, 10]. Let $x \in \mathbb{R}^n$ and let $A(t)$ be an $n \times n$ continuous matrix defined on $\mathbb{R}$. The linear system

$$x'(t) = A(t)x(t) \tag{2.1}$$

admits an exponential dichotomy on $\mathbb{R}$ if there exist positive constants λ, k and a projection P such that the fundamental solution matrix $X(t)$ of (2.1) satisfies

$$\|X(t)PX^{-1}(s)\| \leq \lambda e^{-k(t-s)} \quad \text{for } t \geq s, \quad \|X(t)(I-P)X^{-1}(s)\| \leq \lambda e^{-k(s-t)} \quad \text{for } t \leq s.$$

LEMMA 2.4 [6, 10]. *Let $c_i(t)$ be an almost periodic function on $\mathbb{R}$ and let*

$$M[c_i] = \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} c_i(s)\, ds > 0 \quad \textit{for } i = 1, 2, \ldots, n.$$

Then the linear system $x'(t) = C(t)x(t)$ admits an exponential dichotomy on $\mathbb{R}$, where $C(t) = \operatorname{diag}(-c_1(t), -c_2(t), \ldots, -c_n(t))$.

LEMMA 2.5 [1]. *Let f be an almost periodic function. If the linear system (2.1) admits an exponential dichotomy, then the almost periodic system*

$$x'(t) = A(t)x(t) + f(t)$$

has a unique almost periodic solution $x(t)$ and

$$x(t) = \int_{-\infty}^{t} X(t)PX^{-1}(s)f(s)\, ds - \int_t^{+\infty} X(t)(I-P)X^{-1}(s)f(s)\, ds,$$

where $X(t)$ is the fundamental solution matrix of (2.1).

2.2. The Hilbert projective metric and fixed point theorem in a cone.

Let X be a real Banach space. A closed convex set P in X is called a convex cone if the following conditions are satisfied:

(i) if $x \in P$, then $\lambda x \in P$ for all $\lambda \geq 0$; and
(ii) if $x \in P$ and $-x \in P$, then $x = 0$.

A partial ordering $\leq$ in X is induced by P: for all $x, y \in X$, $x \leq y$ if and only if $y - x \in P$. Given $u, v \in P$, define the interval $[u, v] := \{x \in X : u \leq x \leq v\}$.

A cone P is called normal if there exists a constant $k > 0$ such that, for all $x, y \in P$, $0 \leq x \leq y$ implies that $\|x\| \leq k\|y\|$, where $\|\cdot\|$ is the norm on X.

If P is now a general cone in a Banach space X and x and y are elements of $P^* = P - 0_X$, we say that x and y are comparable if there exist real numbers $\alpha > 0$ and $\beta > 0$ such that $\alpha x \leq y \leq \beta x$. This defines an equivalence relation on P^* and divides P^*

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into disjoint subsets which we call components of P. If x and y are comparable, then we define the numbers $m(y/x)$ and $M(y/x)$ by

$$m(y/x) := \sup\{\alpha > 0 : \alpha x \le y\} \quad \text{and} \quad M(y/x) := \inf\{\beta > 0 : y \le \beta x\}.$$

Thompson [11] introduced a metric d defined as follows: if x and y in P^* are comparable, then

$$d(x, y) := \max(\log M(y/x), \log M(x/y)) = \max(\log M(y/x), -\log m(y/x)). \tag{2.2}$$

If C is a component of P, it is easy to see that d gives a metric on C. Moreover, Thompson proved the following results.

Theorem 2.6 [11]. *Let P be a normal cone in a Banach space X and let C be a component of P. Then C is a complete metric space with respect to the metric d.*

Proposition 2.7 [11]. *Let P be a normal cone in a Banach space X with nonempty interior P°. Then P° is a component of P.*

Thus, the nonempty interior P° of a normal cone P, is a complete metric space with respect to the metric d. We have the following theorem.

Theorem 2.8 [3]. *Let E be a complete space with respect to the metric d. Suppose there is a mapping f from E into E satisfying*

$$d(f(x), f(y)) \le \phi(d(x, y)) \quad \textit{for all } x \textit{ and } y \textit{ in } E,$$

where ϕ is a positive nondecreasing function continuous on $[0, +\infty[$, such that $\phi(r) < r$ for all $r > 0$ and $\phi(0) = 0$. Then f has exactly one fixed point in E.

For a bounded continuous function $h(t)$, we introduce the notation

$$h^+ = \sup_{t\in\mathbb{R}} h(t), \quad h^- = \inf_{t\in\mathbb{R}} h(t). \tag{2.3}$$

3. Positive almost periodic solutions

This section contains the proof of our main result on the existence and uniqueness of a positive almost periodic solution of (1.2). The proof is based on a fixed point theorem in a cone endowed with the Hilbert projective metric.

3.1. Assumptions and main result. For $t \in \mathbb{R}$ and $x \in \mathbb{R}^+$, put

$$F(t, x) = \sum_{i=1}^{k} b_i(t) f[x(t - \tau_i(t))] = \sum_{i=1}^{k} \frac{b_i(t) x^m(t - \tau_i(t))}{1 + x^n(t - \tau_i(t))}.$$

We make two sets of assumptions. The first gives some preliminary estimates and the second deals with the behaviour of flux term. Here, a^+, a^-, b^+ and b^- are as in (2.3).

(H1) $a^- > 0, b_i^- > 0, \tau_i \ge 0$ for $i = 1, 2, \ldots, m$.
(H2) $(\sum_{i=1}^{k} b_i^+)/a^- \le (n/(n - m))(m/(n - m))^{(1-m)/n}$.

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(H3) $(\sum_{i=1}^{k} b_i^-)/a^+ > 1$, that is, the flux is always greater then the loss of blood cells in circulation.

(H4) τ_i and b_i for $i = 1, 2, \ldots, k$ are almost periodic.

Define

$$f(x) = \begin{cases} \dfrac{x^m}{1 + x^n} & \text{for } 0 \le x \le \left(\dfrac{m}{n-m}\right)^{1/n}, \\ \dfrac{n-m}{n}\left(\dfrac{m}{n-m}\right)^{m/n} & \text{for } x > \left(\dfrac{m}{n-m}\right)^{1/n}. \end{cases} \tag{3.1}$$

The following theorem is our main result.

Theorem 3.1. *Suppose that* **(H1)–(H4)** *hold. Then (1.2) has exactly one positive almost periodic solution.*

3.2. Proof of the main result.

Lemma 3.2. *Suppose that φ and σ are in $AP(\mathbb{R})$. Then the function $t \mapsto \varphi(t - \sigma(t))$ is also in $AP(\mathbb{R})$.*

Proof. Let $\epsilon > 0$ and let τ be a common almost period for φ and σ: that is $\tau \in \tilde{\mathcal{T}}(\varphi, \sigma, \epsilon) = \tilde{\mathcal{T}}(\varphi, \epsilon) \cap \tilde{\mathcal{T}}(\sigma, \epsilon)$, where $\tilde{\mathcal{T}}(\varphi, \epsilon)$ and $\tilde{\mathcal{T}}(\sigma, \epsilon)$ are the sets of ϵ-almost periods associated, respectively, to φ and σ. Note that $\tilde{\mathcal{T}}(\varphi, \sigma, \epsilon) = \tilde{\mathcal{T}}(\varphi, \epsilon) \cap \tilde{\mathcal{T}}(\sigma, \epsilon)$ is relatively dense in $\mathbb{R}$, by Lemma 2.2. Consider

$$\begin{aligned} |\varphi(t - \sigma(t)) - \varphi(t + \tau - \sigma(t + \tau))| &\le |\varphi(t - \sigma(t)) - \varphi(t - \sigma(t) + \tau)| \\ &\quad + |\varphi(t + \tau - \sigma(t)) - \varphi(t + \tau - \sigma(t + \tau))|. \end{aligned}$$

Let $\tilde{\epsilon} > 0$. By the uniform continuity of φ, there exists $\epsilon > 0$ such that, for all $t \in \mathbb{R}$, $|\sigma(t) - \sigma(t + \tau)| \le \epsilon$ implies that $|\varphi(t + \tau - \sigma(t)) - \varphi(t + \tau - \sigma(t + \tau))| \le \tilde{\epsilon}$. By the almost periodicity of φ, also $|\varphi(t - \sigma(t)) - \varphi(t - \sigma(t) + \tau)| \le \epsilon$. Thus we deduce the almost periodicity of $t \mapsto \varphi(t - \sigma(t))$. □

Lemma 3.3. *Suppose* **(H1)**, **(H2)** *and* **(H4)** *hold. Then (1.2) has a nonnegative almost periodic solution x which is given for $t \in \mathbb{R}$ by*

$$x(t) = \int_{-\infty}^{t} g(s) \sum_{i=1}^{k} b_i(s) f(x(s - \tau_i(s)))\, ds \quad \text{where } g(s) = \exp\left(-\int_{-s}^{t} a(r)\, dr\right). \tag{3.2}$$

In fact, every nonnegative almost periodic solution φ of (1.2) is also a nonnegative almost periodic solution of (3.2) and vice versa.

Proof. If φ is a positive almost periodic solution of (1.2), then, by hypothesis **(H4)** and Lemma 3.2, $\varphi(\cdot - \tau_i(\cdot))$ is almost periodic for $i = 1, 2, \ldots, k$. Therefore, the function $\sum_{i=1}^{k} b_i(\cdot)\varphi^m(\cdot - \tau_i(\cdot))/(1 + \varphi^n(\cdot - \tau_i(\cdot))) \in AP(\mathbb{R})$. Since $a^- > 0$, from **(H1)**, Lemmas 2.4 and 2.5 yield

$$\varphi(t) = \int_{-\infty}^{t} g(s) \sum_{i=1}^{k} \frac{b_i(s)\varphi^m(s - \tau_i(s))}{1 + \varphi^n(s - \tau_i(s))}\, ds \quad \text{for } t \in \mathbb{R}.$$

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Note that

$$\sup_{x\geq 0}\frac{x^m}{1+x^n}=\frac{n-m}{n}\Big(\frac{m}{n-m}\Big)^{m/n}.$$

So, by **(H2)**,

$$\begin{aligned}\varphi(t)&\leq\int_{-\infty}^{t}e^{-a^-(t-s)}\Big(\sum_{i=1}^{k}b_i^+\frac{n-m}{n}\Big(\frac{m}{n-m}\Big)^{m/n}\Big)\,ds\\&=\frac{\sum_{i=1}^{k}b_i^+}{a^-}\,\frac{n-m}{n}\Big(\frac{m}{n-m}\Big)^{m/n}\leq\Big(\frac{m}{n-m}\Big)^{1/n}.\end{aligned}$$

By (3.1), $f(s-\tau-i(s))=\varphi^m(s-\tau_i(s))/(1+\varphi^n(s-\tau_i(s))$ for $s\in\mathbb{R}$ and $i=1,2,\dots,k$. Thus,

$$\varphi(t)=\int_{-\infty}^{t}g(s)\sum_{i=1}^{k}b_i(s)f(\varphi(s-\tau_i(s)))\,ds\quad\text{for } t\in\mathbb{R}$$

is an almost periodic solution of (3.2).

Similarly, we can show that every nonnegative almost periodic solution φ of (3.2) is also an almost periodic solution of (1.2). □

In the subsequent work, $Q=\{x\in AP(\mathbb{R}):x(t)\geq 0,$ for all $t\in\mathbb{R}\}$ denotes the normal solid cone in $AP(\mathbb{R})$ and $Q^\circ=\{x\in AP(\mathbb{R}):$ there is $\epsilon>0$ such that $x(t)>\epsilon$, for all $t\in\mathbb{R}\}$ denotes its interior. Let $\mathcal{T}$ be an operator on Q° defined by

$$\mathcal{T}(x)(t)=\int_{-\infty}^{t}g(s)\sum_{i=1}^{k}b_i(s)f[x(s-\tau_i(s))]\,ds\quad\text{for } t\in\mathbb{R},$$

$$\text{where } g(s)=\exp\Big(-\int_{-s}^{t}a(r)\,dr\Big).\tag{3.3}$$

Proposition 3.4. *$\mathcal{T}$ maps Q° into itself.*

Proof. Let x be in Q°. By Lemma 3.3, $\mathcal{T}(x)$ is an almost periodic function. In addition, there exists $\epsilon_0>0$ such that $x(t)\geq\epsilon_0$ for all $t\in\mathbb{R}$. Thus

$$\begin{aligned}\mathcal{T}(x)(t)&\geq\int_{-\infty}^{t}e^{-a^+(t-s)}\sum_{i=1}^{k}b_i^-\cdot\min\Big\{\frac{\epsilon_0^m}{1+\epsilon_0^n},\frac{n-m}{n}\Big(\frac{m}{n-m}\Big)^{m/n}\Big\}\,ds\\&=\frac{\sum_{i=1}^{k}b_i^-}{a^+}\cdot\min\Big\{\frac{\epsilon_0^m}{1+\epsilon_0^n},\frac{n-m}{n}\Big(\frac{m}{n-m}\Big)^{m/n}\Big\}=\frac{\sum_{i=1}^{k}b_i^-}{a^+}\cdot\frac{\epsilon_0^m}{1+\epsilon_0^n}.\end{aligned}$$

By **(H3)**, $\mathcal{T}(x)(t)>0$ for all $t\in\mathbb{R}$, which implies that $\mathcal{T}(x)\in Q^\circ$. □

Next, we will prove the fixed point theorem for the operator $\mathcal{T}$.

Proposition 3.5. *$\mathcal{T}$ is a nondecreasing operator on Q°.*

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Proof. Choose x and y in Q° such that $x(t) \geq y(t)$ for all $t \in \mathbb{R}$ and define $g(s)$ as in (3.3). Then, for all $t \in \mathbb{R}$,

$$\mathcal{T}(x)(t) - \mathcal{T}(y)(t) = \int_{-\infty}^{t} g(s) \sum_{i=1}^{k} b_i(s)(f[x(s-\tau_i(s))] - f[y(s-\tau_i(s))])\, ds.$$

From (3.1), f is nondecreasing on $[0, (m/(n-m))^{1/n}]$ and remains constant on $[(m/(n-m))^{1/n}, \infty)$. Therefore, $\mathcal{T}(x)(t) - \mathcal{T}(y)(t) \geq 0$ for all $t \in \mathbb{R}$. □

Proposition 3.6. *Define the metric d as in (2.2). There exists a positive nondecreasing function ϕ defined on $\mathbb{R}^+$, satisfying*

$$\phi(0) = 0, \quad \phi(r) < r \quad \textit{for } r > 0 \quad \textit{and} \quad d(\mathcal{T}(x), \mathcal{T}(y)) \leq \phi(d(x,y)) \quad \textit{for } x, y \in Q^\circ.$$

Proof. Let x and y be two comparable functions in Q° and let $\alpha = m(y/x), \beta = M(y/x)$. Then $\alpha x \leq y \leq \beta x$ and, from (2.2), $d(x,y) = \max(\log(\beta), -\log(\alpha))$. By Proposition 3.5, the operator $\mathcal{T}$ is nondecreasing and so

$$\mathcal{T}(\alpha x) \leq \mathcal{T}(y) \leq \mathcal{T}(\beta x). \tag{3.4}$$

Thus, we have the following cases.

Case 1. $\beta \in (0,1)$. Then $\alpha \in (0,1)$ and $f(\alpha x) \geq \alpha^n f(x)$. Therefore, $\mathcal{T}(\alpha x) \geq \alpha^m \mathcal{T}(x)$ and the left-hand side of (3.4) gives $\varphi(\alpha)\mathcal{T}(x) \leq \mathcal{T}(y)$, where $\varphi(\alpha) = \alpha^m$. For the right-hand side (3.4), consider the nondecreasing function

$$\chi(x) = \frac{x^n}{1 + \beta^n x^n} \quad \text{for } 0 < x \leq \Big(\frac{m}{n-m}\Big)^{1/n},$$

which attains its maximum $\psi(\beta) = [(n/m)(1-\beta^n) - 1]^{-1}$ when $x = (m/(n-m))^{1/n}$. Then,

$$\frac{f(\beta x)}{f(x)} = \beta^m \frac{1 + x^n}{1 + \beta^n x^n} \leq \beta^m [1 + \chi(x)] \leq \psi(\beta)$$

and we conclude that $\mathcal{T}(y) \leq \psi(\beta)\mathcal{T}(x)$, Therefore, $\alpha x \leq y \leq \beta x$ implies that

$$\varphi(\alpha)\mathcal{T}(x) \leq \mathcal{T}(y) \leq \psi(\beta)\mathcal{T}(x). \tag{3.5}$$

Note from (3.5) that, for all $\alpha, \beta \in (0,1)$ with $\alpha \leq \beta$, $\varphi(\alpha) \leq \psi(\beta)$. To compute the metric, note that $M(\mathcal{T}(x), \mathcal{T}(y)) \leq \psi(\beta)$ and $m(\mathcal{T}(x), \mathcal{T}(y)) \geq \varphi(\alpha)$, and hence

$$d(\mathcal{T}(x), \mathcal{T}(y)) \leq \max(\log(\Psi(\beta)), -\log(\varphi(\alpha))).$$

Define the function ϕ by $\phi(0) = 0$ and, for $u > 0$,

$$\phi(u) = \max[-\log(\varphi(e^{-u})), \log(\psi(e^{u}))]. \tag{3.6}$$

Then ϕ is a nondecreasing function and we obtain

$$\begin{aligned}\phi(-\log(\alpha)) &= \max[-\log(\varphi(e^{-(-\log\alpha)})), \log(\psi(e^{-\log\alpha}))]\\ &= \max[-\log(\varphi(\alpha)), \log(\psi(\alpha^{-1}))]\end{aligned}$$

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and

$$\begin{aligned}\phi(\log(\beta)) &= \max[-\log(\varphi(e^{-\log\beta})), \log(\psi(e^{\log\beta}))]\\ &= \max[-\log(\varphi(\beta^{-1})), \log(\psi(\beta))].\end{aligned}$$

Thus,

$$d(\mathcal{T}(x), \mathcal{T}(y)) \le \phi(d(x, y)).$$

Case 2. $\alpha > 1$. Rewrite $\alpha x \le y \le \beta x$ as $\beta^{-1} y \le x \le \alpha^{-1} y$ and apply Case 1. This yields

$$\psi^{-1}(\alpha^{-1})\mathcal{T}(x) \le \mathcal{T}(y) \le \varphi^{-1}(\beta^{-1})\mathcal{T}(x).$$

Next, $M(\mathcal{T}(x), \mathcal{T}(y)) \le \varphi^{-1}(\beta^{-1})$ and $m(\mathcal{T}(x), \mathcal{T}(y)) \ge \psi^{-1}(\alpha^{-1})$, so that

$$d(\mathcal{T}(x), \mathcal{T}(y)) \le \max(\log(\varphi^{-1}(\beta^{-1})), -\log(\psi^{-1}(\alpha^{-1}))).$$

Define the function ϕ as in (3.6). Then

$$\begin{aligned}\phi(-\log(\alpha)) &= \max[-\log(\varphi(e^{-(-\log\alpha)})), \log(\psi(e^{-\log\alpha}))]\\ &= \max[-\log(\varphi(\alpha)), \log(\psi(\alpha^{-1})] = \max[\log(\varphi^{-1}(\alpha)), -\log(\psi^{-1}(\alpha^{-1}))]\end{aligned}$$

and

$$\begin{aligned}\phi(\log(\beta)) &= \max[-\log(\varphi(e^{-\log\beta})), \log(\psi(e^{\log\beta}))]\\ &= \max[-\log(\varphi(\beta^{-1})), \log(\psi(\beta))] = \max[\log(\varphi^{-1}(\beta^{-1})), -\log(\psi^{-1}(\beta))].\end{aligned}$$

Thus,

$$d(\mathcal{T}(x), \mathcal{T}(y)) \le \phi(d(x, y)).$$

Case 3. $\alpha \le 1$ and $\beta \ge 1$. This case is easily deduced from the previous cases giving

$$\varphi(\alpha)\mathcal{T}(x) \le \mathcal{T}(y) \le \varphi^{-1}(\beta^{-1})\mathcal{T}(x).$$

Here, $M(\mathcal{T}(x), \mathcal{T}(y)) \le \varphi^{-1}(\beta^{-1})$ and $m(\mathcal{T}(x), \mathcal{T}(y)) \ge \varphi(\alpha)$. Thus,

$$d(\mathcal{T}(x), \mathcal{T}(y)) \le \max(\log(\varphi^{-1}(\beta^{-1})), -\log(\varphi(\alpha))).$$

Define the function ϕ by $\phi(u) = -\log(\varphi(e^{-u}))$ for $u > 0$, and $\phi(0) = 0$. Then $-\log(\varphi(\alpha)) = \phi(-\log(\alpha))$ and $\log(\varphi^{-1}(\beta^{-1})) = \phi(\log(\beta))$ and, by the monotonicity of ϕ, we conclude that

$$d(\mathcal{T}(x), \mathcal{T}(y)) \le \phi(d(x, y)).$$

This completes the proof of the proposition. □

PROOF OF THEOREM 3.1. By Proposition 3.6, the operator $\mathcal{T}$ satisfies all assumptions of Theorem 2.8 and so it has exactly one fixed point $z \in Q^\circ$. By Lemma 3.3, this gives the unique almost periodic solution of (1.2). □

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4. Application

The following example illustrates our results. Consider the hematopoiesis model

$$x'(t) = -\frac{1}{2}\Big(1 + \frac{1}{2}\cos t\Big)x(t) + \frac{1}{2}\Big(2 + \frac{1}{2}|\cos\sqrt{2}t|\Big)\frac{x^{1/4}(t - 2e^{\sin^2 t})}{1 + x^{1/2}(t - 2e^{\sin^2 t})} + \frac{1}{2}\Big(2 + \frac{1}{2}|\sin\sqrt{3}t|\Big)\frac{x^{1/4}(t - 2e^{\sin^2 t})}{1 + x^{1/2}(t - 2e^{\sin^2 t})}. \tag{4.1}$$

The first term on the right-hand side, $a(t) = \frac{1}{2}(1 + \frac{1}{2}\cos t)$, represents the loss rate from the circulation, the flux rate is $F(t, x) = (b_1(t) + b_2(t))x^m/(1 + x^n)$, where $m = \frac{1}{4}$, $n = \frac{1}{2}$, $b_1(t) = \frac{1}{2}(2 + \frac{1}{2}|\cos\sqrt{2}t|)$ and $b_2(t) = \frac{1}{2}(2 + \frac{1}{2}|\sin\sqrt{3}t|)$, and the delays are $\tau_1(t) = \tau_2(t) = 2e^{\sin^2 t}$. It follows that

$$\lim_{T\to\infty}\frac{1}{T}\int_t^{t+T} a(s)\,ds = \frac{1}{2},$$

which implies that the equation

$$x'(t) = -\frac{1}{2}\Big(1 + \frac{1}{2}\cos t\Big)x(t) \quad t \in \mathbb{R},$$

has an exponential dichotomy. We know that $a^+ = \frac{3}{4}$, $a^- = \frac{1}{4}$ and, for $i = 1, 2$, $b_1^- = b_2^- = 1$ and $\tau_i \geq 0$, so hypothesis **(H1)** holds. Also, $b_1^+ = b_2^+ = \frac{5}{4}$, so hypotheses **(H2)** and **(H3)** hold as well. For $i = 1, 2$, b_i and τ_i are almost periodic, so hypothesis **(H4)** holds. In addition, from (3.1), $F(t, x)$ is a nondecreasing function in x and, since $m = \frac{1}{4} < n = \frac{1}{2}$, the density of cells in the blood is always less than one (that is, $x \leq 1$). Consequently, we can apply Theorem 3.1 to this example.

THEOREM 4.1. *The Hematopoiesis model (4.1) has a unique nonnegative almost periodic solution.*

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