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Computation of eigenvectors corresponding to multiple eigenvalues

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This paper examines a class of common numerical methods for computing eigenvectors of a compact linear operator. A necessary and sufficient condition is established for every element of an arbitrary given eigenspace to be the limit of a sequence of the approximate eigenvectors obtained by any given method in this class.

This paper studies convergence of approximate operator methods [*1*] (which include all projection methods as special cases) for numerical computation of the eigenvectors of a compact linear operator in Hilbert space. For the sake of generality and simplicity, results are proved in an abstract setting. Application of results of this type to specific numerical methods and to differential equations is described in [*1*].

In the case of simple nonzero eigenvalues, approximate operator methods are known to be convergent for both eigenvalues and eigenvectors. In the case of a (nonzero) multiple eigenvalue it is known that, although the sequence $\{\psi_n\}$ of approximate eigenvectors is not always convergent, the component of ψ_n outside the corresponding eigenspace converges to zero [*1*]. In the case of projection methods, Vaĭnikko [*3*] has proved a quantitative version of this result. However Pol'skiĭ [*2*] has shown, by means of an example, that a compact linear operator may have an eigenvector which is not a limit point of any sequence of approximate eigenvectors obtained by a projection method.

This paper establishes a necessary and sufficient condition for every

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element of a given eigenspace of a compact linear operator to be the limit of a sequence of the approximate eigenvectors obtained by an approximate operator method. Note that, by definition of U_n below, $\dim U_n$, the dimension of U_n, satisfies $1 \le \dim U_n \le m$ for all n. Hence Theorem 1 of [1] is the special case $m = 1$ of the theorem proved below.

THEOREM. *Let T_n $(n = 1, 2, \ldots)$ and T be compact linear operators, mapping a (real or complex) Hilbert space H into itself. Let E be the set of all unit vectors in H. Let μ be a nonzero eigenvalue of T and let the corresponding eigenspace V have dimension m. For each positive integer n, let μ_n be an eigenvalue of T_n with corresponding eigenspace V_n. Let*

(1) $$\|T_n-T\| \to 0 \text{ and } |\mu_n-\mu| \to 0 \text{ as } n \to \infty .$$

For all n for which $\dim V_n > m$ let U_n be an arbitrary given m-dimensional subspace of V_n. For all other n let $U_n = V_n$. Then the following conditions are equivalent:

(a) for all but finitely many n, $\dim V_n \ge m$;

(b) for all ψ in $V \cap E$, there is a sequence $\{\psi_n\}$ with ψ_n in $U_n \cap E$ and $\psi_n \to \psi$ as $n \to \infty$;

(c) for all ψ in $V \cap E$, there is a sequence $\{\psi_n\}$ with ψ_n in $V_n \cap E$ and $\psi_n \to \psi$ as $n \to \infty$.

Proof. *(a) ⇒ (b)*. Suppose the contrary. Then there must be an infinite strictly increasing sequence S of positive integers such that

(2) $$(\exists\, d > 0)(\forall\, n \in S)\left(\forall\, \psi_n \in U_n \cap E\right) \|\psi_n-\psi\| > d .$$

By *(a)*, S contains an infinite subsequence S_0 such that for each n in S_0, U_n contains an m-tuple $\{\psi_{n1}, \ldots, \psi_{nm}\}$ of orthonormal vectors. Since T is compact, the image of any sequence of vectors from E must contain a convergent subsequence. Since m is finite (as T is compact), a simple inductive argument on the components of the above m-tuples yields

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an infinite subsequence S_1 of S_0 such that

(3) $$\mu^{-1}T\psi_{ni} \to \psi_{0i} \ , \quad i = 1, \ldots, m$$

when n is restricted to S_1 (which is independent of i). Restricting n to S_1 gives

$$|\mu| \|\psi_{0i}-\psi_{ni}\| \le \|\mu\psi_{0i}-T\psi_{ni}\| + \|T-T_n\| \|\psi_{ni}\| + |\mu_n-\mu| \|\psi_{ni}\| \to 0 \quad \text{as} \quad n \to \infty$$

by (1) and (3). Hence

(4) $$\psi_{ni} \to \psi_{0i} \quad (n \in S_1) \quad i = 1, \ldots, m \ .$$

Since T is continuous, and, for each n , $\{\psi_{n1}, \ldots, \psi_{nm}\}$ is an orthonormal set, (3) and (4) show that $\{\psi_{01}, \ldots, \psi_{0m}\}$ forms an orthonormal basis for V . Hence there exist constants $c_1, \ldots, c_m$ such that $\psi = \sum_{i=1}^{m} c_i\psi_{0i}$. Define $\psi_n = \sum_{i=1}^{m} c_i\psi_{ni}$ for $n \in S_1$. Clearly $\psi_n \in U_n \cap E$ and from (4)

$$(\forall\, d > 0)(\exists\, n \in S_1)(\exists\, \psi_n \in U_n \cap E) \ \|\psi_n-\psi\| > d \ .$$

This contradicts (2), as $S_1 \subseteq S$. The result follows.

(b) $\Rightarrow$ *(c)*. Trivial, as $U_n \subseteq V_n$.

(c) $\Rightarrow$ *(a)*. Let $\psi_{01}, \ldots, \psi_{0m}$ be m orthonormal vectors in V . Then, by *(c)*, there are m sequences $\{\psi_{ni}\}$, $i = 1, \ldots, m$, each with its n-th element in $V_n \cap E$, and with $\psi_{ni} \to \psi_{0i}$ as $n \to \infty$. It is sufficient to prove that, for sufficiently large n , none of the m vectors ψ_{ni} $(i = 1, \ldots, m)$ is a linear combination of the others. Suppose the contrary. Then without loss of generality we may suppose $\psi_{n1} = \sum_{i=2}^{m} c_i\psi_{ni}$ where $|c_i| \le 1$, $i = 2, \ldots, m$. Now

$$\begin{aligned} |(\psi_{ni}, \psi_{nj})| \le |(\psi_{0i}, \psi_{0j})| + |(\psi_{ni}-\psi_{0i}, \psi_{0j})| + |(\psi_{0i}, \psi_{nj}-\psi_{0j})| \\ + |(\psi_{ni}-\psi_{0i}, \psi_{nj}-\psi_{0j})| < 1/m \end{aligned}$$

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for $i \neq j$ and n sufficiently large (since $(\psi_{0i}, \psi_{0j}) = 0$ and $\psi_{ni} \to \psi_{0i}$).

Hence, for n sufficiently large,

$$1 = (\psi_{n1}, \psi_{n1}) = \sum_{i=2}^{m} c_i(\psi_{ni}, \psi_{n1}) \leq \sum_{i=2}^{m} |c_i||(\psi_{ni}, \psi_{n1})| < (m-1)/m < 1 .$$

The result follows.

COROLLARY. *The theorem remains true when (a), (b) and (c) are replaced by (a'), (b') and (c') below.*

(a'). For infinitely many n *,* $\dim V_n \geq m$.

(b'). [respectively (c')]. For all ψ *in* $V \cap E$ *there is a sequence* $\{\psi_n\}$ *with* ψ_n *in* $U_n \cap E$ *[respectively* $V_n \cap E$ *] such that* ψ *is a limit point of* $\{\psi_n\}$.

Proof. *(a')* $\Rightarrow$ *(b')*. From the sequence $\{V_n\}$ delete those terms with $\dim V_n < m$ and apply the theorem.

(b') $\Rightarrow$ *(c')*. Trivial, as before.

(c') $\Rightarrow$ *(a')*. The sequence $\{\psi_n\}$ contains a subsequence converging to ψ . Apply the theorem to this subsequence.

References

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