Nagoya Math. J., **230** (2018), 72–82
DOI 10.1017/nmj.2017.3

OPERATING FUNCTIONS ON MODULATION AND WIENER AMALGAM SPACES

MASAHARU KOBAYASHI AND ENJI SATO

Dedicated to Professor Satoru Igari on his 80th birthday

Abstract. The goal of this paper is to characterize the operating functions on modulation spaces $M^{p,1}(\mathbb{R})$ and Wiener amalgam spaces $W^{p,1}(\mathbb{R})$. This characterization gives an affirmative answer to the open problem proposed by Bhimani (Composition Operators on Wiener amalgam Spaces, arXiv: 1503.01606) and Bhimani and Ratnakumar (J. Funct. Anal. **270** (2016), pp. 621–648).

§1. Introduction

Wiener [15] studied the class $A(\mathbb{T})$ of all continuous functions on the torus $\mathbb{T}$ with the absolutely convergent Fourier series, and proved that $F(z) = 1/z$ operates on $A(\mathbb{T})$. Lévy [9] gave an extension of this result by showing that an analytic function operates on $A(\mathbb{T})$, which is called the Wiener–Lévy Theorem. After that, there are many papers about operating functions on the same function spaces with respect to Fourier series by Helson, Kahane, Katznelson, Rudin, and so forth (see [6, 10]).

In this paper, we give the characterization of operating functions on modulation spaces $M^{p,1}(\mathbb{R})$ and Wiener amalgam spaces $W^{p,1}(\mathbb{R})$. The modulation spaces $M^{p,q}$ and the Wiener amalgam spaces $W^{p,q}$ are two function spaces introduced by Feichtinger [4]. The precise definition of these spaces will be given in Section 2, but the main idea of these spaces is to consider the space variable and the variable of its Fourier transform simultaneously. Let F be a complex-valued function on $\mathbb{R}^2$ and $X = M^{p,1}(\mathbb{R})$ or $W^{p,1}(\mathbb{R})$. If $F(\mathrm{Re} f, \mathrm{Im} f) \in X$ for every $f \in X$, then we say that F operates on X.

Concerning modulation spaces, Wiener amalgam spaces and operating functions, the following theorem is known.

Received February 24, 2016. Revised February 10, 2017. Accepted February 10, 2017.
2010 Mathematics subject classification. 42B35, 47B38.

THEOREM A. (Bhimani [2], Bhimani and Ratnakumar [3]) *Let* $1 \leqslant p < \infty$ *and* F *be a complex-valued function on* $\mathbb{R}^2$. *Suppose* $X = M^{p,1}(\mathbb{R})$ *or* $W^{p,1}(\mathbb{R})$. *If* F *operates on* X, *then* F *is a real analytic function on* $\mathbb{R}^2$ *with* $F(0) = 0$. *Conversely, if* F *is a real analytic on* $\mathbb{R}^2$ *with* $F(0) = 0$, *then* F *operates on* $M^{1,1}(\mathbb{R})(= W^{1,1}(\mathbb{R}))$.

We remark that Theorem A answers negatively the open problem posed by Ruzhansky–Sugimoto–Wang [11] about the general power type nonlinearity of the form $|u|^\alpha u$. In [2] and [3], they also propose an open problem: is the condition in Theorem A sufficient or not for $p > 1$? This paper gives an affirmative answer to this problem. Our result is as follows.

THEOREM 1.1. *Let* $1 \leqslant p < \infty$ *and* F *be a real analytic function on* $\mathbb{R}^2$ *with* $F(0) = 0$. *Suppose* $X = M^{p,1}(\mathbb{R})$ *or* $W^{p,1}(\mathbb{R})$. *Then* F *operates on* X.

Since $F(s,t) = \frac{1}{(1+s^2)(1+t^2)}$ is a real analytic function on $\mathbb{R}^2$, we obtain $F(\mathrm{Re} f, \mathrm{Im} f) \in X$ for all $f \in X$ by Theorem 1.1.

Combining Theorems A and 1.1, we have the following characterization of operating functions on $M^{p,1}(\mathbb{R})$ and $W^{p,1}(\mathbb{R})$.

COROLLARY 1.2. *Let* $1 \leqslant p < \infty$ *and* F *be a complex-valued function on* $\mathbb{R}^2$. *Suppose* $X = M^{p,1}(\mathbb{R})$ *or* $W^{p,1}(\mathbb{R})$. *Then* F *operates on* X *if and only if* F *is a real analytic function with* $F(0) = 0$.

§2. Preliminaries

The following notation will be used throughout this article. We write $\mathcal{S}(\mathbb{R})$ to denote the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}$ and $\mathcal{S}'(\mathbb{R})$ to denote the space of tempered distributions on $\mathbb{R}$, that is, the topological dual of $\mathcal{S}(\mathbb{R})$. The Fourier transform is defined by $\widehat{f}(\xi) = \int_{\mathbf{R}} f(x) e^{-ix\cdot\xi} dx$ and the inverse Fourier transform by $f^\vee(x) = (2\pi)^{-1} \widehat{f}(-x)$. We also write $C_c^\infty(\mathbb{R})$ to denote the set of all complex-valued infinitely differentiable functions on $\mathbb{R}$ with compact support.

2.1 Real analytic function

A complex-valued function F on $\mathbb{R}^2$ is said to be real analytic on $\mathbb{R}^2$ if for each $(s_0, t_0) \in \mathbb{R}^2$, F has a power series expansion

$$F(s,t) = \sum_{m,n=0}^{\infty} a_{mn}(s - s_0)^m (t - t_0)^n$$

which converges absolutely in a neighborhood of (s_0, t_0).

https://doi.org/10.1017/nmj.2017.3 Published online by Cambridge University Press

2.2 Short-time Fourier transform

Let $f \in \mathcal{S}'(\mathbb{R})$ and $g \in \mathcal{S}(\mathbb{R})$. Then the short-time Fourier transform $V_g f$ of f with respect to the window g is defined by

$$V_g f(x,\xi) = \langle f(t), g(t-x)e^{it\cdot\xi}\rangle = \int_{\mathbb{R}} f(t)\overline{g(t-x)}e^{-it\cdot\xi}\,dt.$$

2.3 Modulation spaces

Let $1 \leqslant p, q \leqslant \infty$ and $g \in \mathcal{S}(\mathbb{R}) \setminus \{0\}$. Then the modulation space $M^{p,q}(\mathbb{R}) = M^{p,q}$ consists of all $f \in \mathcal{S}'(\mathbb{R})$ such that the norm

$$\|f\|_{M^{p,q}(\mathbb{R})} = \left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}} |V_g f(x,\xi)|^p dx\right)^{q/p} d\xi\right)^{1/q}$$

is finite (with usual modifications if $p = \infty$ or $q = \infty$).

We note that since $V_g\overline{f}(x,\xi) = \overline{V_{\overline{g}}f(x,-\xi)}$, we have

$$\|\overline{f}\|_{M^{p,q}} = \|f\|_{M^{p,q}},\ \ \|\mathrm{Re} f\|_{M^{p,q}} \leqslant \|f\|_{M^{p,q}},\ \ \|\mathrm{Im} f\|_{M^{p,q}} \leqslant \|f\|_{M^{p,q}}.$$

We collect basic properties of modulation spaces in the following lemma (see [4, 5, 11–14] for more details).

LEMMA 2.1.

(1) *The space $M^{p,q}(\mathbb{R})$ is a Banach space, whose definition is independent of the choice of g. More precisely, we have*

$$\|f\|_{M^{p,q}_{[g_0]}(\mathbb{R})} \leqslant C\|g\|_{M^{1,1}_{[g_0]}(\mathbb{R})}\|f\|_{M^{p,q}_{[g]}(\mathbb{R})}$$

for $f \in M^{p,q}(\mathbb{R})$ and $g_0, g \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$, where

$$\|f\|_{M^{p,q}_{[g]}(\mathbb{R})} = \left\|\|V_g f(x,\xi)\|_{L^p(\mathbb{R}_x)}\right\|_{L^q(\mathbb{R}_\xi)}.$$

(2) $M^{p,min\{p,p'\}}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n) \hookrightarrow M^{p,max\{p,p'\}}(\mathbb{R}^n)$. *In particular, we have* $M^{2,2}(\mathbb{R}^n) = L^2(\mathbb{R}^n)$.

(3) $M^{p,1}(\mathbb{R}) \subset C(\mathbb{R})$, *that is, f is continuous on $\mathbb{R}$ if $f \in M^{p,1}(\mathbb{R})$.*

(4) *If $p_1 \leqslant p_2$ and $q_1 \leqslant q_2$, then $M^{p_1,q_1}(\mathbb{R}) \hookrightarrow M^{p_2,q_2}(\mathbb{R})$.*

(5) *(Density and duality) If $p, q < \infty$, then $\mathcal{S}(\mathbb{R})$ is dense in $M^{p,q}(\mathbb{R})$ and $(M^{p,q}(\mathbb{R}))' = M^{p',q'}(\mathbb{R})$.*

(6) *(Multiplication) If* $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p}$ *and* $\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{q}+1$, *then*

$$\|fg\|_{M^{p,q}(\mathbb{R})} \leqslant C\|f\|_{M^{p_1,q_1}(\mathbb{R})}\|g\|_{M^{p_2,q_2}(\mathbb{R})}, \quad f, g \in \mathcal{S}(\mathbb{R}).$$

Moreover, we have

$$\|fg\|_{M^{p,1}(\mathbb{R})} \leqslant C\|f\|_{M^{p,1}(\mathbb{R})}\|g\|_{M^{p,1}(\mathbb{R})}, \quad f, g \in M^{p,1}(\mathbb{R}),$$

that is, $M^{p,1}(\mathbb{R})$ *is a multiplication algebra.*

(7) *(Convolution) If* $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p}+1$ *and* $\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{q}$, *then*

$$\|f * g\|_{M^{p,q}(\mathbb{R})} \leqslant C\|f\|_{M^{p_1,q_1}(\mathbb{R})}\|g\|_{M^{p_2,q_2}(\mathbb{R})}, \quad f, g \in \mathcal{S}(\mathbb{R}).$$

(8) *(Dilation property) There exists constants* $C, C' > 0$ *such that*

$$\|f_\lambda\|_{M^{\infty,1}(\mathbb{R})} \leqslant C\|f\|_{M^{\infty,1}(\mathbb{R})}, \quad f \in M^{\infty,1}(\mathbb{R}),$$
$$\|(f_\lambda)^\wedge\|_{M^{1,\infty}(\mathbb{R})} \leqslant C'\|\widehat{f}\|_{M^{1,\infty}(\mathbb{R})}, \quad \widehat{f} \in M^{1,\infty}(\mathbb{R})$$

for $0 < \lambda \leqslant 1$. *Here we denote* $f_\lambda(x) = f(\lambda x)$.

2.4 Wiener amalgam spaces

Let $1 \leqslant p, q \leqslant \infty$ and $g \in \mathcal{S}(\mathbb{R}) \setminus \{0\}$. Then the Wiener amalgam space $W^{p,q}(\mathbb{R}) = W^{p,q}$ consists of all $f \in \mathcal{S}'(\mathbb{R})$ such that the norm

$$\|f\|_{W^{p,q}(\mathbb{R})} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |V_g f(x,\xi)|^q d\xi\right)^{p/q} dx\right)^{1/p}$$

is finite (with usual modifications if $p = \infty$ or $q = \infty$).

We remark that since $V_g f(x,\xi) = (2\pi)^{-1} e^{-ix\cdot\xi} V_{\widehat{g}}\widehat{f}(\xi, -x)$, we have

$$C_1\|\widehat{f}\|_{M^{q,p}(\mathbb{R})} \leqslant \|f\|_{W^{p,q}(\mathbb{R})} \leqslant C_2\|\widehat{f}\|_{M^{q,p}(\mathbb{R})}$$

for some positive constants C_1 and C_2. This implies that the definition of $W^{p,q}$ is independent of the choice of g since the modulation space $M^{q,p}$ is independent of the choice of g. For the same reason, $W^{p,q}$ has the following properties.

LEMMA 2.2. *Let* $1 \leqslant p, p_1, p_2, q, q_1, q_2 \leqslant \infty$ *and* $\frac{1}{p}+\frac{1}{p'}=1=\frac{1}{q}+\frac{1}{q'}$.

(1) $M^{p,1}(\mathbb{R}) \hookrightarrow W^{p,1}(\mathbb{R})$.
(2) $W^{p,1}(\mathbb{R}) \subset C(\mathbb{R})$, *that is,* f *is continuous on* $\mathbb{R}$ *if* $f \in W^{p,1}(\mathbb{R})$.

https://doi.org/10.1017/nmj.2017.3 Published online by Cambridge University Press

(3) *If* $p_1 \leqslant p_2$ *and* $q_1 \leqslant q_2$*, then* $W^{p_1,q_1}(\mathbb{R}) \hookrightarrow W^{p_2,q_2}(\mathbb{R})$*.*
(4) (*Multiplication*) *If* $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ *and* $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q} + 1$*, then*

$$\|fg\|_{W^{p,q}(\mathbb{R})} \leqslant C\|f\|_{W^{p_1,q_1}(\mathbb{R})}\|g\|_{W^{p_2,q_2}(\mathbb{R})}, \quad f, g \in \mathcal{S}(\mathbb{R}).$$

Moreover, we have

$$\|fg\|_{W^{p,1}(\mathbb{R})} \leqslant C\|f\|_{W^{p,1}(\mathbb{R})}\|g\|_{W^{p,1}(\mathbb{R})}, \quad f, g \in W^{p,1}(\mathbb{R}),$$

that is, $W^{p,1}(\mathbb{R})$ *is a multiplication algebra.*
(5) (*Dilation property*) *There exists constant* $C > 0$ *such that*

$$\|f_\lambda\|_{W^{\infty,1}(\mathbb{R})} \leqslant C\|f\|_{W^{\infty,1}(\mathbb{R})}, \quad 0 < \lambda \leqslant 1, \ f \in W^{\infty,1}(\mathbb{R}).$$

Here we denote $f_\lambda(x) = f(\lambda x)$*.*

We also recall the following characterization of modulation spaces $M^{p,1}(\mathbb{R})$ and Wiener amalgam spaces $W^{p,1}(\mathbb{R})$.

DEFINITION 2.3. Let $1 \leqslant p \leqslant \infty$ and f be a function defined on $\mathbb{R}$. Suppose $X = M^{p,1}(\mathbb{R})$ or $W^{p,1}(\mathbb{R})$.

(1) Let $x_0 \in \mathbb{R}$. If there exist a neighborhood V of x_0 and a function $g \in X$ satisfying $f(x) = g(x)$ for every $x \in V$, then we say f belongs to X locally at a point $x_0 \in \mathbb{R}$.
(2) If there exist a compact set $K \subset \mathbb{R}$ and $h \in X$ satisfying $f(x) = h(x)$ for all $x \in \mathbb{R} \setminus K$, then we say f belongs to X at ∞.

We denote by X_{loc}, the space of functions that are locally in X at each point $x_0 \in \mathbb{R}$.

LEMMA 2.4. (cf. [2, Lemma 4.3], [3, Propositions 3.12 and 3.13], [8]) *Let* $1 \leqslant p \leqslant \infty$ *and* f *be a function defined on* $\mathbb{R}$*. Suppose* $X = M^{p,1}(\mathbb{R})$ *or* $W^{p,1}(\mathbb{R})$*.*

(1) $f \in X_{loc}$*, if and only if* $\varphi f \in X$ *for every* $\varphi \in C_c^\infty(\mathbb{R})$*.*
(2) f *belongs to* X *at* ∞*, if and only if there exists a function* $\varphi \in C_c^\infty(\mathbb{R})$ *such that* $(1 - \varphi)f \in X$*.*
(3) *If* $f \in X_{loc}$ *and* f *belongs to* X *at* ∞*, then* $f \in X$*.*

2.5 The space $A(\mathbb{T})$

Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ be the torus. Then the space $A(\mathbb{T})$ consists of all continuous function on $\mathbb{T}$ having an absolutely convergent Fourier series,

https://doi.org/10.1017/nmj.2017.3 Published online by Cambridge University Press

that is, the function f for which

$$\|f\|_{A(\mathbb{T})} := \sum_{n=-\infty}^{\infty} |\widehat{f}(n)| < \infty \quad \text{with } \widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}dt.$$

LEMMA 2.5. [7, pp. 56–57] *Let $\lambda \in (0,1)$ and define the 2π-periodic function $\mathbf{V}_\lambda \in C(\mathbb{T})$ by*

$$\mathbf{V}_\lambda(x) = 2\Delta_{2\lambda}(x) - \Delta_\lambda(x), \quad x \in [-\pi, \pi],$$

where $\Delta_\lambda(x) = \max\{0, 1 - \frac{|x|}{\lambda}\}$. Moreover, we define

$$\mathbf{V}_\lambda^{x_0}(x) = \mathbf{V}_\lambda(x - x_0)$$

for $x_0 \in \mathbb{R}$. Then for every $g \in A(\mathbb{T})$ with $g(x_0) = 0$, we have $\|\mathbf{V}_\lambda^{x_0} g\|_{A(\mathbb{T})} \to 0$ as $\lambda \to 0$.

We state the following lemma whose proof is almost a repetition of arguments in Bényi and Oh [1, Proposition B.1] and Bhimani [2, Proposition 3.3].

LEMMA 2.6. *Let $1 \leqslant p \leqslant \infty$ and $\phi \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp} \phi \subset (k\pi, (k+2)\pi)$ for some $k \in \mathbb{Z}$. Suppose $X = M^{p,1}(\mathbb{R})$ or $W^{p,1}(\mathbb{R})$. Then there exist positive constants C_ϕ^1 and C_ϕ^2 (which depend on ϕ but do not depend on k) such that*

$$\|\phi f\|_{A(\mathbb{T})} \leqslant C_\phi^1 \|f\|_X, \quad f \in X,$$

and

$$\|\phi f\|_X \leqslant C_\phi^2 \|f\|_{A(\mathbb{T})}, \quad f \in A(\mathbb{T}).$$

§3. The Proof of Theorem 1.1

We first prove that if F is real analytic, then $F(\mathrm{Re} f, \mathrm{Im} f) \in X_{loc}$ for every $f \in X$.

PROPOSITION 3.1. *Let $1 \leqslant p < \infty$ and F be a complex-valued real analytic function on $\mathbb{R}^2$ with $F(0) = 0$. If $f \in X$, then $F(\mathrm{Re} f, \mathrm{Im} f)$ belongs to X locally at x_0 for all $x_0 \in \mathbb{R}$.*

Proof. Let $f \in X$ and $x_0 \in \mathbb{R}$. Set $f(x_0) = s_0 + it_0$ with $s_0, t_0 \in \mathbb{R}$. Since F is real analytic, for some $\delta > 0$

$$F(s,t) = \sum_{m,n=0}^{\infty} a_{mn}(s - s_0)^m (t - t_0)^n$$

https://doi.org/10.1017/nmj.2017.3 Published online by Cambridge University Press

and

$$\sum_{m,n=0}^{\infty} |a_{mn}|\ |s-s_0|^m|t-t_0|^n < \infty$$

if $|s-s_0|<\delta$ and $|t-t_0|<\delta$.

Let $\phi \in C_c^\infty(\mathbb{R})$ such that $\phi(x)=1$ near x_0 and supp $\phi \subset (x_0-\frac{1}{10}, x_0+\frac{1}{10})$, and define g_j $(j=1,2)$ by

$$g_1(x)=(\mathrm{Re} f)(x)-s_0, \quad g_2(x)=(\mathrm{Im} f)(x)-t_0.$$

We note that $\phi g_j \in X$ and $(\phi g_j)(x_0)=0$ for $j=1,2$. Thus by Lemma 2.6 we can easily see $\phi g_j \in A(\mathbb{T})$. So, by Lemma 2.5 we obtain $\|\mathbf{V}_\lambda^{x_0}\phi g_j\|_{A(\mathbb{T})} \to 0$ as $\lambda \to 0$. Thus, for any $\varepsilon \in (0,\delta)$ there exists $\lambda_0>0$ such that

$$\|\mathbf{V}_\lambda^{x_0}\phi g_j\|_{A(\mathbb{T})} < \frac{\varepsilon}{(1+C_\phi^2)(1+C_X)}$$

for $\lambda<\lambda_0$, where C_X denotes the constant with

$$\|f\cdot g\|_X \leqslant C_X\|f\|_X\|g\|_X, \quad f,g\in X.$$

Hence by Lemma 2.6 again, we obtain

$$\|\phi\mathbf{V}_\lambda^{x_0}\phi g_j\|_X \leqslant C_\phi^2\|\mathbf{V}_\lambda^{x_0}\phi g_j\|_{A(\mathbb{T})} < \frac{\varepsilon}{1+C_X}$$

for $\lambda<\lambda_0$. Now we define

$$G(x)=\sum_{m,n=0}^{\infty} a_{mn}(\phi\mathbf{V}_\lambda^{x_0}\phi g_1)^m(x)(\phi\mathbf{V}_\lambda^{x_0}\phi g_2)^n(x)$$

for $\lambda<\lambda_0$. Since $\varepsilon<\delta$ and $a_{00}=0$, by Lemmas 2.1 and 2.2 we have

$$\begin{aligned}
&\sum_{m,n=0}^{\infty}\|a_{mn}(\phi\mathbf{V}_\lambda^{x_0}\phi g_1)^m(\phi\mathbf{V}_\lambda^{x_0}\phi g_2)^n\|_X \\
&\qquad \leqslant C_X\sum_{m,n=0}^{\infty}|a_{mn}|\ \|(\phi\mathbf{V}_\lambda^{x_0}\phi g_1)^m\|_X\|\phi(\mathbf{V}_\lambda^{x_0}\phi g_2)^n\|_X \\
&\qquad \leqslant C_X\sum_{m,n=0}^{\infty}|a_{mn}|\varepsilon^{m+n}<\infty,
\end{aligned}$$

https://doi.org/10.1017/nmj.2017.3 Published online by Cambridge University Press

and thus $G \in X$. Moreover, since $\phi(x) = 1$ near x_0, we have

$$\phi(x)\mathbf{V}_\lambda^{x_0}(x)\phi(x)g_j(x) = g_j(x)$$

near x_0, and thus

$$\begin{aligned} G(x) &= \sum_{m,n=0}^{\infty} a_{mn}(\mathrm{Re}f(x) - s_0)^m(\mathrm{Im}f(x) - t_0)^n \\ &= F(\mathrm{Re}f, \mathrm{Im}f) \end{aligned}$$

near x_0. This completes the proof. ∎

Next, we prove that if F is real analytic, then $F(\mathrm{Re}f, \mathrm{Im}f)$ belongs X at ∞ for every $f \in X$. For this, we prepare the following proposition.

PROPOSITION 3.2. *Let* $1 \leqslant p < \infty$ *and* $f \in X$. *For any* $\varepsilon > 0$, *there exists a real-valued function* $\Psi \in C_c^\infty(\mathbb{R})$ *such that*

$$\|(1-\Psi)f\|_X < \varepsilon.$$

Proof. Let $f \in X$, $\varepsilon > 0$ and ϕ be a real-valued function in $C_c^\infty(\mathbb{R})$ with $\phi(0) = 1$. Since $\mathcal{S}(\mathbb{R})$ is dense in X, there exists $g \in \mathcal{S}(\mathbb{R})$ such that

$$\|f - g\|_X < \frac{\varepsilon}{2(1 + C_0\|\phi\|_X)},$$

where C_0 is decided later. We also recall the fact that for any $g \in \mathcal{S}(\mathbb{R})(\subset M^{1,1}(\mathbb{R}) = W^{1,1}(\mathbb{R}))$, there exists $\lambda_0 \in (0,1)$ such that

$$\|(1-\phi_\lambda)g\|_{M^{1,1}(\mathbb{R})} < \frac{\varepsilon}{2}$$

for $\lambda \in (0, \lambda_0)$, where $\phi_\lambda(x) = \phi(\lambda x)$ (see for example [3, Proposition 3.14]).

Now we define $\Psi \in C_c^\infty(\mathbb{R})$ by $\Psi(x) = \phi(\lambda x)$. Then we have

$$\begin{aligned} \|(1-\Psi)f\|_X &\leqslant \|(1-\Psi)(f-g)\|_X + \|(1-\Psi)g\|_X \\ &= \|f - g - \Psi(f-g)\|_X + \|(1-\phi_\lambda)g\|_X \\ &\leqslant \|f-g\|_X + C\|\Psi\|_Y\|f-g\|_X + \|(1-\phi_\lambda)g\|_{M^{1,1}(\mathbb{R})} \\ &< (1 + C\|\phi_\lambda\|_Y)\|f-g\|_X + \frac{\varepsilon}{2}, \end{aligned}$$

for $\lambda \in (0, \lambda_0)$, where $Y = M^{\infty,1}$ (if $X = M^{p,1}$) or $= W^{\infty,1}$ (if $X = W^{p,1}$). By Lemmas 2.1 and 2.2 we have

$$C\|\phi_\lambda\|_Y \leqslant C_0\|\phi\|_Y$$

https://doi.org/10.1017/nmj.2017.3 Published online by Cambridge University Press

for $\lambda \in (0, \lambda_0)$. Hence we have

$$\|(1-\Psi)f\|_X < \varepsilon. \qquad \square$$

COROLLARY 3.3. *Let $1 \leqslant p < \infty$ and $f_1, \ldots, f_N \in X$. For any $\varepsilon > 0$, there exists a real-valued function $\Psi \in C_c^\infty(\mathbb{R})$ such that*

$$\|(1-\Psi)f_i\|_X < \varepsilon, \quad i = 1, \ldots, N.$$

PROPOSITION 3.4. *Let $1 \leqslant p < \infty$ and F be a real analytic function on $\mathbb{R}^2$ with $F(0) = 0$. If $f \in X$, then there exists $H \in X$ such that*

$$H(x) = F(\mathrm{Re}f(x), \mathrm{Im}f(x))$$

except for some compact set in $\mathbb{R}$.

Proof. Let $f \in X$. Since F is real analytic, for some $\delta > 0$

$$F(s,t) = \sum_{m,n=0}^{\infty} a_{mn} s^m t^n$$

and

$$\sum_{m,n=0}^{\infty} |a_{mn}|\ |s|^m |t|^n < \infty$$

if $|s| < \delta$ and $|t| < \delta$. By Corollary 3.3 there exists a real-valued function $\Psi \in C_c^\infty(\mathbb{R})$ such that

$$\|(1-\Psi)\mathrm{Re}f\|_X < \frac{\delta}{1+C_X}, \quad \|(1-\Psi)\mathrm{Im}f\|_X < \frac{\delta}{1+C_X},$$

where C_X denotes the constant with

$$\|f \cdot g\|_X \leqslant C_X \|f\|_X \|g\|_X, \quad f, g \in X.$$

From this and Lemmas 2.1 and 2.2,

$$H(x) = \sum_{m,n=0}^{\infty} a_{mn}((1-\Psi(x))\mathrm{Re}f(x))^m((1-\Psi(x))\mathrm{Im}f(x))^n$$

converges in X. Since $\Psi \in C_c^\infty(\mathbb{R})$, $H(x) = F(\mathrm{Re}f(x), \mathrm{Im}f(x))$ except for some compact set in $\mathbb{R}$. $\square$

The proof of Theorem 1.1. By Propositions 3.1, 3.4 and Lemma 2.4, we have the desired result. $\square$

Acknowledgments. The authors wish to thank the anonymous referee for carefully reading their paper and for providing some helpful comments. This work was supported by JSPS KAKENHI, Grant Numbers JP16K17606, JP26400129.

References

[1] A. Bényi and T. Oh, *Modulation spaces, Wiener amalgam spaces, and Brownian motions*, Adv. Math. **228** (2011), 2943–2981.
[2] D. G. Bhimani, *Composition operators on Wiener amalgam spaces*, preprint, 2015, arXiv:1503.01606.
[3] D. G. Bhimani and P. K. Ratnakumar, *Functions operating on modulation spaces and nonlinear dispersive equations*, J. Funct. Anal. **270** (2016), 621–648.
[4] H. G. Feichtinger, "*Modulation spaces on locally compact abelian groups*", in *Wavelets and their Applications*, (eds. M. Krishna, R. Radha and S. Thangavelu) Chennai, India, Allied Publishers, New Delhi, 2003, 99–140. Updated version of a technical report, University of Vienna, 1983.
[5] K. Gröchenig, *Foundations of Time-Frequency Analysis*, Birkhäuser, Boston, 2001.
[6] H. Helson, J. P. Kahane, Y. Katznelson and W. Rudin, *The functions which operate on Fourier transforms*, Acta Math. **102** (1959), 135–157.
[7] J. P. Kahane, *Series de Fourier Absolument Convergentes*, Springer, Berlin–Heidelberg–New york, 1970.
[8] Y. Katznelson, *An Introduction to Harmonic Analysis*, Second corrected edition Dover Publications, Inc, New York, 1976.
[9] P. Lévy, *Sur la convergence absolue des series de Fourier*, Compos. Math. **1** (1935), 1–14.
[10] W. Rudin, *Fourier Analysis on Groups*, Interscience Tracts in Pure and Applied Mathematics, Interscience Publishers, New York–London, 1962.
[11] M. Ruzhansky, M. Sugimoto and B. Wang, "*Modulation spaces and nonlinear evolution equations*", in *Evolution Equations of Hyperbolic and Schrodinger Type, 267–283*, Progr. Math., **301**, Springer Basel AG, Basel, Birkhäuser, 2012.
[12] M. Sugimoto and N. Tomita, *The dilation property of modulation spaces and their inclusion relation with Besov spaces*, J. Funct. Anal. **248** (2007), 79–106.
[13] J. Toft, *Continuity properties for modulation spaces, with applications to pseudo-differential calculus. I*, J. Funct. Anal. **207** (2004), 399–429.
[14] J. Toft, *Pseudo-differential operators with smooth symbols on modulation spaces*, Cubo **11** (2009), 87–107.
[15] N. Wiener, *Tauberian theorems*, Ann. of Math. (2) **33** (1932), 1–100.

Masaharu Kobayashi
Department of Mathematics
Hokkaido University
Kita 10, Nishi 8, Kita-Ku, Sapporo
Hokkaido, 060-0810
Japan
m-kobayashi@math.sci.hokudai.ac.jp

Enji Sato
Department of Mathematical Sciences, Faculty of Science
Yamagata University
Kojirakawa 1-4-12, Yamagata-City
Yamagata 990-8560
Japan
esato@sci.kj.yamagata-u.ac.jp