Canad. J. Math. Vol. **55** (5), 2003 pp. 1080–1099

# Quaternions and Some Global Properties of Hyperbolic 5-Manifolds

Ruth Kellerhals

*Abstract.* We provide an explicit thick and thin decomposition for oriented hyperbolic manifolds $M$ of dimension 5. The result implies improved universal lower bounds for the volume $\mathrm{vol}_5(M)$ and, for $M$ compact, new estimates relating the injectivity radius and the diameter of $M$ with $\mathrm{vol}_5(M)$. The quantification of the thin part is based upon the identification of the isometry group of the universal space by the matrix group $\mathrm{PS}_\Delta\,\mathrm{L}(2,\mathbb{H})$ of quaternionic $2\times 2$-matrices with Dieudonné determinant $\Delta$ equal to 1 and isolation properties of $\mathrm{PS}_\Delta\,\mathrm{L}(2,\mathbb{H})$.

## 0 Introduction

The Margulis lemma for discrete groups of hyperbolic isometries has important consequences for the geometry and topology of hyperbolic manifolds of dimensions $n \geq 2$. There is a universal constant $\varepsilon = \varepsilon_n$ such that for each oriented hyperbolic $n$-manifold $M$ of finite volume there is a thick and thin decomposition

$$M = M_{\leq\varepsilon} \cup M_{>\varepsilon} \tag{0.1}$$

of $M$ as follows. The thick part $M_{>\varepsilon}$ having at each point an injectivity radius bigger than $\varepsilon/2$ is compact. The thin part $M_{\leq\varepsilon}$ of all points $p \in M$ with injectivity radius smaller than or equal to $\varepsilon/2$ consists of connected components of the following types. The bounded components are neighborhoods of simple closed geodesics in $M$ of length $\leq \varepsilon$ homeomorphic to ball bundles over the circle. The unbounded components are cusp neighborhoods homeomorphic to products of compact flat manifolds with a real half line.

Estimates for the constant $\varepsilon_n$ induce universal bounds for various characteristic invariants of $M$ such as volume. Explicit values for $\varepsilon_n$ are known for $n = 2$ by work of P. Buser [Bu2, Chapter 4] and for $n = 3$ by work of R. Meyerhoff [M]. For $n = 4$, partial results are contained in [K3].

The aim of this work is to estimate the constant $\varepsilon_5$ and to derive some global properties such as new lower volume bounds for hyperbolic 5-manifolds $M$ (*cf.* Section 2 and Section 3). We show that for $\varepsilon \leq \sqrt{3}/9\pi$ there is a decomposition of $M$ according to (0.1). Moreover, we prove the universal bound $\mathrm{vol}_5(M) > 0.000083$.

To this end, we analyse the thin part of $M$ and construct embedded tubes around simple closed geodesics of length $l \leq \sqrt{3}/8\pi$ of radius given by (*cf.* Section 2.1)

$$\cosh(2r) = \frac{1-3k}{k}, \quad \text{where } k = \frac{2\pi l}{\sqrt{3}}. \tag{0.2}$$

Received by the editors February 27, 2002; revised July 2, 2002.
Partially supported by Schweizerischer Nationalfonds No. 20-61379.00.
AMS subject classification: 53C22, 53C25, 57N16, 57S30, 51N30, 20G20, 22E40.

The tubes around distinct closed geodesics of lengths $\leq \sqrt{3}/9\pi \simeq 0.0612$ are pairwise disjoint. In the non-compact case, they are also distinct from the canonical cusps associated to parabolic elements in the fundamental group of $M$.

Our considerations are based upon the identification of hyperbolic space $H^5$ and its boundary through quaternions such that $\mathrm{Iso}^+(H^5)$ equals the group $\mathrm{PS}_\Delta\mathrm{L}(2;\mathbb{H})$ of quaternionic $2 \times 2$-matrices with Dieudonné determinant $\Delta = 1$ as described by [H] and [Wil] (*cf.* Section 1.2). In this context, we characterise the isolation of the identity in $\mathrm{PS}_\Delta\mathrm{L}(2;\mathbb{H})$ (*cf.* Section 1.3). The strategies involved are standard and go back to [J], [Be] and [Wat].

The explicit tube construction (0.2) implies comparison results between injectivity radius, diameter and volume of compact hyperbolic 5-manifolds $M$ (*cf.* Section 3.2). For example, we prove that the injectivity radius $i(M)$ of $M$ satisfies $i(M) \geq \text{const} \cdot \mathrm{vol}_5(M)^{-1}$ improving results of P. Buser [Bu1] and A. Reznikov [Re].

In [CW, Section 9], C. Cao and P. Waterman constructed tubes around closed geodesics in hyperbolic $n$-manifolds $M$ for $n \geq 2$ and give a lower bound for the in-radius of $M$ by viewing isometries of hyperbolic $n$-space as Clifford matrices of pseudo-determinant 1. By different methods, Buser [Bu1, Section 4] obtained analogous results for compact hyperbolic manifolds of dimensions $> 2$. Both contributions provide clearly weaker bounds than ours when specialized to $n = 5$. As an illustration, the in-radius $r(M)$ measuring the radius of a largest embeddable ball in $M$ is bounded from below by $1/65536$ according to [Bu1, Theorem 4.11] and by $1/544$ according to [CW, Theorem 9.8] while we obtained the bound $1/30$ (*cf.* Lemma 5).

**Acknowledgment** The work was completed during a short stay at the Max-Planck-Institute for Mathematics in Bonn. The author expresses her thanks to the Director, Professor G. Harder, and to Professor F. Hirzebruch for the invitation and the hospitality.

## 1 The Quaternion Formalism for Isometries of $H^5$

### 1.1 Loxodromic Isometries of Hyperbolic $n$-Space

Let $\hat{E}^n := E^n \cup \{\infty\}$. A Möbius transformation of $\hat{E}^n$ is a finite composition of reflections in spheres or hyperplanes of $\hat{E}^n$ and preserves cross ratios

$$[x, y; u, v] = \frac{|x-u| \cdot |y-v|}{|x-y| \cdot |u-v|}$$

for distinct points $x, y, u, v \in \hat{E}^n$. The group of all Möbius transformations of $\hat{E}^n$ is denoted by $M(\hat{E}^n)$, or by $M(n)$ for short.

Consider hyperbolic space $H^n$ in the upper half space $E^n_+$, that is,

$$H^n = \left( E^n_+ \, , \; ds^2 = \frac{1}{x_n^2}(dx_1^2 + \cdots + dx_n^2) \right) \tag{1.1}$$

with distance between two points $x, y \in H^n$ given by

$$\cosh d(x, y) = 1 + \frac{|x-y|^2}{2x_n y_n}. \tag{1.2}$$

By Poincaré extension, every Möbius transformation $T \in M(n-1)$ gives rise to an element in $M(E^n_+)$ again denoted by $T$. In fact, $T \in \mathrm{Iso}(H^n)$ since it leaves invariant the hyperbolic metric (1.2).

According to the fixed point behavior a Möbius transformation is either elliptic, parabolic, or loxodromic. For example, if $T \in M(E^n_+)$ has precisely one, resp. two, fixed points in $\hat{E}^{n-1}$ and none in $E^n_+$, then $T$ is parabolic, resp. loxodromic.

Let $T \in \mathrm{Iso}(H^n)$ be a loxodromic element, and denote by $q_1, q_2 \in \partial H^n$ its two different fixed points. They determine a unique geodesic $a_T \subset H^n$, the axis of $T$, along which $T$ acts as a translation. For $p \in a_T$, $d\big(p, T(p)\big) =: \tau$ is constant and called the *translational length* of $T$. Besides, $T$ consists of a rotational part $R$ such that—after a suitable conjugation—we obtain the representation

$$T = rA, \quad \text{where } r = e^{\tau},\ A \in O(E^{n-1}). \tag{1.3}$$

For later purpose, we prove the following very useful property of $T$ (for $n = 4$, see [K3, Lemma 1.3]).

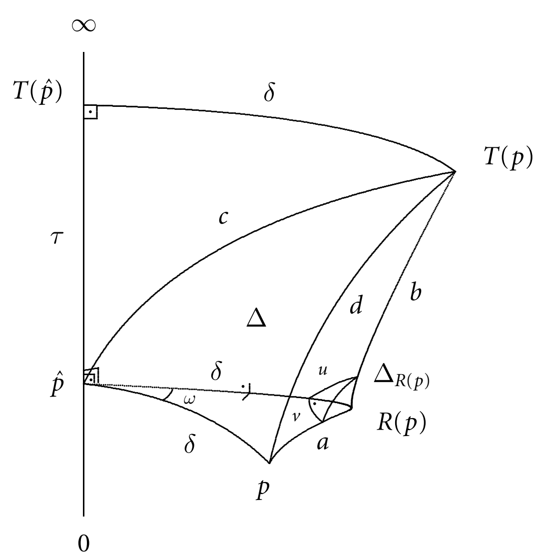


*Figure 1*

***Proposition 1*** *Let $T \in \mathrm{Iso}(H^n)$ be a loxodromic element with axis $a_T$, with rotational part $R$ and with translational length $\tau$. Let $p \in H^n$ be such that $p \notin a_T$, and assume that the foot of the perpendicular from $p$ to $a_T$ is $\hat{p}$. Denote by $\omega = \omega(p)$ the angle at $\hat{p}$ in the triangle $\big(p, \hat{p}, R(p)\big)$. Let $d = d\big(p, T(p)\big)$ and $\delta = d(p, a_T)$. Then,*

$$\cosh d = \cosh \tau + \sinh^2 \delta \cdot (\cosh \tau - \cos \omega). \tag{1.4}$$

https://doi.org/10.4153/CJM-2003-042-4 Published online by Cambridge University Press

**Proof** Without loss of generality, we may assume that $a_T = (0, \infty)$. Then, $\hat{p} = |p| e_n$. Let $a := d\big(p, R(p)\big)$, $b := d\big(R(p), T(p)\big)$, and $c := d\big(\hat{p}, T(p)\big)$ (*cf.* Figure 1).

Hyperbolic trigonometry yields with respect to the triangle $\big(p, \hat{p}, R(p)\big)$

$$\cosh a = \cosh^2\delta - \sinh^2\delta\cos\omega = 1 + \sinh^2\delta(1 - \cos\omega), \tag{1.5}$$

and with respect to the Saccheri quadrangle $\big(\hat{p}, T(\hat{p}), T(p), R(p)\big)$

$$\cosh b = \cosh\tau\cosh^2\delta - \sinh^2\delta, \tag{1.6}$$

and finally with respect to the right-angled triangle $\big(\hat{p}, T(\hat{p}), T(p)\big)$

$$\cosh c = \cosh\tau\cosh\delta. \tag{1.7}$$

Next, consider the hyperbolic tetrahedron $\Delta = \Delta\big(\hat{p}, p, R(p), T(p)\big)$. The dihedral angle formed by the facets opposite to $p$ and $T(p)$, respectively, and attached at the edge $\big(\hat{p}, R(p)\big)$ equals $\pi/2$. Denote by $\Delta_{R(p)}$ the spherical vertex figure of $\Delta$ at the vertex $R(p)$. $\Delta_{R(p)}$ is a right-angled triangle with hypotenuse $\beta$, say. Furthermore, let $u$ (resp. $v$) be the edge of $\Delta_{R(p)}$ in the facet opposite to $p$ (resp. $T(p)$) in $\Delta$. Then, $\cos\beta = \cos u\cos v$.

By hyperbolic trigonometry, we deduce

$$\cosh d = \cosh a\cosh b - \sinh a\sinh b\cos\beta, \tag{1.8}$$

as well as

$$\begin{aligned} \cosh c &= \cosh b\cosh\delta - \sinh b\sinh\delta\cos u, \\ \cosh\delta &= \cosh a\cosh\delta - \sinh a\sinh\delta\cos v. \end{aligned} \tag{1.9}$$

Hence, by (1.7) and (1.9),

$$\begin{aligned} \cos\beta = \cos u\cos v &= \frac{\cosh b\cosh\delta - \cosh\tau\cosh\delta}{\sinh b\sinh\delta} \cdot \frac{\cosh a\cosh\delta - \cosh\delta}{\sinh a\sinh\delta} \\ &= \coth^2\delta \cdot \frac{\cosh b - \cosh\tau}{\sinh b} \cdot \frac{\cosh a - 1}{\sinh a}. \end{aligned}$$

By using (1.5), (1.6) and (1.8), we obtain

$$\begin{aligned} \cosh d &= \cosh a\cosh b - \coth^2\delta(\cosh b - \cosh\tau)(\cosh a - 1) \\ &= \cosh a\cosh b(1 - \coth^2\delta) + \coth^2\delta\cdot[\cosh b + (\cosh a - 1)\cosh\tau] \\ &= -\frac{1}{\sinh^2\delta}[\cosh^2\delta - \sinh^2\delta\cos\omega]\cdot\cosh b \\ &\qquad + \coth^2\delta\cdot[\cosh b + \sinh^2\delta(1 - \cos\omega)\cosh\tau] \\ &= \cosh b\cos\omega + \cosh^2\delta\cosh\tau(1 - \cos\omega) \\ &= \cosh^2\delta\cosh\tau\cos\omega - \sinh^2\delta\cos\omega + \cosh^2\delta\cosh\tau(1 - \cos\omega) \\ &= \cosh\tau + \sinh^2\delta(\cosh\tau - \cos\omega). \end{aligned}$$

■

***Remark*** Let $0 \leq \alpha_0, \ldots, \alpha_r < 2\pi$, $0 \leq r < [\frac{n}{2}]$, with $\cos\alpha_0 \geq \cdots \geq \cos\alpha_r$ denote the rotation angles of the loxodromic element $T \in \operatorname{Iso}(H^n)$. Then,

$$\cos\alpha_0 \geq \cos\omega \geq \cos\alpha_r.$$

To see this, pass to the normal form of the orthogonal part $R \in O(n-1)$ of $T$ and express $p = (p_0, \ldots, p_{n-2}, t) \in H^n$ with respect to the new basis in $E^{n-1} = \{t = 0\}$. Then, project the triangle $\big(p, \hat{p}, R(p)\big)$ orthogonally down to $\{t = 0\}$ in order to compute

$$\begin{aligned}\cos\omega &= \frac{(p_0^2 + p_1^2)\cos\alpha_0 + \cdots + (p_{2r}^2 + p_{2r+1}^2)\cos\alpha_r + p_{2r+2}^2 + \cdots + p_{n-2}^2}{p_0^2 + \cdots + p_{n-2}^2} \\ &\geq \frac{(p_0^2 + \cdots + p_{n-2}^2)\cos\alpha_r}{p_0^2 + \cdots + p_{n-2}^2} = \cos\alpha_r.\end{aligned}$$

### 1.2 Quaternions and $\operatorname{Iso}^+(H^5)$

Consider the quaternion algebra $\mathbb{H} = \{q = q_0 + q_1 i + q_2 j + q_3 k \mid q_l \in \mathbb{R}\}$ with generators $i$, $j$, where $k = ij$ as usual. $\mathbb{H}$ is a Euclidean vector space with basis $1, i, j, k$. Decompose a quaternion $q = q_0 + q_1 i + q_2 j + q_3 k$ into scalar part $Sq := q_0$ and vector part $Vq := q_1 i + q_2 j + q_3 k$ so that $q = Sq + Vq$. The (quaternionic) conjugate of $q$ is given by $\bar{q} = Sq - Vq$ and satisfies $|q|^2 = q\bar{q} = \bar{q}q$. For a unit quaternion $a$, we can write

$$a = \exp(I\alpha) := \cos\alpha + I\sin\alpha \quad \text{for some } \alpha \in [0, 2\pi), \tag{1.10}$$

where $I$ is a pure unit quaternion, *i.e.*, the scalar part of $I$ vanishes and therefore $I = -\bar{I}$, or equivalently $I^2 = -1$. Furthermore, write $q =: u + vj$ with $u = q_0 + q_1 i$, $v = q_2 + q_3 i \in \mathbb{C}$. Then, there is the correspondence

$$q = (q_0 + q_1 i) + (q_2 + q_3 i)j = u + vj \sim Q := \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} \in \operatorname{Mat}(2; \mathbb{C}). \tag{1.11}$$

Consider a matrix $M \in \operatorname{Mat}(2; \mathbb{H})$ and associate to $M$ the complex block matrix

$$\mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \operatorname{Mat}(4; \mathbb{C})$$

according to (1.11). The trace $\operatorname{Tr} M$ of $M$ is defined by

$$\operatorname{Tr} M := \frac{1}{2}\operatorname{tr}\mathcal{M} = S(a + d) \quad \text{for } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and is obviously conjugacy invariant. In order to establish a determinant of $M$ we adopt the point of view of J. Dieudonné (*cf.* [D], [As]) and consider again $\mathcal{M}$. By exploiting the correspondence (1.11), one calculates (*cf.* [Wil, Section 3])

$$\det \mathcal{M} = |l_{ij}|^2 = |r_{ij}|^2, \qquad 1 \le i, j \le 2, \text{ where} \tag{1.12}$$

$$\begin{aligned} l_{11} &= da - dbd^{-1}c, & l_{12} &= bdb^{-1}a - bc,\\ l_{21} &= cac^{-1}d - cb, & l_{22} &= ad - aca^{-1}b;\\ r_{11} &= ad - bd^{-1}cd, & r_{12} &= db^{-1}ab - cb,\\ r_{21} &= ac^{-1}dc - bc, & r_{22} &= da - ca^{-1}ba. \end{aligned} \tag{1.13}$$

In particular, $\det \mathcal{M} \ge 0$, and

$$\det \mathcal{M} = |ad - aca^{-1}b|^2 = |ad|^2 + |bc|^2 - 2S(a\bar{c}d\bar{b}). \tag{1.14}$$

The quantity

$$\Delta = \Delta(M) :=_{+} \sqrt{\det \mathcal{M}} \tag{1.15}$$

is called the *Dieudonné determinant* of $M$.

***Proposition 2*** **[Wil, Theorem 1]** *Let* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}(2;\mathbb{H})$ *be such that* $\Delta(M) \neq 0$. *Then, $M$ is invertible, and*

$$M^{-1} = \begin{pmatrix} l_{11}^{-1}d & -l_{12}^{-1}b \\ -l_{21}^{-1}c & l_{22}^{-1}a \end{pmatrix} = \begin{pmatrix} dr_{11}^{-1} & -br_{12}^{-1} \\ -cr_{21}^{-1} & ar_{22}^{-1} \end{pmatrix}.$$

In order to abbreviate, we write

$$\begin{pmatrix} {}^{\sim}d & {}^{\sim}b \\ {}^{\sim}c & {}^{\sim}a \end{pmatrix} := \begin{pmatrix} l_{11}^{-1}d & l_{12}^{-1}b \\ l_{21}^{-1}c & l_{22}^{-1}a \end{pmatrix}, \qquad \begin{pmatrix} d_{\sim} & b_{\sim} \\ c_{\sim} & a_{\sim} \end{pmatrix} := \begin{pmatrix} dr_{11}^{-1} & br_{12}^{-1} \\ cr_{21}^{-1} & ar_{22}^{-1} \end{pmatrix}. \tag{1.16}$$

By coefficient comparison in $MM^{-1} = I = M^{-1}M$, one obtains the following useful identities.

***Lemma 1*** *Let* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}(2;\mathbb{H})$ *be invertible. Then,*

(i) $ad_{\sim} - bc_{\sim} = da_{\sim} - cb_{\sim} = 1; \quad {}^{\sim}da - {}^{\sim}bc = {}^{\sim}ad - {}^{\sim}cb = 1.$
(ii) $a{}^{\sim}d - b{}^{\sim}c = d{}^{\sim}a - c{}^{\sim}b = 1; \quad d_{\sim}a - b_{\sim}c = a_{\sim}d - c_{\sim}b = 1.$
(iii) $ab_{\sim} = ba_{\sim}, \quad cd_{\sim} = dc_{\sim}; \quad {}^{\sim}ac = {}^{\sim}ca, \quad {}^{\sim}bd = {}^{\sim}db.$
(iv) $a{}^{\sim}b = b{}^{\sim}a, \quad c{}^{\sim}d = d{}^{\sim}c; \quad a_{\sim}c = c_{\sim}a, \quad b_{\sim}d = d_{\sim}b.$

By Lemma 1, the group $S_{\Delta}L(2;\mathbb{H})$ of all quaternionic $2 \times 2$-matrices with Dieudonné determinant $\Delta = 1$ can be identified according to[1]

$$S_{\Delta}L(2;\mathbb{H}) = \left\{ T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}(2;\mathbb{H}) \;\middle|\; ad_{\sim} - bc_{\sim} = 1 \right\}.$$

[1]Following L. Ahlfors [Al], $\mathrm{SL}(2;\mathbb{H})$ is used to denote the group of quaternionic Clifford matrices of pseudo-determinant equal to 1.

There is a close relationship to the group $\mathrm{Iso}^+(H^5)$ of orientation preserving isometries of $H^5$ in the following way (*cf.* [H], [Wil]). Take the hyperbolic 5-space $H^5$ with its canonical orientation and parametrize the space with the aid of $\mathbb{H}$ by writing $E^5_+ = \mathbb{H} \times \mathbb{R}_+$ so that $\partial H^n = \hat{\mathbb{H}}$ (*cf.* (1.1)). The group $S_\Delta L(2;\mathbb{H})$ acts on $\hat{\mathbb{H}}$ by linear fractional transformations

$$T(x) = (ax+b)(cx+d)^{-1}$$

with $T(\infty) = \infty$ for $c = 0$, and with $T(\infty) = ac^{-1}$ and $T(-c^{-1}d) = \infty$ for $c \neq 0$. By passing to the projectivized group

$$\mathrm{PS}_\Delta\mathrm{L}(2;\mathbb{H}) := S_\Delta L(2;\mathbb{H})/\{\pm E\},$$

one gets the isomorphism

$$\mathrm{PS}_\Delta\mathrm{L}(2;\mathbb{H}) \simeq \mathrm{Iso}^+(H^5).$$

In the following, we do not distinguish in the notation between elements of these groups.

Let $T \in \mathrm{Iso}^+(H^5)$ be a loxodromic element with rotational part $R$ (*cf.* (1.3)). Since $T$ is orientation preserving, $R$ is the Poincaré extension of the composition of either one or two rotations in planes of $\mathbb{H}$. In fact, $R \in \mathrm{SO}(4)$ is given by (*cf.* [C2, (6.78)], [C1], [Po])

$$R(x) = axb \quad \text{with } a, b \in \mathbb{H}, |a| = |b| = 1.$$

In particular, the rotation through the angles $\pm\alpha + \beta \in [0, 2\pi)$, $0 \leq \alpha \leq \beta < \pi$, about two completely orthogonal planes is given by

$$\begin{pmatrix} \exp(\alpha I) & 0 \\ 0 & \exp(-\beta J) \end{pmatrix} \tag{1.17}$$

for some unit pure elements $I, J \in \mathbb{H}$. Finally, consider a parabolic element $X \in \mathrm{Iso}^+(H^5)$ which acts as a translation. Modulo conjugation in $\mathrm{PS}_\Delta\mathrm{L}(2;\mathbb{H})$, $X$ can be written in the form

$$X = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \quad \text{with } \mu \in \mathbb{H} \cong E^4.$$

### 1.3 Isolation of the Identity in $\mathrm{PS}_\Delta\mathrm{L}(2,\mathbb{H})$

Consider a non-elementary discrete two generator subgroup $\langle S, T\rangle$ of $\mathrm{PSL}(2,\mathbb{C})$. By Jørgensen's trace inequality [J],

$$|\operatorname{tr}^2 T - 4| + |\operatorname{tr}[S,T] - 2| \geq 1, \tag{1.18}$$

where $[S,T] = STS^{-1}T^{-1}$. By specializing, for example to an element

$$T = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad \text{with } |\lambda| \neq 1,$$

https://doi.org/10.4153/CJM-2003-042-4 Published online by Cambridge University Press

the inequality (1.18) takes the form

$$|\lambda - \lambda^{-1}|^2 \cdot (1 + |bc|) \geq 1. \tag{1.19}$$

By writing $\lambda =: e^{\frac{1}{2}(\tau + i\alpha)}$, (1.19) turns into

$$2(\cosh\tau - \cos\alpha) \cdot (1 + |bc|) \geq 1. \tag{1.20}$$

Formulas avoiding trace such as (1.19) and (1.20) allow generalizations for $\mathrm{Iso}^+(H^n)$ of geometrical relevance. In [Wat], P. Waterman presents various versions of (1.19) for the group $\mathrm{PSL}(2; C_{n-2})$ of Clifford matrices associated to the Clifford algebra $C_{n-2}$ with $n-2$ generators.

Here, we derive a formula analogous to (1.20) for $\mathrm{PS}_\Delta\mathrm{L}(2; \mathbb{H}) \simeq \mathrm{Iso}^+(H^5)$ and for an element

$$T = \begin{pmatrix} e^{\tau/2}\exp(I\alpha) & 0 \\ 0 & e^{-\tau/2}\exp(-J\beta) \end{pmatrix}$$

with rotational part according to (1.17) by adapting suitably standard methods (*cf.* [Be], [Wat] and [K3]).

***Proposition 3*** *Let*

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, T = \begin{pmatrix} e^{\tau/2}\exp(I\alpha) & 0 \\ 0 & e^{-\tau/2}\exp(-J\beta) \end{pmatrix} \in \mathrm{PS}_\Delta\mathrm{L}(2; \mathbb{H})$$

*be loxodromic elements generating a non-elementary discrete subgroup. Then,*

$$2\big(\cosh\tau - \cos(\alpha+\beta)\big) \cdot (1 + |bc|) \geq 1. \tag{1.21}$$

**Proof** We follow the strategy of [Wat, Theorem I]. Suppose that

$$\mu := 2\big(\cosh\tau - \cos(\alpha+\beta)\big) \cdot (1 + |bc|) < 1, \tag{1.22}$$

and write $\rho := e^{\tau/2}$ for short, as well as

$$T =: \begin{pmatrix} A & 0 \\ 0 & B^{-1} \end{pmatrix}.$$

Consider the Shimizu-Leutbecher sequence defined inductively by

$$S_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} := S = \begin{pmatrix} a & b \\ c & d \end{pmatrix};$$

$$S_{n+1} = \begin{pmatrix} a_{n+1} & b_{n+1} \\ c_{n+1} & d_{n+1} \end{pmatrix} := S_n T S_n^{-1} \quad \text{for } n \geq 0.$$

By Section 1.2, Proposition 2 and (1.16), one computes

$$S_{n+1} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} {}^{\sim}d_n & -{}^{\sim}b_n \\ -{}^{\sim}c_n & {}^{\sim}a_n \end{pmatrix}$$
$$= \begin{pmatrix} a_n A^{\sim} d_n - b_n B^{-1\sim} c_n & -a_n A^{\sim} b_n + b_n B^{-1\sim} a_n \\ c_n A^{\sim} d_n - d_n B^{-1\sim} c_n & -c_n A^{\sim} b_n + d_n B^{-1\sim} a_n \end{pmatrix}.$$

Since $\Delta(S_n) = 1$, we deduce that $|a_n| = |a_{n\sim}| = |{}^{\sim}a_n|$ and so forth. Therefore,

(1.23)
$$\begin{aligned} |b_{n+1}c_{n+1}| &= |(-a_n A^{\sim} b_n + b_n B^{-1\sim} a_n) \cdot (c_n A^{\sim} d_n - d_n B^{-1\sim} c_n)| \\ &= |a_n b_n c_n d_n| \cdot |A - a_n^{-1} b_n B^{-1\sim} a_n{}^{\sim} b_n^{-1}| \cdot |A - c_n^{-1} d_n B^{-1\sim} c_n{}^{\sim} d_n^{-1}|. \end{aligned}$$

For the middle factor in (1.23), for example, one gets the estimate (*cf.* Section 1.2)

$$\begin{aligned} |A - a_n^{-1} b_n B^{-1\sim} a_n{}^{\sim} b_n^{-1}| &= |SA + VA - (SB^{-1}) \cdot a_n^{-1} b_n{}^{\sim} a_n{}^{\sim} b_n^{-1} \\ &\qquad - a_n^{-1} b_n (VB^{-1})^{\sim} a_n{}^{\sim} b_n^{-1}| \\ &= |S(A - B^{-1}) + VA - a_n^{-1} b_n (VB^{-1})^{\sim} a_n{}^{\sim} b_n^{-1}| \\ &= \{S(A - B^{-1})^2 + |VA - a_n^{-1} b_n (VB^{-1})^{\sim} a_n{}^{\sim} b_n^{-1}|^2\}^{1/2} \\ &\leq \{(\rho \cos\alpha - \rho^{-1} \cos\beta)^2 + (|VA| + |VB^{-1}|)^2\}^{1/2} \\ &= \{(\rho \cos\alpha - \rho^{-1} \cos\beta)^2 + (\rho |\sin\alpha| + \rho^{-1} |\sin\beta|)^2\}^{1/2} \\ &= \{\rho^2 + \rho^{-2} - 2c(\alpha, \beta)\}^{1/2} \\ &= \Big\{ 2(\cosh\tau - c\big(\alpha, \beta)\big) \Big\}^{1/2}, \end{aligned}$$

where we used the notation

$$c(\alpha, \beta) := \begin{cases} \cos(\alpha + \beta) & \text{if } \alpha, \beta \in [0, \pi] \text{ or } \alpha, \beta \in [\pi, 2\pi), \\ \cos(\alpha - \beta) & \text{else.} \end{cases}$$

Hence, $c(0, \beta) = \cos\beta$, and by (1.17), $c(\alpha, \beta) \geq \cos(\alpha + \beta)$. The same estimate results for the third factor in (1.23). Therefore,

$$|b_{n+1}c_{n+1}| \leq |a_n b_n c_n d_n| \cdot \Big\{ 2\big( \cosh\tau - \cos(\alpha + \beta)\big) \Big\}.$$

Since $|a_n d_n| \leq 1 + |b_n c_n|$ by Lemma 1(i), we obtain by induction

$$|b_{n+1}c_{n+1}| \leq \mu^n |bc|,$$

and therefore, by (1.22), $b_n c_{n\sim} \to 0$ and $a_n d_{n\sim} \to 1$. Since

$$|a_{n+1}| = |a_n A^{\sim} d_n - b_n B^{-1\sim} c_n|, \quad |d_{n+1}| = |-c_n A^{\sim} b_n - d_n B^{-1\sim} a_n|,$$

we deduce that $|a_n| \to \rho$ and $|d_n| \to \rho^{-1}$. Moreover, we get the estimate

$$|b_{n+1}| \leq |a_n b_n| \cdot \left\{ 2(\cosh \tau - \cos(\alpha + \beta)) \right\},$$

and by induction

$$\frac{|b_n|}{\rho^n}, \quad |c_n| \cdot \rho^n \to 0.$$

Next, consider the elements

$$\begin{aligned} T_n := T^{-n} S_{2n} T^n &= \begin{pmatrix} A^{-n} & 0 \\ 0 & B^n \end{pmatrix} \begin{pmatrix} a_{2n} & b_{2n} \\ c_{2n} & d_{2n} \end{pmatrix} \begin{pmatrix} A^n & 0 \\ 0 & B^{-n} \end{pmatrix} \\ &= \begin{pmatrix} A^{-n} a_{2n} A^n & A^{-n} b_{2n} B^{-n} \\ B^n c_{2n} A^n & B^n d_{2n} B^{-n} \end{pmatrix} \\ &=: \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix} \quad \text{for } n \geq 0. \end{aligned}$$

The sequence $\{T_n\}_{n \geq 0}$ has a convergent subsequence since

$$\begin{aligned} |\alpha_n| &= |a_{2n}| \to \rho \\ |\delta_n| &= |d_{2n}| \to \rho^{-1} \\ |\beta_n| &= \frac{|b_{2n}|}{\rho^{2n}} \to 0 \\ |\gamma_n| &= |c_{2n}| \cdot \rho^{2n} \to 0. \end{aligned}$$

If we can show that the elements $T_n$ are all distinct, then the group $\langle S, T \rangle$ is not discrete which yields the desired contradiction.

Suppose on the contrary that the sequence $\{T_n\}_{n \geq 0}$ stabilises, that is, $\beta_n = \gamma_n = 0$. Then, $b_{2n} = c_{2n} = 0$. Let $T_{n+1}$ be the first element such that $b_{n+1} = c_{n+1} = 0$. Since $\rho \neq 1$, (1.23) yields $a_n b_n = 0$ and $c_n d_n = 0$. But $\det S_n = |a_n d_n - a_n c_n a_n^{-1} b_n| = 1$, which leaves only two possibilities. In the first case, $b_n = c_n = 0$ which is impossible. In the second case, $a_n = d_n = 0$. For $n > 0$, $0 = \operatorname{Tr} S_n = S(a_n + d_n) = S(A + B^{-1}) = \rho \cos \alpha + \rho^{-1} \cos \beta$. It is easy to see that this contradicts $2\left(\cosh \tau - \cos(\alpha + \beta)\right) < 1$ given by the assumption (1.22). Therefore, $n = 0$ and $a = d = 0$. This is impossible since the group $\langle S, T \rangle$ is supposed to be non-elementary. ∎

***Proposition 4*** *Let* $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $T = \begin{pmatrix} A & 0 \\ 0 & B^{-1} \end{pmatrix} \in \mathrm{PS}_\Delta \mathrm{L}(2; \mathbb{H})$ *be loxodromic elements such that* $2r := \operatorname{dist}(a_T, a_{STS^{-1}}) > 0$. *Then,*

$$\cosh r \geq |bc|^{1/2}. \tag{1.24}$$

**Proof** Denote by $p$ the common perpendicular of the axes $a_T, a_{STS^{-1}}$ whose end points equal $0, \infty, S(0), S(\infty)$ in $\partial H^5$. Choose a Möbius transformation

$$V = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{PS}_\Delta \mathrm{L}(2, \mathbb{H})$$

such that $0, \infty, S(0), S(\infty)$ are mapped to $-w, w, -1, 1$ with $|w| > 1$, say. That is, $p$ is mapped to the positive $t$-axis, and $2r = \operatorname{dist}(a_T, a_{STS^{-1}}) = \log|w|$. For the cross ratios, we obtain

$$\frac{|1-w|^2}{4|w|} = [-1, 1, -w, w] = [bd^{-1}, ac^{-1}, 0, \infty] = \frac{|bd^{-1}|}{|bd^{-1} - ac^{-1}|}.$$

By (1.12) and (1.13), this means that

$$\frac{|1-w|^2}{4|w|} = |bc|.$$

By (1.10), we can write $w = \rho \exp(I\omega)$ in $E^4$ for some $\omega \in [0, 2\pi)$ and a unit pure element $I \in \mathbb{H}$. Hence, $2r = \log \rho$. Putting $z := (2r + I\omega)/2$, we deduce

$$w = e^{2r} \exp(I\omega) =: \exp(2r + I\omega) = \exp(2z).$$

Next, define

$$\sinh z := \frac{1}{2}\{\exp(z) - \exp(-z)\}.$$

It follows that

$$|\sinh z|^2 = \frac{1}{4}|(1-w)^2 w^{-1}| = \frac{1}{2}\big(\cosh(2r) - \cos\omega\big) \leq \frac{1}{2}\big(\cosh(2r) + 1\big).$$

Thus,

$$\cosh^2 r = \frac{1}{2}\big(\cosh(2r) + 1\big) \geq |\sinh z|^2 = |bc|. \qquad \blacksquare$$

***Proposition 5*** *Let* $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and* $T = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \in \mathrm{PS}_\Delta \mathrm{L}(2; \mathbb{H})$ *with* $\mu \in E^4$ *generate a non-elementary discrete subgroup. Then,*

$$|c| \cdot |\mu| \geq 1. \tag{1.25}$$

The proof is a slight modification of the proof of [K3, Theorem 1.2] by using Lemma 1.

## 2 A Thick and Thin Decomposition for Hyperbolic 5-manifolds

Let $M$ denote an oriented complete hyperbolic 5-manifold of finite volume which consequently will be called *hyperbolic* 5*-manifold* for short. That is, $M$ is a Clifford-Klein space form $H^5/\Gamma$ where $\Gamma < \mathrm{PS}_\Delta L(2, \mathbb{H})$ is discrete, torsion-free and cofinite. In particular, $\Gamma$ is non-elementary. Denote by $i_p(M)$ the injectivity radius of $M$ at $p$. By the Margulis Lemma for discrete groups of hyperbolic isometries (*cf.* [BGS, Section 9–10], [T], [R1]), there is a universal positive constant $\varepsilon$ such that there is a thick and thin decomposition

$$M = M_{\leq\varepsilon} \cup M_{>\varepsilon} \tag{2.1}$$

https://doi.org/10.4153/CJM-2003-042-4 Published online by Cambridge University Press

of $M$ as follows. The thick part $M_{>\varepsilon} = \{p \in M \mid i_p(M) > \frac{\varepsilon}{2}\}$ of $M$ is compact.

The thin part $M_{\leq\varepsilon} = \{p \in M \mid i_p(M) \leq \frac{\varepsilon}{2}\}$ in (2.1) consists of connected components of the following types. The bounded components are neighborhoods $N$ of simple (*i.e.* with no self-intersection) closed geodesics $g$ through $p \in M_{\leq\varepsilon}$ in $M$ of length $l(g) \leq \varepsilon$ homeomorphic to ball bundles over the circle. In fact, $N$ is a quotient $U/\Gamma_U$ by an infinite cyclic group $\Gamma_U < \Gamma$ of loxodromic type with common axis projecting to $g$ and leaving precisely invariant some component $U \subset H^5$ lying above $N$. The unbounded components are cusp neighborhoods homeomorphic to products of compact flat manifolds with a real half line. Each cusp neighborhood can be written in the form $C = C_q = V_q/\Gamma_q$ with $\Gamma_q < \Gamma$ of parabolic type fixing some point $q \in \partial H^5$ and leaving precisely invariant some horoball $V_q \subset H^5$ based at $q$.

In fact, to each subgroup $\Gamma_q < \Gamma$ of parabolic type corresponds a particular extremal horoball $B_q$ such that $B_q/\Gamma_q$ embeds in $M$. We describe it for the case $q = \infty$, only. Denote by $\mu \neq 0$ a shortest vector in the translational lattice $\Lambda < \Gamma_\infty$ here identified with $E^4$. Then,

$$B(\mu) = B_\infty(\mu) := \left\{ x \in H^5 \;\middle|\; x_5 > |\mu| \right\}$$

is called the *canonical horoball* of $\Gamma_\infty$. $B(\mu)$ is precisely invariant with respect to $\Gamma_\infty$ and gives rise to a cusp neighborhood in $M$. Moreover, canonical horoballs associated to inequivalent parabolic transformations in $\Gamma$ are disjoint. The proofs are slight variations of those of [K3, Lemma 2.7] and [K3, Lemma 2.8].

## 2.1 The Thin Part of a Hyperbolic 5-manifold

In the following, we construct neighborhoods of sufficiently small simple closed geodesics in $M$ such that they are disjoint from canonical cusp neighborhoods. If $g$ is a simple closed geodesic in $M$, denote by $r_g$ the injectivity radius for the exponential map of the normal bundle of $g$ into $M$. For $r \leq r_g$, the set $T_g(r) = \{p \in M \mid \operatorname{dist}(p, g) < r\}$ is called a *tube around* $g$ of radius $r$. By making use of the description $\operatorname{Iso}^+(H^5) \simeq \mathrm{PS}_\Delta L(2, \mathbb{H})$, we construct tubes as follows.

***Proposition 6*** *Let $l_0 = \frac{\sqrt{3}}{8\pi} \simeq 0.068916$. Then, each simple closed geodesic $g$ in $M$ of length $l(g) \leq l_0$ has a tube $T_g(r)$ of radius $r$ satisfying*

$$\cosh(2r) = \frac{1-3k}{k}, \quad \text{where } k = \frac{2\pi l(g)}{\sqrt{3}}. \tag{2.2}$$

**Proof** Consider two different lifts $\tilde{g}_1$, $\tilde{g}_2$ of $g$ in $H^5$. They give rise to $\Gamma$-conjugate loxodromic elements $T_1$, $T_2$ with disjoint axes $a_{T_1}$, $a_{T_2}$ but equal translational length $\tau$ and rotational angles $\pm\alpha + \beta$ with $0 \leq \alpha \leq \beta < \pi$. Denote by $p$ the common perpendicular of $a_{T_1}$ and $a_{T_2}$. We have to study the length $2r$ of $p$ in terms of $\tau = l(g)$. Without loss of generality assume that (*cf.* (1.17))

$$T_1 = \begin{pmatrix} e^{\tau/2}\exp(I\alpha) & 0 \\ 0 & e^{-\tau/2}\exp(-J\beta) \end{pmatrix},$$

$$T_2 = ST_1S^{-1} \quad \text{with } S = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

for some unit pure quaternions $I$, $J$. Since $\langle T_1, T_2\rangle$ is non-elementary, $\langle T_1, S\rangle$ is non-elementary as well. By Proposition 3, (1.21), applied to $\langle T_1, S\rangle$, we obtain

$$2k\cdot(1+|bc|)\geq 1, \quad \text{where } k=\cosh\tau-\cos(\alpha+\beta). \tag{2.3}$$

Now, (1.24) of Proposition 4 yields $\cosh^2 r\geq |bc|$, that is,

$$\cosh(2r)\geq\frac{1-3k}{k}, \tag{2.4}$$

which is nontrivial if

$$k=k(\tau;\alpha,\beta)=\cosh\tau-\cos(\alpha+\beta)\leq\frac{1}{4}. \tag{2.5}$$

Next, observe that (2.4) remains valid for $k(n\tau;n\alpha,n\beta)$ by considering $n$-th iterates of $T_1$, $T_2$ for arbitrary $n\in\mathbb{N}$. In this situation, we make use of the modified Zagier inequality [CGM, Lemma 3.4] which says that for arbitrary $0<\rho\leq\pi\sqrt{3}$ and $\nu\in[0,2\pi)$, there exists a number $n_0\in\mathbb{N}$ such that

$$\cosh(n_0\rho)-\cos(n_0\nu)\leq\frac{2\pi\rho}{\sqrt{3}}. \tag{2.6}$$

By choosing $\tau=\rho\leq\frac{\sqrt{3}}{8\pi}$ and $\nu=\alpha+\beta$ according to (2.3), (2.5) and (2.6) imply that

$$k(n_0\tau;n_0\alpha,n_0\beta)\leq\frac{1}{4}.$$ ■

***Lemma 2*** *Let $g$ denote a simple closed geodesic in $M$ of length $l(g)\leq l_0$ with tube $T_g(r)$ of radius $r$ satisfying (2.2). Then,*

(a) *$r=r(l)$ is strictly decreasing.*
(b) *The volume $\mathrm{vol}_5\big(T_g(r)\big)$ is strictly decreasing with respect to $l$.*

**Proof** Part (a) is obvious. As to part (b), observe that the volume of $T_g(r)$ equals the volume of a cylinder $\mathrm{Cyl}(r,l)$ of radius $r$ with axis of length $l$ which in general is given by (*cf.* [K3, Lemma 2.4])

$$\mathrm{vol}_n\big(\mathrm{Cyl}(r,l)\big)=\frac{2\pi}{n-1}\cdot l\cdot\sinh^{n-1}r.$$

Hence,

$$\mathrm{vol}_5\big(T_g(r)\big)=\frac{\pi}{2}\cdot l\cdot\sinh^4 r=\frac{\sinh^2 r}{2}\cdot\mathrm{vol}_3\big(\mathrm{Cyl}(r,l)\big). \tag{2.7}$$

By (2.2),

$$\mathrm{vol}_3\big(\mathrm{Cyl}(r,l)\big)=\pi\cdot l\cdot\sinh^2 r=\frac{\sqrt{3}}{4}-2\pi l,$$

which is a strictly decreasing function of $l$. ∎

***Remark*** Cao and Waterman [CW] obtained tubes around short closed geodesics of lengths $\le l_n$ in hyperbolic manifolds $M$ of arbitrary dimensions $n \ge 2$. They made use of certain extremal values associated to the rotational part of loxodromic elements losing much accuracy when estimating the tube radius. For example, for $n = 5$, a closed geodesic $g$ of length $l_5 \simeq 0.0045$ in $M$ has a tube of radius $\simeq 0.9885$ and volume $\simeq 0.01269$ according to [CW, Corollary 9.5] while $g$ has a tube of radius $\simeq 2.3786$ and volume $\simeq 5.7846$ according to (2.2).

***Lemma 3*** *Let $g$, $g'$ denote two simple closed geodesics in $M$ of lengths $l, l' \le l_1 := \sqrt{3}/9\pi \simeq 0.061258$ which do not intersect. Then, the tubes $T_g, T_{g'}$ of radii $r, r'$ subject to (2.2) are disjoint.*

**Proof** Write $M = H^5/\Gamma$, and let $\tilde{g}, \tilde{g}'$ be lifts to $H^5$ of $g$, $g'$ which are the axes of loxodromic elements $T, T' \in \Gamma$ with translational lengths $\tau$ and $\tau'$ and angles of rotation $\pm\alpha + \beta$ and $\pm\alpha' + \beta'$ as usually. Let $\delta = \operatorname{dist}(\tilde{g}, \tilde{g}')$. We must prove that $\delta \ge r + r'$.

For this, conjugate $T$, $T'$ in $\mathrm{PS_\Delta L}(2, \mathbb{H})$ in order to obtain the elements

$$X = \begin{pmatrix} e^{\tau/2}\exp(I\alpha) & 0 \\ 0 & e^{-\tau/2}\exp(-J\beta) \end{pmatrix}, \quad Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The axis $a_{YXY^{-1}} = Y(a_X)$ of the element $YXY^{-1}$ is disjoint from $a_X$ and $a_Y$. Let $p \in a_X$ denote the point such that $\delta = \operatorname{dist}(a_X, a_Y) = \operatorname{dist}(p, a_Y)$, that is, $p$ is the foot point on $a_X$ of the common perpendicular of $a_X, a_Y$. By construction, $d := \operatorname{dist}\big(p, Y(p)\big) \ge 2r$. Denote by $k' := k(Y) = \cosh\tau' - \cos(\alpha' + \beta')$. Then, Proposition 1 implies that

$$\cosh(2r) \le \cosh d = \cosh\tau' + \sinh^2\delta(\cosh\tau' - \cos\omega).$$

Remark Section 1.1 yields $\cos\omega \ge \cos(\alpha' + \beta')$. Therefore,

$$\begin{aligned}\cosh(2r) &\le \cosh\tau' + \sinh^2\delta\big(\cosh\tau' - \cos(\alpha' + \beta')\big)\\ &\le k' + 1 + \sinh^2\delta \cdot k' = \cosh^2\delta \cdot k' + 1.\end{aligned}$$

By Proposition 6, we deduce that

$$\begin{aligned}\cosh(2\delta) &= 2\cosh^2\delta - 1 \ge 2 \cdot \frac{\cosh(2r) - 1}{k'} - 1 = 2 \cdot \frac{1 - 4k}{kk'} - 1\\ &= \frac{1 - 4k}{kk'} + \frac{1 - 4k - kk'}{kk'}.\end{aligned}$$

Suppose that $k' \ge k$ (otherwise, exchange the role of $X$ and $Y$). Then, we obtain

$$\cosh(2\delta) \ge \frac{\sqrt{1 - 4k}}{k} \cdot \frac{\sqrt{1 - 4k'}}{k'} + \frac{1 - 4k - kk'}{kk'}.$$

https://doi.org/10.4153/CJM-2003-042-4 Published online by Cambridge University Press

By assumption, $l \leq l_1 = \frac{\sqrt{3}}{9\pi}$ so that, by Proposition 6,

$$k = \frac{2\pi l}{\sqrt{3}} < 2/9.$$

Hence,

$$\cosh(2r) = \frac{1-3k}{k} < \frac{\sqrt{1-4k}}{k},$$

and similarly for $\cosh(2r')$. In order to conclude that $\cosh(2\delta) \geq \cosh(2r + 2r')$, it suffices to show that

$$\frac{1-4k-kk'}{kk'} \geq \frac{\sqrt{1-4k-k^2}}{k} \cdot \frac{\sqrt{1-4k'-k'^2}}{k'} \geq \sinh(2r) \cdot \sinh(2r').$$

The verification is left to the reader (for details, *cf.* [K3, p. 64]). ∎

***Lemma 4*** *Let $M$ denote a non-compact hyperbolic 5-manifold. Then, the canonical cusps and the tubes around closed geodesics according to (2.2) do not intersect in $M$.*

The proof of Lemma 4 is basically a consequence of Proposition 5. For details we refer to the analogous proof of [K3, Theorem 2.9].

### 2.2 A Thick and Thin Decomposition

Let $M$ be a hyperbolic 5-manifold, and consider the thin and thick parts

$$M_{\leq\varepsilon} = \{p \in M \mid i_p(M) \leq \varepsilon/2\} \quad \text{and} \quad M_{>\varepsilon} = \{p \in M \mid i_p(M) > \varepsilon/2\}$$

of $M$ as in (2.1).

***Theorem I*** *For $\varepsilon \leq \sqrt{3}/9\pi \simeq 0.0612$, the thin part $M_{\leq\varepsilon}$ is a finite disjoint union of canonical cusps and tubes $T_g(r)$ around simple closed geodesics $g$ of length $\leq \varepsilon$ according to (2.2).*

**Proof** We take up an idea of Meyerhoff [M]. Write $M = H^5/\Gamma$, where $\Gamma < \mathrm{Iso}^+(H^5)$ is discrete, torsion-free and cofinite. The canonical cusps $C$ and the tubes $T$ around simple closed geodesics of lengths $\leq \frac{\sqrt{3}}{9\pi} \simeq 0.0612$ in $M$ as constructed in Section 2 are mutually disjoint. Hence, we must show that any cusp, resp. any bounded component in $M_{\leq\varepsilon}$, $\varepsilon \leq 0.0612$, is contained in a canonical cusp $C$, resp. in a tube $T$. It is easy to verify the assertion for the canonical cusps (*cf.* Section 2).

Let $p \in M_{\leq\varepsilon}$ providing a loxodromic element $X \in \Gamma$ with distance $d := d\big(p, X(p)\big) \leq 0.0612$. Assume without loss of generality that $X$ has axis $a_X$ with end points $0, \infty$, and denote by $\tau > 0$ the translational length and by $\pm\alpha+\beta \in [0, 2\pi)$ the angles of rotation of $X$. Let $R$ be the rotational part of $X$. We show that $p \in T_{a_X}(r)$, where the tube radius is given by (2.3) and (2.4), that is,

$$\cosh(2r) = \frac{1-3k}{k} \quad \text{with } k = k(X) = \cosh\tau - \cos(\alpha+\beta). \tag{2.8}$$

Let $\delta = d(p, a_X)$, and suppose that $\delta > 0$. By Proposition 1,

$$\cosh d = \cosh \tau + (\cosh \tau - \cos \omega) \cdot \sinh^2 \delta, \tag{2.9}$$

where $\omega = \omega(p)$ denotes the angle at the foot point $\hat{p}$ of the perpendicular from $p$ to $a_X$ in the triangle $\big(p, \hat{p}, R(p)\big)$. Observe that $\cos(\alpha + \beta) \le \cos \omega \le \cos(\alpha - \beta)$.

By (2.9), we must show that for $d\big(p, X(p)\big) \le d_0 := 0.0612$

$$\frac{\cosh d - \cosh \tau}{\cosh \tau - \cos \omega} = \sinh^2 \delta \le \sinh^2 r = \frac{1 - 4k}{2k}, \tag{2.10}$$

where we may work with $k = k(X^n) < 1/4$ for any integer $n \ge 1$ (*cf.* proof of Proposition 6) and especially with

$$k = k(X^{n_0}) \le \frac{2\pi\tau}{\sqrt{3}} \tag{2.11}$$

for $n_0 \in \mathbb{N}$ as given by (2.6). Now, write $p = (p_1, \dots, p_5) \in H^5$ and consider the circular locus of all points $q \in H^5$ with $q_5 = p_5$ and $d(q, a_X) = \delta$. Varying over all such $q$, we find $d^-$, $d^+$ such that $0 < \tau < d^- \le d \le d^+ \le d_0$ and (*cf.* (2.9) and Remark, Section 1.1)

$$\frac{\cosh d^+ - \cosh \tau}{\cosh \tau - \cos(\alpha + \beta)} = \sinh^2 \delta = \frac{\cosh d^- - \cosh \tau}{\cosh \tau - \cos(\alpha - \beta)}. \tag{2.12}$$

Therefore, it suffices to check that

$$\frac{\cosh d_0 - \cosh \tau}{\cosh \tau - \cos(\alpha + \beta)} \le \frac{1 - 4k}{2k}. \tag{2.13}$$

In order to verify (2.13), we distinguish between two cases.

Consider first the case $\cos(\alpha + \beta) > 1 - \tau$. Choose $k$ according to (2.8). Then, (2.13) simplifies to

$$\cosh d_0 \le \cosh \tau + \frac{1 - 4k}{2}.$$

Since $k < \cosh \tau + \tau - 1 < \cosh d_0 + d_0 - 1 =: k_0$ with $\cosh d_0 \simeq 1.00187$, we see that the inequality

$$\cosh d_0 \le 1 + \frac{1 - 4k_0}{2}$$

implying (2.13) is verified.

Next, suppose that $\cos(\alpha+\beta) \le 1-\tau$. Choose $k$ according to (2.11). Then, (2.13) turns into

$$\cosh d_0 \le \cosh \tau + (\cosh \tau + \tau - 1) \cdot \frac{\sqrt{3} - 8\pi\tau}{4\pi\tau}.$$

Since $\cosh \tau + \tau - 1 > \tau$, it suffices to verify

$$1.0019 < 1 + \tau \cdot \frac{\sqrt{3} - 8\pi\tau}{4\pi\tau}. \tag{2.14}$$

The last term in (2.14) is strictly decreasing. Since $\tau < d_0$, we obtain the bound

$$\frac{\sqrt{3} - 8\pi\tau}{4\pi} > \frac{\sqrt{3} - 8\pi d_0}{4\pi} \simeq 0.0154,$$

which proves (2.14). ■

# 3 Consequences

## 3.1 Volume Bounds

As first application, we derive some volume bounds.

***Proposition 7*** *Let $M$ be a hyperbolic 5-manifold $M$ with $m$ cusps and $n$ distinct simple closed geodesics of lengths $\leq 0.059$. Then,*

$$\mathrm{vol}_5(M) > \frac{m+n}{96}. \tag{3.1}$$

**Proof** Replace each of the $m$ cusps by the canonical cusp neighborhood $C_i$, $1 \leq i \leq m$, as described above. $C_1, \dots, C_m$ are pairwise disjoint. By methods based on results of Bieberbach and a sphere packing argument including the lattice constant computation $\delta_4 = \pi^2/16$ of Korkine-Zolotareff (*cf.* [K2, Remark (a), p. 726]), one has

$$\mathrm{vol}_5(C_i) > \frac{1}{96} \quad \text{for } i = 1, \dots, m,$$

whence

$$\mathrm{vol}_5\Big(\bigcup_{i=1}^{m} C_i\Big) = \sum_{i=1}^{m} \mathrm{vol}_5(C_i) > \frac{m}{96}.$$

Suppose that $M$ carries $n \geq 1$ distinct simple closed geodesics of lengths $\leq 0.059$ ($< l_1 < l_0$). By Proposition 6, Lemma 2, Lemma 3 and (2.7), $M$ contains $n$ mutually disjoint tubes $T_j$, $1 \leq j \leq n$, of total volume

$$\mathrm{vol}_5\Big(\bigcup_{j=1}^{n} T_j\Big) = \sum_{j=1}^{n} \mathrm{vol}_5(T_j) > n \cdot 0.01042 > \frac{n}{96}.$$

Finally, by Lemma 4, the canonical cusps and the tubes are pairwise disjoint. This finishes the proof. ∎

***Remark*** Let $M$ be a (possibly non-orientable) hyperbolic 5-manifold $M$ with $m \geq 1$ cusps. In [K2] and by methods based on the theory of (horo-)sphere packings, we deduced the much better bound

$$\mathrm{vol}_5(M) > m \cdot 0.3922. \tag{3.2}$$

Adjusting suitably the estimate (3.1) requires to lower the upper length bound 0.059.

***Lemma 5*** *Let $M$ be a hyperbolic 5-manifold. Then, there is a point $p \in M$ such that the injectivity radius $i_p(M)$ of $M$ at $p$ satisfies*

$$i_p(M) > 0.0343 > 1/30. \tag{3.3}$$

**Proof** Suppose that a shortest closed geodesic of $M$ has length $l \leq l_2 := 0.0687526 < l_0$. Then, by Proposition 6, there is a tube $T$ embedded in $M$ of radius $r = r(l)$ according to (2.2). By a result of A. Przeworski (*cf.* [Pr, Proposition 4.1]), there

https://doi.org/10.4153/CJM-2003-042-4 Published online by Cambridge University Press

is an embedded ball $B_p(\rho)$ centered at some point $p \in M$ which is of radius $\rho = \operatorname{arcsinh}\big(\tanh(r)/2\big)$. Since $r(l)$ is strictly monotonically decreasing, it follows that $\rho \geq \rho(l_2) \simeq 0.03439$ and hence $i_p(M) \geq 0.03439$. If a shortest closed geodesics on $M$ is of length $> l_2$, then $i_p(M) > l_2/2 \simeq 0.03437$ for all $p \in M$. By comparison, the result (3.3) follows. ■

***Theorem II*** *For a hyperbolic 5-manifold M,*

$$\operatorname{vol}_5(M) > 0.000083. \tag{3.4}$$

**Proof** If $M$ is non-compact, the estimate follows from (3.2). Suppose that $M$ is compact. By Lemma 5, $M$ contains a ball $B$ of radius at least 0.0343. This yields the estimate

$$\operatorname{vol}_5(M) \geq \operatorname{vol}_5(B) > 0.00000025, \tag{3.5}$$

which we improve as follows. Consider the in-radius $r(M) = \max_{p\in M} i_p(M)$ of $M$. Let $S_{\mathrm{reg}} \subset H^5$ denote a regular hyperbolic simplex of edge length $2r(M)$ with spherical vertex simplex $s_{\mathrm{reg}}$ of dimension 4. By [K1, Theorem], there is the volume bound

$$\operatorname{vol}_5(M) \geq \frac{4\pi^2}{9} \cdot \frac{\operatorname{vol}_5(S_{\mathrm{reg}})}{\operatorname{vol}_4(s_{\mathrm{reg}})}. \tag{3.6}$$

By means of [K1, Lemma 4] and [K1, Lemma 5], the quotient $\operatorname{vol}_5(S_{\mathrm{reg}})/\operatorname{vol}_4(s_{\mathrm{reg}})$ in (3.6) can be estimated in terms of the dihedral angle $2\alpha$ as given by the edge length $2r(M)$ (*cf.* [K1, (3)]). Since $r(M) > 0.0343$, this leads to the asserted volume bound $\operatorname{vol}_5(M) > 0.000083$. ■

***Remarks*** (a) Cao and Waterman derived the bound $r(M) \geq 1/544$ for the in-radius of a hyperbolic 5-manifold $M$ (*cf.* [CW, Theorem 9.8]). By exploiting (3.6) as above, this yields the volume bound $\operatorname{vol}_5(M) > 0.00000023$.

(b) Ratcliffe and Tschantz (*cf.* [R2]) announced a geometrical construction of a non-orientable hyperbolic 5-manifold with 10 cusps which is of volume $28\zeta(3) \simeq 33.6576$. By passing to its oriented double cover one obtains a hyperbolic 5-manifold of volume $56\zeta(3)$ which to our knowledge represents the smallest known volume hyperbolic 5-manifold. Therefore, a smallest volume hyperbolic 5-manifold $M_0$ satisfies $0.000083 < \operatorname{vol}_5(M_0) \leq 67.3152$. Moreover, by Proposition 6 and Lemma 2, a shortest closed geodesic in $M_0$ has length $> 0.00043$.

## 3.2 Injectivity Radius Versus Volume and Diameter

Let $M$ be a compact hyperbolic 5-manifold. Denote by $i(M) = \min_{p\in M} i_p(M)$ the injectivity radius of $M$ and by $\operatorname{diam}(M) = \max_{p,q\in M} \operatorname{dist}(p,q)$ the diameter of $M$. The injectivity radius $i(M)$ equals one half of the length of a shortest simple closed geodesic in $M$. By results of P. Buser [Bu1, Corollary 4.15] and A. Reznikov [R, Theorem],

$$i(M) \geq \mathrm{const} \cdot \operatorname{vol}_5(M)^{-3}.$$

We improve this estimate as follows.

***Proposition 8*** *For a compact hyperbolic* 5*-manifold M,*

$$i(M) \geq \text{const} \cdot \text{vol}_5(M)^{-1}. \tag{3.7}$$

**Proof** Assume that there is a short simple closed geodesic $g$ of length $l$ in $M$. Then, there is a tube $T_g(r)$ around $g$ of radius $r$ given by (*cf.* Proposition 6, (2.2))

$$\sinh^2 r = \frac{1}{2k} - 2, \quad \text{where } k = \frac{2\pi l}{\sqrt{3}}.$$

This implies

$$\text{vol}_5(M) \geq \text{vol}_5\big(T_g(r)\big) = \frac{\pi}{2} \cdot l \cdot \sinh^4 r.$$

Since $\sinh^4 r \sim \text{const} \cdot l^{-2}$ for small $l$, we deduce $l \geq \text{const} \cdot \text{vol}_5(M)^{-1}$ as desired. ■

***Proposition 9*** *For a compact hyperbolic* 5*-manifold M,*

$$i(M) \geq \text{const} \cdot \sinh(\text{diam}(M))^{-2}. \tag{3.8}$$

**Proof** Let $g$ denote a simple closed geodesic in $M$. By a result due to E. Heintze and H. Karcher [HK, Corollary 2.3.2], the length $l$ of $g$ is bounded from below as follows.

$$l \geq \frac{2}{\pi^2} \cdot \frac{\text{vol}_5(M)}{\sinh^4\big(\text{diam}(M)\big)}.$$

This together with Proposition 8, (3.7), yields

$$l \geq \text{const} \cdot \frac{1}{i(M) \cdot \sinh^4\big(\text{diam}(M)\big)},$$

which implies the desired result. ■

## References

[Al] L. V. Ahlfors, *Möbius transformations and Clifford numbers*. In: Differential geometry and complex analysis. A volume dedicated to the memory of Harry Ernest Rauch. Eds. I. Chavel and H. M. Farkas. Springer-Verlag, Berlin, 1985.

[As] H. Aslaksen, *Quaternionic determinants*. Math. Intelligencer, **18**(1996), 57–65.

[BGS] W. Ballmann, M. Gromov and V. Schroeder, *Manifolds of nonpositive curvature*. Birkhäuser, 1985.

[Be] A. F. Beardon, *The geometry of discrete groups*. Springer-Verlag, 1983.

[Bu1] P. Buser, *On Cheeger's inequality,* $\lambda_1 \geq h^2/4$. In: Geometry of the Laplace operator, 29–77, Proc. Sympos. Pure Math. XXXVI, Amer. Math. Soc., Providence, R.I., 1980.

[Bu2] ———, *Geometry and spectra of compact Riemann surfaces*. Birkhäuser, 1992.

[CGM] C. Cao, F. W. Gehring and G. J. Martin, *Lattice constants and a lemma of Zagier*. In: Lipa's legacy. Proceedings of the 1st Bers Colloquium. Eds. J. Dodziuk *et al.* Contemp. Math. **211**(1997), 107–120.

[CW] C. Cao and P. L. Waterman, *Conjugacy invariants of Möbius groups*. In: Quasiconformal Mappings and Analysis, A collection of papers honoring F. W. Gehring. Eds. Peter Duren, *et al.*, Springer-Verlag, New York, 1998.
[C1] H. S. M. Coxeter, *Quaternions and reflections*. Amer. Math. Monthly **53**(1946), 136–146.
[C2] ———, *Regular complex polytopes*. Cambridge University Press, 1974.
[D] J. Dieudonné, *Les déterminants sur un corps non-commutatif*. Bull. Soc. Math. France **71**(1943), 27–45.
[HK] E. Heintze and H. Karcher, *A general comparison theorem with applications to volume estimates for submanifolds*. Ann. Sci. École Norm. Sup. **11**(1978), 451–470.
[H] Y. Hellegouarch, *Quaternionic homographies: application to Ford hyperspheres*. C. R. Math. Rep. Acad. Sci. Canada **11**(1989), 165–170.
[J] T. Jørgensen, *On discrete groups of Möbius transformations*. Amer. J. Math. **98**(1976), 739–749.
[K1] R. Kellerhals, *Regular simplices and lower volume bounds for hyperbolic n-manifolds*. Ann. Global Anal. Geom. **13**(1995), 377–392.
[K2] ———, *Volumes of cusped hyperbolic manifolds*. Topology **37**(1998) 719–734.
[K3] ———, *Collars in* $\mathrm{PSL}(2, \mathbb{H})$. Ann. Acad. Sci. Fenn. Math. **26**(2001), 51–72.
[M] R. Meyerhoff, *A lower bound for the volume of hyperbolic* 3*-manifolds*. Canad. J. Math. **39**(1987), 1038–1056.
[Po] I. R. Porteous, *Topological geometry*. Cambridge University Press, 1981.
[Pr] A. Przeworski, *Cones embedded in hyperbolic manifolds*. J. Differential Geom. **58**(2001), 219–232.
[R1] J. G. Ratcliffe, *Foundations of hyperbolic manifolds*. Springer-Verlag, 1994.
[R2] ———, *Hyperbolic manifolds*. In: Handbook of geometric topology. Eds., R. J. Daverman and R. B. Sher, Elsevier, 2001.
[Re] A. Reznikov, *The volume and the injectivity radius of a hyperbolic manifold*. Topology **34**(1995), 477–479.
[T] W. P. Thurston, *Three-dimensional geometry and topology, vol. 1*. Princeton University Press, 1997.
[Wat] P. L. Waterman, *Möbius transformations in several dimensions*. Adv. in Math. **101**(1993), 87–113.
[Wil] J. B. Wilker, *The quaternion formalism for Möbius groups in four and fewer dimensions*. Linear Algebra Appl. **190**(1993), 99–136.

*University of Fribourg*
*Department for Mathematics*
*CH–1700 Fribourg*
*Switzerland*
*e-mail: Ruth.Kellerhals@unifr.ch*