MINIMAL DECOMPOSITIONS OF COMPLETE GRAPHS INTO SUBGRAPHS WITH EMBEDDABILITY PROPERTIES

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Although the problem of finding the minimum number of planar graphs into which the complete graph can be decomposed remains partially unsolved, the corresponding problem can be solved for certain other surfaces. For three, the torus, the double-torus, and the projective plane, a single proof will be given to provide the solutions. The same questions will also be answered for bicomplete graphs.

I. Complete graphs. For a given surface S and graph G, define the *S-thickness* of G to be the minimum number of S-embeddable graphs whose union is G. If the characteristic of S is χ, it follows from Euler's polyhedron formula that a triangulation of S using p vertices has $3(p - \chi)$ edges. This implies that the S-thickness of a graph G with p vertices and q edges satisfies the inequality

$$t_S(G) \geqq \left\{\frac{q}{3(p-\chi)}\right\}. \tag{1}$$

(Here, $\{x\}$ denotes the least integer greater than or equal to x. It is readily seen that $\{x\} = -[-x]$. Also, if a and b are positive integers,

$$\{a/b\} = [(a + b - 1)/b].)$$

The surfaces which will be considered here are the orientable surfaces S_n obtained from the sphere by adding n handles and the non-orientable surfaces $S^{[n]}$ obtained from the sphere by making n crosscuts. The corresponding characteristics are of course $2 - 2n$ and $2 - n$. The corresponding thicknesses of the graph G will be denoted by $t_n(G)$ and $t^{[n]}(G)$.

For the plane (or sphere), the inequality (1) has been shown (**1**) to be equality for "five-sixths" of the complete graphs K_p. The known results are summarized in the following theorem.

THEOREM 1. *If $p \neq 9$ and $p \not\equiv 4 \pmod 6$, then the planar thickness of the complete graph is*

$$t_0(K_p) = \left[\frac{p+7}{6}\right].$$

Furthermore, $t_0(K_4) = 1$, $t_0(K_9) = t_0(K_{10}) = 3$, $t_0(K_{22}) = 4$, $t_0(K_{28}) = 5$, *and* $t_0(K_{34}) = 6$.

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For the projective plane S', the torus S_1, and the double-torus S_2, the lower bound of (1) gives the corresponding thickness of all complete graphs. These expressions can be simplified in the following way.

For the projective plane:

$$t'(K_p) \geqq \left\{\frac{p(p-1)}{2\cdot 3(p-1)}\right\} = \left\{\frac{p}{6}\right\} = \left[\frac{p+5}{6}\right].$$

For the torus:

$$t_1(K_p) \geqq \left\{\frac{p(p-1)}{6p}\right\} = \left\{\frac{p-1}{6}\right\} = \left[\frac{p+4}{6}\right].$$

For the double-torus:

$$t_2(K_p) \geqq \left\{\frac{p(p-1)}{6(p+2)}\right\} = \left[\frac{p^2+5p+11}{6(p+2)}\right] \geqq \left[\frac{p+3}{6}\right].$$

For the reverse inequalities, constructive decompositions will be provided to show that for any positive integer n,

$$t'(K_{6n}) \leqq n, \quad t_1(K_{6n+1}) \leqq n, \quad \text{and} \quad t_2(K_{6n+2}) \leqq n.$$

The second of these has also been shown by Ringel (**4**).

The device which will be used is the matrix $A_n = (a_{ij})$, whose entries are the integers $1, 2, \ldots, n$ as defined by

$$a_{ij} \equiv (-1)^i\left[\frac{i}{2}\right] + (-1)^j\left[\frac{j}{2}\right] \pmod{n}.$$

For $n = 5$ and $n = 6$, these matrices are:

$$A_5 = \begin{bmatrix} 5 & 1 & 4 & 2 & 3 \\ 1 & 2 & 5 & 3 & 4 \\ 4 & 5 & 3 & 1 & 2 \\ 2 & 3 & 1 & 4 & 5 \\ 3 & 4 & 2 & 5 & 1 \end{bmatrix}, \qquad A_6 = \begin{bmatrix} 6 & 1 & 5 & 2 & 4 & 3 \\ 1 & 2 & 6 & 3 & 5 & 4 \\ 5 & 6 & 4 & 1 & 3 & 2 \\ 2 & 3 & 1 & 4 & 6 & 5 \\ 4 & 5 & 3 & 6 & 2 & 1 \\ 3 & 4 & 2 & 5 & 1 & 6 \end{bmatrix}.$$

This matrix A_n was essential to the proof of the results on the planar thickness, by means of some interesting properties. Several of the same properties, as set forth in the following lemma, will be used here. Its proof can be found elsewhere (**1**).

LEMMA. *Each integer* $1, 2, \ldots, n$ *appears exactly once in every row and column of the matrix* A_n. *Furthermore, any two distinct integers are consecutive entries of exactly two columns. In one, both are on or above the main diagonal and in the other, both are on or below it.*

Figures 1, 2, and 3 show K_6 embedded in the projective plane, K_7 in the torus, and K_8 in the double-torus. The models of the surfaces used here are the

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standard polygons with sides being identified as labeled: $\alpha\alpha$ for the projective plane, $\alpha\beta\alpha^{-1}\beta^{-1}$ for the torus, and $\alpha\beta\gamma\delta\alpha^{-1}\beta^{-1}\gamma^{-1}\delta^{-1}$ for the double-torus.

All these graphs have the same six distinguished faces $H(x)$ with vertices x, y, and $-y$. These faces will be modified by adding vertices and edges, so that when appropriately labeled, n copies of each graph will have a complete graph as their union. First consider the case of the projective plane.

Form n graphs G_r, for $r = 1, 2, \ldots, n$, from the graph of Figure 1 by inserting the graph $H_n(x)$ of Figure 4 in each of the faces labeled $H(x)$ for $x = \pm u, \pm v, \pm w$. Each of the resulting graphs is clearly S'-embeddable and has $6n$ vertices. Other than the dual base vertices, the graph $H_n(x)$ has n vertices. These are to be labeled using that column, say the jth, whose leading entry is r, as follows: If $a_{i,j} = s$, the ith vertex is $+x_s$ or $-x_s$ according as $\min(i, j)$ is odd or even. (For example, when $n = 5$, the non-base vertices of $H(u_2)$ are u_2, $-u_3$, u_1, $-u_4$, u_5, and when $n = 6$, those of $H(-v_2)$ are $-v_2$, v_3, $-v_1$, v_4, v_6, v_5.)

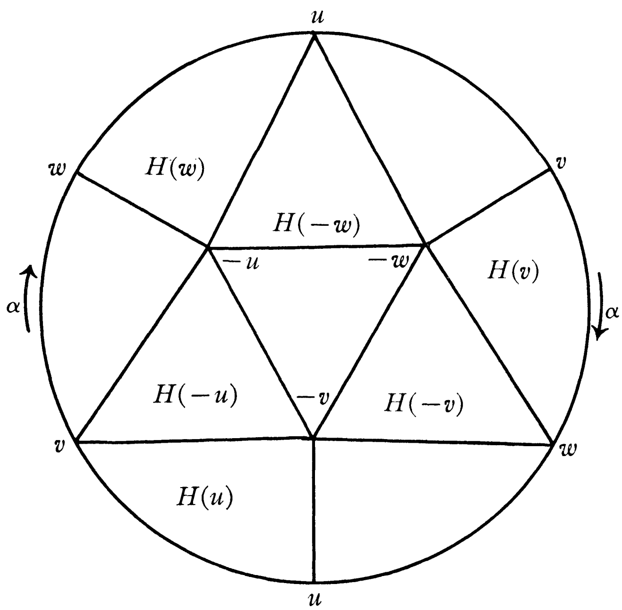

FIGURE 1

That the union of these n graphs is K_{6n} will now be verified. By symmetry, it is sufficient to show that u_r is adjacent to each of the other $6n - 1$ vertices in one of the graphs. First, it is clearly adjacent to $-u_r$, v_r, $-v_r$, w_r, and $-w_r$ in G_r. Now assume that $s \neq r$. Then u_r is adjacent to v_s and $-v_s$ in G_s since they are the base vertices of $H(u_s)$ and $H(-u_s)$ and u_r appears in one of these. It is adjacent to w_s and $-w_s$ in G_r since it is a base vertex of $H(w_r)$ and $H(-w_r)$ which contain those vertices. Finally, by the lemma, r and s are consecutive entries in two columns of A_n. If h is the leading entry of the column in which

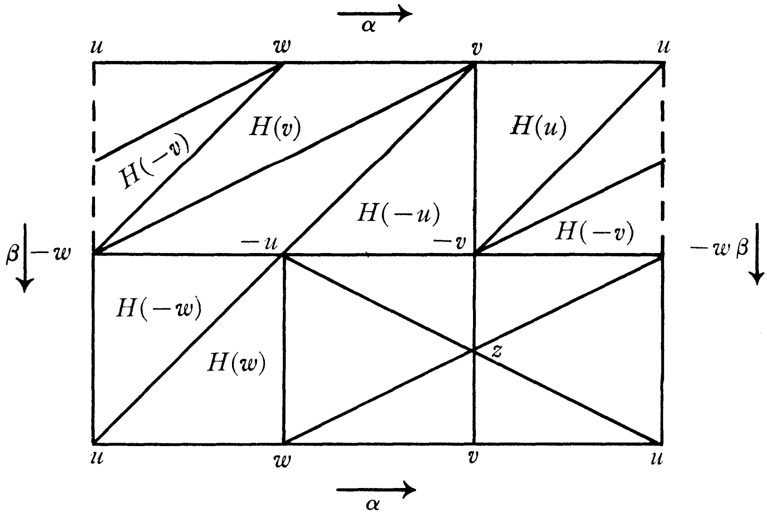

FIGURE 2

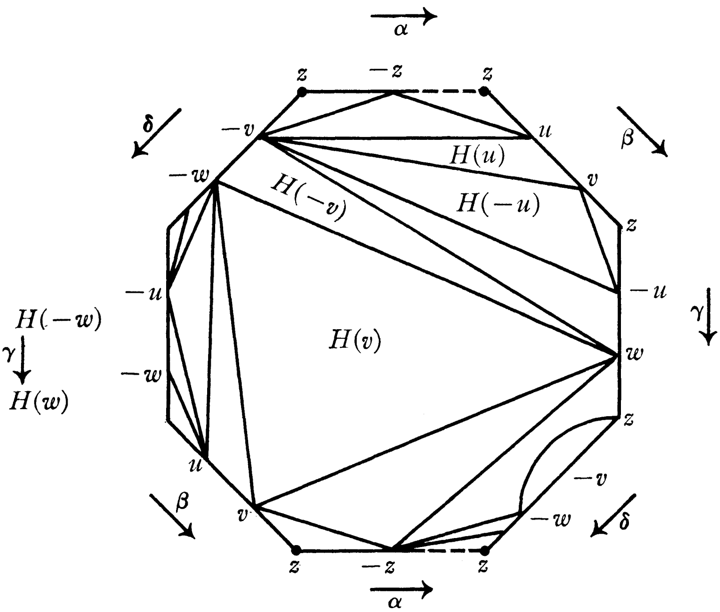

FIGURE 3

they are on and above the diagonal and if k heads the other column, then u_r is adjacent to $-u_s$ in G_h and to u_s in G_k. Therefore, u_r is adjacent to all of the other vertices, and the union of the n graphs is K_{6n}.

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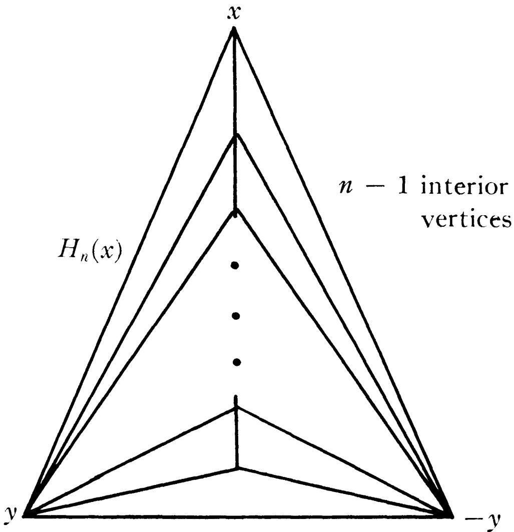

FIGURE 4

In the graph of Figure 2, z is adjacent to the other six vertices and in Figure 3, both z and $-z$ are adjacent to all other vertices. Since these graphs have the same six distinguished faces $H(x)$, it follows that K_{6n+1} can be decomposed into n toroidal graphs and K_{6n+2} into n graphs embeddable in the double-torus. This completes the proof of the following theorem.

THEOREM 2. *For three surfaces, the S-thickness of the complete graph K_p is the following:*

Projective plane: $$t'(K_p) = \left[\frac{p+5}{6}\right],$$

Torus: $$t_1(K_p) = \left[\frac{p+4}{6}\right],$$

Double-torus: $$t_2(K_p) = \left[\frac{p+3}{6}\right].$$

After these, the next surfaces to be considered are of course the Klein bottle S'' and the "triple-torus" S_3. In both cases, difficulties appear quickly. Although its characteristic is the same as that of the torus, the Klein bottle cannot have an embedding of K_7. This was shown by Franklin (**3**) and means that in this case the equality in (1) does not always hold. The next question here is whether K_{13} is the union of two S''-embeddable graphs. For the triple-torus, the first unanswered question is whether K_{16} is the union of two S_3-embeddable graphs. It is noteworthy that in both cases the two subgraphs in a decomposition would have to be triangulations of the corresponding surfaces. Incidentally, it is not difficult to establish the results for "five-sixths" of the complete graphs for both of these surfaces, as has been done for the plane.

II. Bicomplete graphs. The planar thickness of "nearly all" complete bipartite graphs has been found (**2**), as given in the following theorem.

THEOREM 3. *The planar thickness of the complete $m \times n$ bipartite graph is*

$$t_0(K_{m,n}) = \left\{\frac{mn}{2(m+n-2)}\right\},$$

except possibly when m and n are both odd, $m < n$, and there is an integer k, with $\frac{1}{4}(m+5) \leqq k \leqq \frac{1}{2}(m-3)$, such that

$$n = \left[\frac{2k(m-2)}{m-2k}\right].$$

There are no known exceptions to the equality of the theorem, and the unanswered cases are quite rare. For instance, when $m \leqq 30$, there are only six. Unfortunately, such cases also exist when other surfaces are considered. However, when consideration is restricted to the one-parameter family of regular bicomplete graphs $K_{n,n}$, exact results can be found for a number of surfaces.

A quadrangulation of a surface S with characteristic χ using p vertices has $2(p-\chi)$ edges. Therefore, a bipartite graph G with p vertices and q edges has S-thickness satisfying the inequality

$$t_S(G) \geqq \left\{\frac{q}{2(p-\chi)}\right\}. \tag{2}$$

This gives lower bounds for various S-thicknesses of $K_{n,n}$ as follows:

$$\chi = 1\colon \left\{\frac{n^2}{4n-2}\right\} = \left[\frac{n^2+4n-3}{4n-2}\right] \geqq \left[\frac{n^2+7/2n-4}{4n-2}\right] = \left[\frac{n+4}{4}\right],$$

$$\chi = 0, -1, -2\colon \left\{\frac{n^2}{4n}\right\} \geqq \left\{\frac{n^2}{4n+2}\right\} \geqq \left\{\frac{n^2}{4n+4}\right\}$$
$$= \left[\frac{n^2+4n+3}{4(n+1)}\right] = \left[\frac{n+3}{4}\right].$$

Constructive decompositions will next be given to show that $t'(K_{4r-1,4r-1}) \leqq r$, $t''(K_{4r,4r}) \leqq r$, and $t_1(K_{4r,4r}) \leqq r$. These will determine the S-thickness of every $K_{n,n}$ for the torus, double-torus, projective plane, Klein bottle, and other surfaces.

Projective plane. First, it will be shown that given four vertices u_1, u_2, u_3, and u_4 and $4r-1$ vertices v_j (for $j = 1, 2, \ldots, 4r-1$) one can join each u_i to $2r+1$ of the vertices v_j in a graph with a particular S'-embedding. Place the $4r-1$ vertices v_j consecutively on one semicircle α of the projective plane model $\alpha\alpha$. See Figure 5 for the case $r = 3$. Join u_1 to the first $2r+1$ of these vertices; beginning with v_{2r+1}, join u_2 to the next $2r+1$; and beginning with the last of these, join u_3 to the next $2r+1$. That is, u_1 is adjacent to $v_1, v_2, \ldots, v_{2r+1}$; u_2 to $v_{2r+1}, \ldots, v_{4r-1}, v_1, v_2$; and u_3 to $v_2, v_3, \ldots, v_{2r+2}$. One

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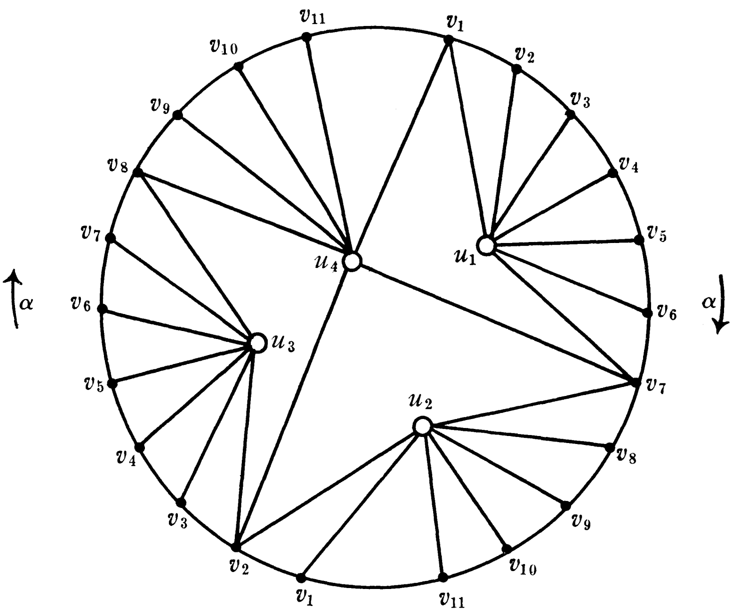

FIGURE 5

face of the graph now has, in addition to u_1, u_2, and u_3, the vertices $v_{2r+1}, v_{2r+2}, \ldots, v_{4r-1}, v_1, v_2$. Join u_4 to all of these. The resulting graph has each of the four vertices u_i adjacent to $2r + 1$ consecutive vertices v_j, and it is

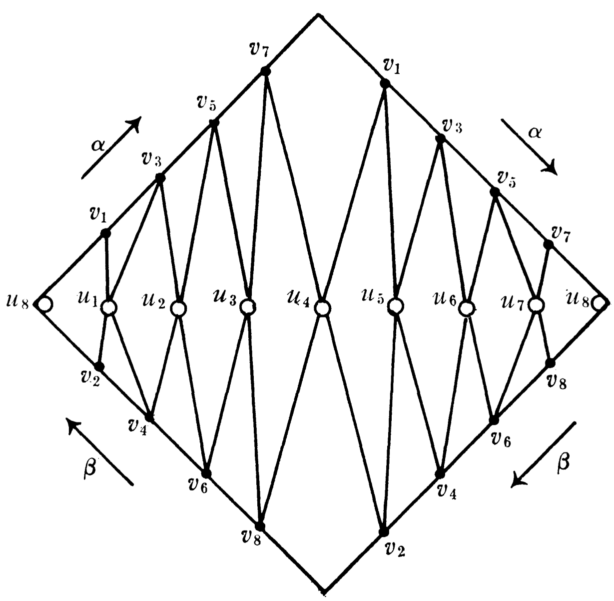

FIGURE 6

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embeddable in the projective plane model in such a way that no edge crosses the boundary $\alpha\alpha$. Therefore, new vertices can be added adjacent to any pair of consecutive vertices v_j, v_{j+1}. Take r copies of this graph with u_i being replaced by $u_{i+4}, u_{i+8}, \ldots$ in the other graphs. In the $r - 1$ graphs in which u_1 does not appear, it can be made adjacent to v_{2r+2} and v_{2r+3}, v_{2r+4} and v_{2r+5}, etc. However, the same thing can clearly be done for the other vertices u_i, so that one can form r graphs whose union contains the bicomplete graph $K_{4r-1,4r-1}$.

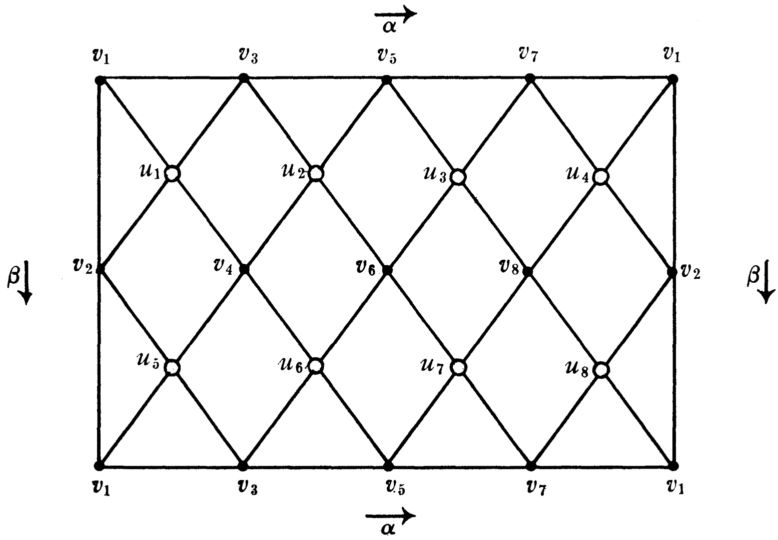

FIGURE 7

Klein bottle. A construction will be given to show that the graph G_1 in which each pair of vertices u_i and u_{2r+i} (for $i = 1, 2, \ldots, 2r$) are adjacent to the four vertices $v_{2i-1}, v_{2i}, v_{2i+1}, v_{2i+2}$ (with all sums modulo $4r$) is embeddable in the Klein bottle. Then by taking r isomorphic graphs G_k relabeled from G_1 by changing each u_{i+2k-2} to u_i but keeping each v_j fixed, one has r S''-embeddable graphs whose union is $K_{4r,4r}$.

The construction is made in the model $\alpha\alpha\beta\beta$ in the following way. Along α (not at corners) place $v_1, v_3, \ldots, v_{4r-1}$ and along β place $v_{4r}, v_{4r-2}, \ldots, v_2$. Along the diagonal joining the corners where α and β meet, put $u_{4r}, u_1, u_2, \ldots, u_{4r-1}, u_{4r}$. The joins required to form G_1 can now be made without intersections. Figure 6 illustrates the construction for $r = 2$.

For the non-orientable surfaces with three and four crosscuts, the S-thicknesses of $K_{n,n}$ are also determined by this construction since the same lower bound applies. That is,

$$r \geqq t''(K_{4r,4r}) \geqq t'''(K_{4r,4r}) \geqq t^{[4]}(K_{4r,4r}) \geqq r.$$

Torus and double-torus. The same graph G_1 defined above is embeddable in the torus as follows. Take the model $\alpha\beta\alpha^{-1}\beta^{-1}$ and place $v_1, v_3, \ldots, v_{4r-1}, v_1$ along α. Place $v_2, v_4, \ldots, v_{4r}, v_2$ on the segment parallel to α joining the midpoints of β. Put $u_1, u_2, \ldots, u_{2r}$ in the upper half and $u_{2r+1}, \ldots, u_{4r}$ in the lower half. The necessary joins are easily made. The case of $r = 2$ is shown in Figure 7. Since the S_1-thickness is at least as large as the S_2-thickness, this determines both.

This construction can also be used to determine the result for the triple-torus. By appropriately adding two handles and two vertices u_{4r+1} and v_{4r+1} to each of the r toroidal graphs, one can form r S_3-embeddable graphs whose union is $K_{4r+1,4r+1}$. That the S_3-thickness of this graph is at least r follows from (2). Further details of this result will not be given here.

The S-thickness results for bicomplete graphs are summarized in the following theorem.

THEOREM 4. *For various surfaces, the S-thickness of the regular bicomplete graph $K_{n,n}$ is the following:*

Plane: $$t_0(K_{n,n}) = \left[\frac{n+5}{4}\right],$$

Torus: $$t_1(K_{n,n}) = \left[\frac{n+3}{4}\right],$$

Double-torus: $$t_2(K_{n,n}) = \left[\frac{n+3}{4}\right],$$

Triple-torus: $$t_3(K_{n,n}) = \left[\frac{n+2}{4}\right],$$

Projective plane: $$t'(K_{n,n}) = \left[\frac{n+4}{4}\right],$$

Klein bottle: $$t''(K_{n,n}) = \left[\frac{n+3}{4}\right].$$

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