and therefore x_n is a factor in $\psi(x_1, x_2, \dots x_{n-1}, x_n)$. Since ψ is a symmetric function, $x_1, x_2, \dots x_{n-1}$, must also be factors; and therefore $x_1x_2 \dots x_n$, which is equal to ${}_np_n$, is a factor. If this factor be divided out, the quotient will be a symmetric function, the degree of which will be less by n that of the given function. The above process may then be repeated with this quotient; and so on, till the degree is reduced to zero.

Since every (symmetric) function of a single x_1 is a function of ${}_1p_1(=x_1)$, it follows by induction that every symmetric function of n variables is expressible in terms of the n elementary symmetric functions.

The ordinary propositions about the weight and order of symmetric functions may easily be obtained from the above.

On laboratory work in electricity in large classes.

By Messrs A. Y. Fraser, J. T. Morrison, and W. Wallace.

Seventh Meeting, May 10th, 1889.

George A. Gibson, Esq., M.A., President, in the Chair.

Solutions of two geometrical problems.

By J. S. Mackay, LL.D.

The two problems are :—

1. *To divide a given straight line internally and externally so that the ratio between its segments may be equal to a given ratio.*

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2. *To divide a given straight line internally and externally so that the rectangle under its segments may be equal to a given rectangle.*

The solutions of these problems commonly given in text-books of geometry, whether home or foreign, do not bring out so clearly as might be wished the correspondence which exists between them. Nor in the case of the second problem do the usual construction and proof for the external section run, as they ought to do, on all fours with the construction and proof for the internal section. Where correspondences between problems or between theorems exist, it cannot surely be instructive to ignore them.

It should be added that the solutions themselves are not new. The construction for Problem 1 will be found in Pappus's *Mathematical Collection*, Book III., Prop. 9; that for Problem 2 in Willebrord Snel's *Apollonius Batavus* (1608) or Edmund Halley's *Apollonii Pergaei Conica* (1710), Book VIII., Prop. 18, Scholion. Leslie, in the third and later editions of his *Elements of Geometry*, gives both Snel and Pappus's solutions of Problem 2.

Problem 1.

To divide a given straight line internally and externally so that the ratio between its segments may be equal to a given ratio.

Figure [8].

Let AB be the given straight line, K : L the given ratio.

Draw AE and BF perpendicular to AB and equal respectively to K and L. If AB is to be divided internally, AE and BF are drawn on opposite sides of it; if externally, on the same side of it. Let the straight line drawn through E and F cut AB at C.

C is the required point of division.

For triangles ACE, BCF are similar;

therefore $AC : AE = BC : BF$;

therefore $AC : BC = AE : BF$,

$= K : L$

It may be noted that:

The straight line EF whose direction gives the point of internal or external section for the ratio K : L is the diameter of the circle whose intersection with AB gives the point of external or internal section for the rectangle K·L.

If AB, measured from A to B, is to be divided in the ratio K : L, the construction gives only one point of section, and at the same time that the ratio K : L is not equivalent to the ratio L : K.

PROBLEM 2.

To divide a given straight line internally and externally so that the rectangle under its segments may be equal to a given rectangle.

Figure [9].

Let AB be the given straight line, K·L the given rectangle.

Draw AE and BF perpendicular to AB and equal respectively to K and L. If AB is to be divided internally, AE and BF are drawn on the same side of it; if externally, on opposite sides of it. Let the circle whose diameter is EF cut AB at C.

C is the required point of division.

For triangles ACE, BFC are similar;

therefore $AC : AE = BF : BC$;

therefore $AC \cdot BC = AE \cdot BF$,

$= K \cdot L$.

It may be noted that:

The straight line EF which is the diameter of the circle whose intersection with AB gives the point of internal or external section for the rectangle K·L is the direction which gives the point of external or internal section for the ratio K : L.

If AB, measured from A to B, is to be divided in the rectangle K·L, the construction gives two points of section, and at the same time that the rectangle K·L is equivalent to the rectangle L·K.

While the preceding solutions of Problems 1 and 2 would seem to be worthy of occupying the position of "classical" solutions, it may not be out of place to call attention to two other modes of resolving Problem 2.

Figure [10].

Let AB be the given straight line, K·L the given rectangle.

Draw AE and BF perpendicular to AB and equal respectively to K and L. If AB is to be divided internally, AE and BF are drawn on the same side of it; if externally, on opposite sides of it. Let the circle whose diameter is AB cut EF at G, and let GC drawn perpendicular to EF meet AB at C.

C is the required point of division.

Join AG, BG.

Because	$\angle EAG = \angle CBG$,
and	$\angle AGE = \angle BGC$;
therefore triangles	AEG and BCG are similar.
Hence also triangles	ACG and BFG are similar;
therefore quadrilaterals	ACGE and BFGC are similar;
therefore	$AC : AE = BF : BC$;
therefore	$AC \cdot BC = AE \cdot BF$,
	$= K \cdot L$.

This method of solution, at least in the case of external section, will be found in Pappus's *Mathematical Collection*, Book VII., Prop. 157.

It may be noted that:

When K and L are equal, or when the rectangle $K \cdot L$ is transformed into a square, in the one case EF is parallel to AB, and in the other case EF passes through the middle point of AB; or, as was previously remarked, EF passes through the point where AB is divided externally or internally in a ratio of equality. The constructions for these two cases usually given in foreign text-books (see Rouché et De Comberousse's *Traité de Géométrie*, Livre III., §§ 255, 257, or Vacquant's *Cours de Géométrie Elémentaire*, Livre III., §§ 310, 311) are now seen to be perfectly correspondent, and to be merely modifications of that given long ago by Pappus.

The following method of solution was communicated to me by Mr A. Y. Fraser.

Figure [11].

Let AB be the given straight line, $DE \cdot EF$ the given rectangle.

Describe any circle passing through D and F, and let O be its centre. In this circle place GH equal to AB. With O as centre describe a circle to touch GH, and through E draw the chord KL touching the second circle.

Then	$KL = GH = AB$,
and	$KE \cdot EL = DE \cdot EF$.

Hence the required segments of AB are equal to KE and EL.

For comparison with the above the constructions given in Playfair's *Elements of Geometry*, VI., 28, 29 are annexed.

Figure [12].

Let AB be the given straight line, C a side of the square to which the rectangle under the segments (internal or external) of AB is to be equal.

Case 1. Bisect AB in D; draw DE perpendicular to AB and equal to C. Produce ED to F so that EF be equal to AD or DB; with centre E and radius EF describe a circle cutting AB in G.

G is the required point of division.

Case 2. Bisect AB in D; draw BE perpendicular to AB and equal to C. Join DE; with centre D and radius DE describe a circle cutting AB produced in G.

G is the required point of division.

Kötter's synthetic geometry of algebraic curves.

By NORMAN FRASER, B.D.

In the following paper I propose to give a short account of Dr Ernst Kötter's purely geometrical theory of the algebraic plane curves. This theory is developed in a treatise* which, in 1886, gained the prize of the Berlin Royal Academy; but the contents of my paper are also partly drawn from a course of lectures delivered by Dr Kötter in the University of Berlin, W.S. 1887-88.

The method followed is that of involutions, and we might pass at once to a discussion of these; but it may be of interest to call attention, first of all, to one of the most remarkable features of the theory—Dr Kötter's treatment of imaginary points and lines. He undertakes to show that the projective relationship between two aggregates may be extended so as to embrace their imaginary elements, and, hence, that imaginary elements may take their place with real elements in the theory of involutions. One method of establishing this result has been given by von Staudt;† but it travels beyond the limits of Plane Geometry. Many writers on the subject simply assume the point at issue; a course which has its advantages, and may be recommended to any reader who should chance to find

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* *Grundzüge einer rein geometrischen Theorie der algebraischen ebenen Curven.* (*Transactions of the Royal Academy of Science.* Berlin, 1887. Also published separately in the same year.)

† *Beiträge zur Geometrie der Lage.* Nürnberg, 1856-60.