EXISTENCE OF NON-TRIVIAL DEFORMATIONS OF SOME INSEPARABLE EXTENSION FIELDS

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Dedicated to the memory of Professor Tadasi Nakayama

A deformation theory for rings and algebras was introduced recently by M. Gerstenhaber [1]. Let K be an extension of a field k, and p denotes the characteristic. One of his results is that, if K is separable over k, then it is rigid. It was conjectured in [1] that, if K is not separable over k, then it is not rigid, and if it is further finitely generated, then an integrable element of $H^2_C(K, K)$ (see [2]) will be found in the image of Sq_p. In this note we shall study the above conjecture in certain special case.

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1. Preliminaries. Let K be an extension field of a field k of characteristic p, and V be the underlying vector space of K. Let R and S denote the power series ring $k[[t]]$ over k in one variable t and its quotient field $k((t))$ respectively, and let V_S be $V \otimes_k S$.

Let a bilinear mapping $f_t : V_S \times V_S \longrightarrow V_S$ expressible in the form

$$f_t(a, b) = ab + tF_1(a, b) + t^2F_2(a, b) + \cdots,$$

where F_i is a bilinear mapping defined over k, be a one-parameter family of deformations of K considered as a commutative k-algebra. Then f_t satisfies the conditions;

$$f_t(a, b) = f_t(b, a)$$
$$f_t(f_t(a, b),\ c) = f_t(a, f_t(b, c))$$

for all a, b, c in V_S and F_1 is an element of the group $Z^2_C(K, K)$ of commutative two-cocycles of K with coefficients in K (see [1]). Following [1], we say that f_t is trivial if there is a nonsingular linear mapping Φ_t of V_S onto itself of the form

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$$\Phi_t(a) = a + t\varphi_1(a) + t^2\varphi_2(a) + \cdots,$$

where φ_i is a linear mapping defined over k, such that

$$f_t(a, b) = \Phi_t^{-1}(\Phi_t a \cdot \Phi_t b).$$

K is rigid if and only if there is no non-trivial one-parameter family of deformations of K.

From now on, throughout this note, we assume $p \neq 0$.

The squaring operation is defined for every derivation φ of K over k by $Sq_p\varphi = -\frac{1}{p}\delta\varphi^p = \sum_{i=1}^{p-1}\frac{1}{p}\,{}_pC_i\varphi^{p-i} \cup \varphi^i$, where δ denotes the coboundary operator and $\cup$ denotes the cup product. For example, if $p = 2, 3$, then

$$Sq_2\varphi = \varphi \cup \varphi,$$

and

$$Sq_3\varphi = \varphi^2 \cup \varphi + \varphi \cup \varphi^2.$$

Gerstenharber proved in [1] that $Sq_p\varphi \in Z_C^2(K, K)$. It is known ([1]) that, for any derivation φ of K, there exists a one-parameter family f_t of deformations of K such that

$$f_t(a, b) = ab + tSq_p\varphi(a, b) + t^2F_2(a, b) + \cdots.$$

2. Let K be an inseparable extension field over k such that there exists an inseparable algebraic element θ of exponent one over k such that θ is not contained in $k(K^p)$. Let $f(X) = X^{mp} - a_{m-1}X^{(m-1)p} - \cdots - a_1X^p - a_0$ be the minimum polynomial of θ over k. Then there eixsts $a_i \neq 0$, $1 \leqq i \leqq m$, such that i is not divisible by p (where $a_m = 1$).

Let B be a p-base for K over k such that θ is contained in B ([3]). Then there exists one and only one derivation φ of K over k such that, for $\beta \in B$,

$$\varphi(\beta) = \begin{cases} 1 & (\beta = \theta) \\ 0 & (\beta \neq 0). \end{cases}$$

Then there is a one-parameter family of deformations of K;

$$f_t(a, b) = ab + tSq_p\varphi(a, b) + \cdots,$$

where $Sq_p\varphi = -\frac{1}{p}\delta\varphi^p = \sum_{i=1}^{p-1}\frac{1}{p}\,{}_pC_i\varphi^{p-i} \cup \varphi^i$. We assume that t_t is trivial, i.e.,

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$$f_t(a, b) = \Phi_t^{-1}(\Phi_t a \cdot \Phi_t b),$$

where

$$\Phi_t(a) = a + t\varphi_1(a) + \cdots.$$

By this assumption we obtain $\delta\varphi_1 = Sq_p\varphi$.

The following lemma is trivial.

LEMMA 1. *Let θ, φ be as above. Then*

$$Sq_p\varphi(\theta^t, \theta) = \varphi^{p-1}(\theta^t)$$
$$= \begin{cases} -\theta^{(n-1)p} & (t = np-1 \text{ for some integer } n) \\ 0 & (\text{there is no integer } n \text{ such that } t = np-1). \end{cases}$$

LEMMA 2. *Let θ, φ be as above and φ_1 be a k-linear mapping of K into itself such that $\delta\varphi_1 = Sq_p\varphi$. Then,*

$$\varphi_1(\theta^t) = t\theta^{t-1}\varphi_1(\theta) + n\theta^{t-p},$$

for $np \leqq t < (n+1)p$. In particular,

$$\varphi_1(\theta^{np}) = n\theta^{(n-1)p}.$$

Proof. We shall prove this by induction on t.

1) The case $t = 0$.

$$\delta\varphi_1(1, 1) = Sq_p\varphi(1, 1) = 0.$$

Therefor $\varphi_1(1) = 0$.

2) The case $t = np$.

$$\begin{aligned}\varphi_1(\theta^t) &= \theta^{t-1}\varphi_1(\theta) + \theta\varphi_1(\theta^{t-1}) - Sq_p\varphi(\theta^{t-1}, \theta) \\ &= \theta^{t-1}\varphi_1(\theta) + \theta\{(t-1)\theta^{t-2}\varphi_1(\theta) + (n-1)\theta^{t-1-p}\} \\ &\quad - \varphi^{p-1}(\theta^{t-1}) \\ &= t\theta^{t-1}\varphi_1(\theta) + n\theta^{t-p} = n\theta^{(n-1)p}.\end{aligned}$$

3) The case $np < t < (n+1)p$.

$$\begin{aligned}\varphi_1(\theta^t) &= \theta^{t-1}\varphi_1(\theta) + \theta\{(t-1)\theta^{t-2}\varphi_1(\theta) + n\theta^{t-1-p}\} \\ &\quad - \varphi^{p-1}(\theta^{t-1}) \\ &= t\theta^{t-1}\varphi_1(\theta) + n\theta^{t-p}.\end{aligned}$$

This ends the proof.

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By Lamma 2, we have

$$\varphi_1(\theta^{mp}) = m\theta^{(m-1)p}.$$

On the other hand,

$$\varphi_1(\theta^{mp}) = \varphi_1\left(\sum_{i=0}^{m-1} a_i\theta^{ip}\right) = \sum_{i=1}^{m-1} i a_i \theta^{(i-1)p}.$$

Therefore $m \equiv 0 \pmod{p}$ and if $a_i \neq 0$, then $i \equiv 0 \pmod{p}$. Hence θ is an inseparable element of exponent ≥ 2 over k. This is a contradiction, and we have obtained the following

THEOREM. *Let K be an extension field of a field k of characteristic $p \neq 0$. If there exists an inseparable algebraic element of exponent one over k which is not contained in $k(K^p)$, then K is not rigid, and more precisely a non-trivial integrable element of $H^2_C(K, K)$ is found in the image of Sq_p.*

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