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EXTREMAL PROPERTIES OF CONSTRAINED TCHEBYCHEV POLYNOMIALS

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1. Introduction. In the sequel, π_n will denote the class of real polynomials of degree at most n and $\|f(x)\|_\infty$ the L_∞-norm of a function on $[-1, +1]$.

In a series of recent papers, Saff and Varga studied the properties of the so-called incomplete polynomials; that is to say polynomials of the form

$$(1 - x)^{s_1}(1 + x)^{s_2}q(x)$$

where s_1 and s_2 are fixed integers and $q \in \pi_n$.

In there, they define the constrained Tchebychev polynomial as being, up to a multiplicative constant, the solution of the following minimization problem

$$\min\{ \|p(x)\|_\infty \,|\, p(x) = (1 - x)^{s_1}(1 + x)^{s_2}q(x),\ q \in \pi_n,\ q \text{ monic}\}.$$

These polynomials, which they denote by $T_{s_1,s_2,n}(x)$ exhibit extremal properties very similar to those of the classical Tchebychev polynomials. Indeed, we have the following (see [**3**], Theorem 3.3):

THEOREM A. *Let $p \in \pi_n$ and satisfy*

$$\|(1 - x)^{s_1}(1 + x)^{s_2}p(x)\|_\infty \leqq 1,$$

then, for each integer k and real $x \notin [-1, +1]$,

$$(1) \qquad \left|\frac{d^k}{dx^k}\{(1 - x)^{s_1}(1 + x)^{s_2}p(x)\}\right| \leqq |T^{(k)}_{s_1,s_2,n}(x)|.$$

In the light of this result and in view of the importance of these polynomials in the work of Saff and Varga, it is natural to investigate the growth of the derivatives of incomplete polynomials in the open interval $(-1, +1)$. A first attempt in this direction was made by Lachance [**2**] who obtained asymptotically best possible results in the symmetric case $(s_1 = s_2)$.

In this paper, we study the problem for the class corresponding to

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$s_1 = \lambda/2$, $s_2 = \mu/2$ where λ and μ are either 0, 1 or 2. Doing this, we will be able to extend results of I. Schur and to precise certain inequalities obtained by Lachance in the aforementioned article.

The outline is as follows. In the next section we state the main results and discuss their sharpness. Section 3 contains the necessary lemmas and we conclude with the proof of the theorems in Section 4.

2. The main theorems. The case $s_1 = s_2 = 0$ has been extensively studied. The polynomial $T_{0,0,n}(x)$ is the classical Tchebychev polynomial

$$(2) \qquad T_n(x) = \cos(n \arccos x).$$

If we define the related functions $S_n(x)$ and $M_{k,n}(x)$ by

$$(3) \qquad S_n(x) = \sin(n \arccos x)$$

$$(4) \qquad M_{k,n}(x) = |T_n^{(k)}(x) + iS_n^{(k)}(x)| \quad k = 0, 1, 2, \ldots, n,$$

the extremal properties of $T_n(x)$ are best summarized, at least in our context, by the next theorem due to Duffin and Schaeffer (see [**1**]).

THEOREM B. *For each integer k, let a_k denote the right-most zero of $S_n^{(k)}(x)$ in $(-1, +1)$. If $p \in \pi_n$ and satisfies $\|p(x)\|_\infty \leqq 1$, then, for $k = 1, 2, \ldots, n$,*

$$(5) \qquad |p_n^{(k)}(x)| \leqq |T_n^{(k)}(x)| \textit{ for each } x \notin [-a_k, a_k],$$

$$(6) \qquad |p_n^{(k)}(x)| \leqq M_{k,n}(x) \textit{ for each } x \in (-1, +1)$$

$$(7) \qquad \|p_n^{(k)}(x)\|_\infty \leqq \|T_n^{(k)}(x)\|_\infty = |T_n^{(k)}(1)|.$$

The inequalities (5) and (7) go back to W. Markov who obtained them as a product of the study of a more general problem; they are obviously best possible. Inequality (6), known, for $k = 1$, as Bernstein inequality, is best possible only at the $(n + 1 - k)$ zeros of $S_n^{(k)}(x)$, but asymptotically best possible at each point.

One very important remark is that $M_{k,n}(x)$ is an increasing function on $(0, 1)$ (see [**1**], Lemma 2).

Since (1) already extends (5) in the case where s_1 and s_2 are arbitrary integers we concentrate our attention on (6) and (7).

Our first theorem gives the correct improvement of Theorem B for polynomials having at least one simple zero at the end points.

THEOREM 1. *Let $p \in \pi_n$ be such that $\|p(x)\|_\infty \leqq 1$.*
a) *If $p(1) = 0$, then, for $k = 1, 2, \ldots, n$,*

$$(8) \qquad |p^{(k)}(x)| \leqq \left(\cos\frac{\pi}{4n}\right)^{2k} M_{k,n}\left(x\cos^2\frac{\pi}{4n} - \sin^2\frac{\pi}{4n}\right)$$

for each

$$x \in -1, \left(1 + \sin^2 \frac{\pi}{4n}\right) \Big/ \left(\cos^2 \frac{\pi}{4n}\right)$$

while

$$(9) \quad \| p^{(k)}(x) \|_\infty \leqq \left(\cos \frac{\pi}{4n}\right)^{2k} |T_n^{(k)}(-1)| = \| T_{1,0,n}^{(k)}(x) \|_\infty.$$

b) *If* $p(-1) = p(+1) = 0$, *then, for* $k = 1, 2, \dots, n$,

$$(10) \quad | p^{(k)}(x) | \leqq \left(\cos \frac{\pi}{2n}\right)^{k} M_{k,n}\left(\cos \frac{\pi}{2n}\right)$$

for each

$$x \in \left(-1/\left(\cos \frac{\pi}{2n}\right), 1/\left(\cos \frac{\pi}{2n}\right)\right),$$

while

$$(11) \quad \| p^{(k)}(x) \|_\infty \leqq \left(\cos \frac{\pi}{2n}\right)^{k} \left| T_n^{(k)}\left(\cos \frac{\pi}{2n}\right)\right| = \| T_{1,1,n}^{(k)}(x) \|_\infty.$$

For $k = 1$, this theorem was proven by I. Schur [**7**]. The inequalities (9) and (11) are best possible as shown by the extremals

$$T_{1,0,n}(x) = T_n\left(x \cos^2 \frac{\pi}{4n} - \sin^2 \frac{\pi}{4n}\right) \quad \text{and}$$

$$T_{1,1,n}(x) = T_n\left(x \cos \frac{\pi}{2n}\right).$$

On the other hand, for (8) and (10), the equality will be valid, for the same polynomials, only at the interior zeros of

$$S_n^{(k)}\left(x \cos^2 \frac{\pi}{4n} - \sin^2 \frac{\pi}{4n}\right) \quad \text{and} \quad S_n^{(k)}\left(x \cos \frac{\pi}{2n}\right)$$

respectively.

Let us now define the class $\pi_{\lambda,\mu,n}$ by

$$\pi_{\lambda,\mu,n} = \{p \in \pi_n | \, \| (1 - x)^{\lambda/2}(1 + x)^{\mu/2}p(x) \|_\infty \leqq 1\}.$$

In his paper [**7**], I. Schur considered and solved the problem of estimating $\|p\|_\infty$ for $p \in \pi_{2,0,n}$ and $p \in \pi_{2,2,n}$, while, in [**2**], Lachance was able to get estimates for $\| p^{(k)}(x) \|_\infty$ when $p \in \pi_{\lambda,\lambda,n}$, which, although not exact, give the right order of n.

The next three theorems provide exact results in certain cases. Their statements require definitions which we group in a table.

TABLE 1

(λ, μ)	$t_{\lambda,\mu,n}$	$s_{\lambda,\mu,n}$
(1, 0)	$\dfrac{S_{2n+1}\left(\sqrt{\dfrac{1+x}{2}}\right)}{\sqrt{1-x}}$	$\dfrac{T_{2n+1}\left(\sqrt{\dfrac{1+x}{2}}\right)}{\sqrt{1-x}}$
(1, 1)	$\dfrac{S_{n+1}(x)}{\sqrt{1-x^2}}$	$\dfrac{T_{n+1}(x)}{\sqrt{1-x^2}}$
(2, 0)	$\dfrac{T_{n+1}\left(x\cos^2\dfrac{\pi}{4(n+1)} - \sin^2\dfrac{\pi}{4(n+1)}\right)}{(1-x)}$	$\dfrac{S_{n+1}\left(x\cos^2\dfrac{\pi}{4(n+1)} - \sin^2\dfrac{\pi}{4(n+1)}\right)}{(1-x)}$
(2, 2)	$\dfrac{T_{n+2}\left(x\cos\dfrac{\pi}{2(n+2)}\right)}{(1-x^2)}$	$\dfrac{S_{n+2}\left(x\cos\dfrac{\pi}{2(n+2)}\right)}{(1-x^2)}$

The cases (0, 1) and (0, 2) were not included since they are readily obtained from the case (1, 0) and (2, 0) by changing x to $-x$.

We first consider the problem of obtaining pointwise bounds.

THEOREM 2. *Let $p \in \pi_{\lambda,\mu,n}$ where (λ, μ) takes the values* (1, 0), (1, 1), (2, 0) *and* (2, 2). *For each $k = 0, 1, 2, \dots, n$ and for each $x \in (-1, +1)$, we have*

$$(12) \quad |p^{(k)}(x)| \leqq |t^{(k)}_{\lambda,\mu,n}(x) + is^{(k)}_{\lambda,\mu,n}(x)|.$$

The sharpness of (12) is subject to the same limitations as that of (8) and (10), but its use will lead to sharp global bounds.

In the case $(\lambda, \mu) = (1, 1)$, we were able to use (12) to its full power to obtain:

THEOREM 3. *Let $p \in \pi_{1,1,n}$, then for $k = 1, 2, \dots, n$ we have*

$$(13) \quad \|p^{(k)}(x)\|_\infty \leqq \|t^{(k)}_{1,1,n}(x)\|_\infty = |t^{(k)}_{1,1,n}(+1)|.$$

The study of the class $\pi_{1,1,n}$ goes back to Bernstein for the case $k = 0$. The case $k = 1$ was more recently considered by Pierre and Rahman in **[4]**.

An immediate consequence of Theorem 3 is the following corollary which, in view of the Bernstein inequality, is a nice improvement of the W. Markov inequality (7).

COROLLARY 1. *Let $p \in \pi_n$ and satisfy $|p'(x)| \leqq n/\sqrt{1-x^2}$, for $-1 < x < +1$, then for $k = 2, 3, \dots, n$,*

$$\|p^{(k)}(x)\|_\infty \leqq |T^{(k)}_n(1)|.$$

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For the remaining cases $(\lambda, \mu) = (1, 0)$, $(2, 0)$ and $(2, 1)$, we have the following Theorem which, in the last two cases, generalizes results of I. Schur (see [7], Sections 3 and 4).

THEOREM 4. *Let* (λ, μ) *be equal to* $(1, 0)$, $(2, 0)$ *or* $(2, 1)$. *If* $p \in \pi_{\lambda,\mu,n}$, *then*

(14) $\quad \|p'(x)\|_\infty \leqq \|t'_{\lambda,\mu,n}(x)\|_\infty.$

3. Lemmas. We now prove or simply quote certain results which we shall need later.

First we recall that the polynomial $T_n(x)$ and the function $S_n(x)$ are linearly independent solutions of the equation

(15) $\quad (1 - x^2)y'' - xy' + n^2y = 0.$

Using (15) and the relations between $t_{\lambda,\mu,n}$, $s_{\lambda,\mu,n}$ and the classical functions, it is easy to verify that the former are linearly independent solutions of an equation of the form

(16) $\quad p_k(x)y'' + p_{k-1}(x)y' + p_{k-2}(x)y = 0$

where p_k, p_{k-1}, p_{k-2} are real polynomials of degree k, $k - 1$, $k - 2$ respectively. Moreover, $p_k(x) \neq 0$ on $(-1, +1)$. As a matter of fact this verification will show that $k = 2$ if $(\lambda, \mu) = (1, 0)$ or $(1, 1)$ while $k = 3$ if $(\lambda, \mu) = (2, 0)$ and $k = 4$ if $(\lambda, \mu) = (2, 2)$.

This remark leads us to the first two lemmas.

LEMMA 1. *Let* $p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0$ *be a solution of the differential equation*

(17) $\quad (x^2 + bx + c)y'' + (dx + e)y' + fy = 0,$

where $x^2 + bx + c \neq 0$ *for* $x \in (-1, +1)$. *If* y *is any solution of* (17), *then, either* $y^{(n+1)}(x) \equiv 0$ *on* $(-1, +1)$ *or* $y^{(n+1)}(x) \neq 0$ *for every* $x \in (-1, +1)$.

Proof. If we differentiate both sides of (17) n times with respect to x, we obtain

(18) $$(x^2 + bx + c)y^{(n+2)} + \{(2n + d)x + nb + e\}y^{(n+1)} + \{n(n - 1) + nd + f\}y^{(n)} = 0.$$

Setting $y = p(x)$ in (18) we get

$$n(n - 1) + nd + f = 0$$

which shows that (18) reduces to

(19) $$(x^2 + bx + c)y^{(n+2)} + \{(2n + d)x + nb + e\}y^{(n+1)} = 0.$$

Now, let y be any solution of (17) and x_0 be a point in $(-1, +1)$.

If $y^{(n+1)}(x) \not\equiv 0$, there exist an integer k and a function $z(x)$ analytic in $(-1, +1)$ for which

$$y^{(n+1)}(x) = (x - x_0)^k z(x),$$

where $z(x_0) \neq 0$. Substituting in (19) and dividing throughout by $(x - x_0)^{k-1}$, we find

$$(x^2 + bx + c)(kz(x) + (x - x_0)z'(x)) + z(x)(x - x_0) \times \{(2n + d)x + nb + e\} = 0.$$

Setting $x = x_0$ and remembering that $x_0^2 + bx_0 + c \neq 0$, we get $k = 0$ which is the desired conclusion.

LEMMA 2. *Let $t_n(x)$ and $s_n(x)$ be two linearly independent solutions of the equation*

$$(20) \quad p(x)y'' + q(x)y' + y = 0,$$

where $p(x)$ and $q(x)$ are analytic in $(-1, +1)$ and $p(x) \neq 0$ for $x \in (-1, +1)$. Let us suppose moreover that

(i) $t_n \in \pi_n$,

(ii) $s_n(x)$ *possesses* $(n + 1)$ *simple roots in* $[-1, +1]$,

(iii) $\lim_{|x| \to 1} |s_n'(x)| = +\infty$,

(iv) $s_n^{(n+1)}(x) \neq 0$ *for* $x \in (-1, +1)$.

If $f \in \pi_n$ and satisfies

$$(21) \quad |f'(x)| \leqq |t_n'(x) + is_n'(x)|, \text{ for } x \in (-1, +1),$$

then, for every real α, the first n derivatives of

$$\cos \alpha t_n(x) + \sin \alpha s_n(x) - f(x)$$

have only simple zeros in $(-1, +1)$.

Proof. Let us differentiate both sides of (20) with respect to x. We get,

$$(22) \quad p(x)y''' + (q(x) + p'(x))y'' + (q'(x) + 1)y' = 0.$$

If we apply the Sturm separation Theorem to the solutions t_n and s_n of (20), we deduce from (ii) that t_n possesses n distinct zeros in $(-1, +1)$. In view of Rolle's Theorem and of the Sturm separation Theorem applied to the solutions t_n' and s_n' of (22), we see that there exist $(n - 1)$ points $y_1, y_2, \ldots, y_{n-1}$ and n points $x_1, \ldots, x_n$ for which

$$(23) \quad \begin{cases} t_n'(y_i) = 0 & i = 1, 2, \ldots, (n - 1), \\ s_n'(x_i) = 0 & i = 1, \ldots, n, \end{cases}$$

and

$$(24) \quad -1 < x_1 < y_1 < x_2 < \ldots < x_n < 1.$$

Let us now distinguish two cases.

Case 1. $\alpha = 0$. From (23), (24) and (21), we see that the inequality

$$|f'(x_i)| < |t_n'(x_i)|$$

is valid for $i = 1, 2, \ldots, n$, while

$$\operatorname{sgn}(t_n'(x_i)) = -\operatorname{sgn}(t_n'(x_{i+1}))$$

for $i = 1, 2, \ldots, n - 1$. This implies that the difference $t_n'(x) - f'(x)$ has at least $(n - 1)$ distinct roots in $(-1, +1)$ which, in view of Rolle's Theorem, implies that $t_n^{(k)}(x) - f^{(k)}(x)$ has at least $(n - k)$ distinct roots in $(-1, +1)$ for $k = 1, 2, \ldots, n$. Since this difference is in π_{n-k}, it can have no other root and they are all simple.

Case 2. $\alpha \neq 0$. Set

$$R_\alpha(x) = \cos \alpha t_n(x) + \sin \alpha s_n(x).$$

In order to adapt the preceding reasoning, we first try to show that there exist $(n + 1)$ points $z_0, z_1, \ldots, z_n$ in $(-1, +1)$, for which

$$\operatorname{sgn}(R_\alpha'(z_i)) = -\operatorname{sgn}(R_\alpha'(z_{i+1})) \quad \text{when } i = 0, 1, \ldots, n - 1,$$

while

$$|f'(z_i)| < |R_\alpha'(z_i)| \quad \text{for } i = 0, 1, 2, \ldots, n.$$

Let us remark that, if $Q_\alpha(x) = R_{\alpha+\pi/2}(x)$, then, at every zero z of $Q_\alpha'(x)$, we have

$$(R_\alpha'(z))^2 = (t_n'(z))^2 + (s_n'(z))^2$$

and thus

$$(f'(z))^2 < (R_\alpha'(z))^2.$$

We first restrict our attention to the case $\alpha \in (0, \pi/2)$. The case $\alpha \in (\pi/2, \pi)$ can be dealt with similarly. There are two possibilities, either $t_n'(x)$ and $s_n'(x)$ are of opposite sign in a neighbourhood of -1, or they are of the same sign. We treat the first case, the other one being obtained mutatis mutandis. Let $\epsilon > 0$ be small enough and, without loss of generality,

$$\operatorname{sgn}(t_n'(-1 + \epsilon)) = -1 = -\operatorname{sgn}(s_n'(-1 + \epsilon)).$$

Going back to (23) and (24), we see that, for $k = 1, 2, \ldots, n$ the relations

$$\operatorname{sgn}(t_n'(x_k)) = (-1)^k = \operatorname{sgn}(R_\alpha'(x_k)) = -\operatorname{sgn}(Q_\alpha'(x_k)),$$

$$\operatorname{sgn}(s_n'(y_k)) = (-1)^k = \operatorname{sgn}(R_\alpha'(y_k)) = \operatorname{sgn}(Q_\alpha'(y_k)),$$

are valid. On the other hand, using (iii), we get

$$\operatorname{sgn}(s_n'(-1+\epsilon)) = +1 = \operatorname{sgn}(R_\alpha'(-1+\epsilon))$$
$$= \operatorname{sgn}(Q_\alpha'(-1+\epsilon))$$

and

$$\operatorname{sgn}(s_n'(1-\epsilon)) = (-1)^n = \operatorname{sgn}(R_\alpha'(1-\epsilon)) = \operatorname{sgn}(Q_\alpha'(1-\epsilon)).$$

From this, we readily deduce that Q_α' has n roots $z_1, z_2, \ldots, z_n$ satisfying

$$z_i \in (x_i, y_i) \quad \text{for } i = 1, 2, \ldots, n-1 \text{ and } z_n \in (x_n, 1),$$

whereas R_α' has n roots $\xi_1, \xi_2, \ldots, \xi_n$ satisfying

$$\xi_{i+1} \in (y_i, x_{i+1}), \quad i = 1, \ldots, n-1 \quad \text{and} \quad \xi_1 \in (-1, x_1).$$

Since, in view of (iii), the inequality $|f'(x)| < |R_\alpha'(x)|$ is always valid in a neighbourhood of -1 and $+1$, we can choose $z_0 \in (-1, \xi_1)$ sufficiently near -1, thereby obtaining the desired set of points $\{z_i\}_0^n$.

In the case $\alpha = \pi/2$, we simply choose z_0 near -1, $z_i = y_i$ $i = 1, \ldots, n-1$ and z_n near $+1$. This completes the first step.

To conclude we observe that the conditions imposed on the points $\{z_i\}$ imply that the function $R_\alpha'(x) - f'(x)$ has at least n distinct zeros in $(-1, +1)$. This, in turn, implies that for $p = 1, 2, \ldots, n$ the function $R_\alpha^{(p)}(x) - f^{(p)}(x)$ has at least $n - p + 1$ distinct roots there. If any of these was not simple, the function

$$R_\alpha^{(n+1)}(x) - f^{(n+1)}(x) = \sin \alpha s_n^{(n+1)}(x)$$

would be left with one root in $(-1, +1)$ which contradicts (iv).

We will see that Lemma 2 is the principal ingredient in the proof of Theorem 2. The only problem in using it is to verify conditions (iii). Although we suspect that it should follow from the previous ones, we were unable to check its validity in a general context. This forced us to use a more "ad hoc" reasoning.

LEMMA 3. *Let* $s_n(x) = s_{\lambda,\mu,n}(x)$ *for* (λ, μ) *equal to* $(1, 0)$, $(1, 1)$, $(2, 0)$ *or* $(2, 2)$. *Then*

$$s_n^{(n+1)}(x) \neq 0 \quad \textit{for } x \in (-1, +1).$$

Proof. We have already noticed that, when (λ, μ) is equal to $(1, 0)$ or $(1, 1)$, the function $s_n(x)$ satisfies a differential equation of the form (17). Therefore, in those cases Lemma 3 follows from Lemma 1 since $s_n(x)$ is never a polynomial.

For $(\lambda, \mu) = (2, 0)$ we consider the case $n = 2k$ while for $(\lambda, \mu) = (2, 2)$ we consider the case $n = (2k - 1)$. The two other cases are obtained through obvious modifications. Let us put

$$z(x) = S_{2k+1}(ax + b)$$

where a and b are such that $[b - a, a + b] \subseteq [-1, +1]$. Using (15), we see that $z(x)$ is a solution of the equation

$$\left(\frac{1}{a^2} - \left(x + \frac{b}{a}\right)^2\right) z''(x) - \left(x + \frac{b}{a}\right) z'(x) + (2k + 1)^2 z(x) = 0. \tag{25}$$

Since $S_{2k+1}(x)$ is even, we look for a solution of (25) of the form

$$z(x) = \sum_0^\infty d_j\left(x + \frac{b}{a}\right)^{2j}$$

where the development is valid on

$$\left(-\frac{(b + 1)}{a}, \frac{(1 - b)}{a}\right).$$

Substituting in (25) we obtain the following relationship for the coefficients

$$d_{j+1} = \frac{-a^2 d_j[(2k + 1)^2 - 4j^2]}{(2j + 2)(2j + 1)} \quad j = 0, 1, \dots,$$

from which we deduce that, for $s = 1, 2, \dots,$

$$\operatorname{sgn}(d_{k+s}) = \operatorname{sgn}(d_{k+1}) = (-1)^{k+1}\operatorname{sgn}(d_0) = -1. \tag{26}$$

We now distinguish the two cases.

a) $(\lambda, \mu) = (2, 0)$. In that case

$$s_{2k}(x) = z(x)/(1 - x),$$

with

$$a = \cos^2(\pi/4(2k + 1)) \quad \text{and} \quad b = -\sin^2(\pi/4(2k + 1)).$$

Since

$$(1 - x) = \left(1 + \frac{b}{a}\right) - \left(x + \frac{b}{a}\right)$$

we can write, for $x \in (-1, +1)$

$$s_{2k}(x) = \left(\frac{a}{a + b}\right) \sum_{u=0}^\infty c_u\left(x + \frac{b}{a}\right)^u,$$

where

$$c_u = \left(\frac{a}{a + b}\right)^u \sum_{i=0}^{[u/2]} d_i\left(\frac{a + b}{a}\right)^{2i} \quad \text{for } u = 0, 1, \dots$$

Using the fact that

$$z(1) = S_{2k+1}(\cos(\pi/2(2k + 1))) > 0$$

we infer from (26) that, for $u \geqq 2k + 1$,

$$c_u = \left(\frac{a}{a + b}\right)^u \sum_{i=0}^{[u/2]} \left(\frac{a + b}{a}\right)^{2i} d_i$$

$$= \left(z(1) - \sum_{i=[u/2]+1}^{\infty} d_i\left(\frac{a + b}{a}\right)^{2i}\right)\left(\frac{a}{a + b}\right)^u > 0.$$

Hence $s_{2k}^{(2k+1)}(x)$ is strictly positive for $x > -b/a$. On the other hand, differentiating both sides of the relation

$$(1 - x)s_{2k}(x) = z(x),$$

$(2k + 2)$ times with respect to x, we get

$$(1 - x)s_{2k}^{(2k+2)}(x) - (2k + 2)s_{2k}^{(2k+1)}(x) = z^{(2k+2)}(x).$$

Multiplying both sides by $(1 - x)^{2k+1}$ we can write the last relation as

$$\frac{d}{dx}[(1 - x)^{2k+2}s_{2k}^{(2k+1)}(x)] = z^{(2k+2)}(x)(1 - x)^{2k+1}.$$

Now using (26) again, we see that

$$z^{(2k+2)}(x) < 0 \quad \text{for } x \in (-1, +1),$$

hence that $(1 - x)^{2k+2}{s_{2k}}^{(2k+1)}(x)$ is strictly decreasing there. Since it is strictly positive in $(-b/a, +1)$, it is strictly positive everywhere, and the same is true for $s_{2k}^{(2k+1)}(x)$.

b) $(\lambda, \mu) = (2, 2)$. This time

$$s(x) = z(x)/(1 - x^2),$$

with

$$a = \cos(\pi/2(n + 2)), \quad b = 0.$$

Repeating the above reasoning, we will obtain that, for $x \in (-1, +1)$,

$$s_{2k-1}(x) = \sum_0^{\infty} c_u x^{2u},$$

where

$$c_u = \sum_0^u d_i \quad \text{for } u = 0, 1, 2, \dots .$$

Using (26) and the fact that $z(1) > 0$, we deduce in the same way as in a) that $c_u > 0$ for $u \geqq k$ which imply that

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$$s_{2k-1}^{(2k)}(x) > 0 \quad \text{for } x \in (-1, +1).$$

The last lemma is a refined version of a theorem of Levin. Its proof can be found in [**5**].

LEMMA 4. *Let*

$$\tau_n(z) = \sum_{-n}^{n} d_\mu e^{i\mu z},$$

with $d_n \neq 0$, *be a trigonometric polynomial having all its zeros in* $\operatorname{Im} z \geqq 0$. *If*

$$S_n(z) = \sum_{-n}^{n} c_\mu e^{i\mu z}$$

is a trigonometric polynomial of degree n such that $|S_n(\theta)| \leqq |(\tau_n(\theta)|$ *for all* $\theta \in \mathbf{R}$ *and if a, b, c are real numbers satisfying*

$$nb^2 + n(2n - 1)a^2 - 2ac(2n - 1) \geqq 0,$$

then

$$|a\, S_n''(\theta) + bS_n'(\theta) + cS_n(\theta)| \leqq |a\tau_n''(\theta) + b\tau_n'(\theta) + c\tau_n(\theta)|.$$

4. Proofs of the theorems.

Proof of Theorem 1. We first consider case a). If $p \in \pi_n$ and satisfies the hypotheses, then, according to Theorem A,

$$|p(x)| \leqq |T_{1,0,n}(x)| \quad \text{for } x \notin [-1, +1].$$

But

$$|T_{1,0,n}(x)| \leqq 1 \quad \text{for } x \in \left[1, \sec^2\frac{\pi}{4n} + tg^2\frac{\pi}{4n}\right],$$

thus, the same is true for $p(x)$. Let us put

$$\hat{p}(y) = p\left(\sec^2\frac{\pi}{4n}y + tg^2\frac{\pi}{4n}\right).$$

According to the last remark, $\|\hat{p}(y)\|_\infty \leqq 1$, whence applying Theorem B,

$$|\hat{p}^{(k)}(y)| \leqq |T_n^{(k)}(y) + i\, S_n^{(k)}(y)|$$

for

$$y \in (-1, +1) \quad \text{and} \quad \|\hat{p}^{(k)}(y)\|_\infty \leqq |T_n^{(k)}(-1)|.$$

Going back to x via the change of variable

$$y = \left(\cos^2 \frac{\pi}{4n}\right)x - \sin^2 \frac{\pi}{4n},$$

we obtain the desired inequalities.

The proof of case b) goes along the same lines. Using Theorem A and the change of variable

$$\hat{p}(y) = p\left(y/\cos \frac{\pi}{2n}\right),$$

we will again obtain

$$|\hat{p}^{(k)}(y)| \leqq |T_n^{(k)}(y) + iS_n^{(k)}(y)| \quad \text{for } y \in (-1, +1).$$

Clearly the right-hand side of this last inequality is even. As remarked previously it is also increasing on (0, 1), thus, if we denote by a_k the right-most zero of $S_n^{(k)}(y)$ in $(-1, +1)$, we get

(27) $\quad |\hat{p}^{(k)}(y)| \leqq |T_n^{(k)}(a_k)| \quad \text{for } y \in (-a_k, a_k).$

Now, using the fact that $S_n^{(k)}(x)$ and $T_n^{(k)}(x)$ are linearly independent solutions of a differential equation of the form (17) and the fact the rightmost zero of $S_n'(x)$ is equal to $\cos(\pi/2n)$, we may argue as in the first part of the proof of Lemma 2 to deduce that

$$a_k \leqq \cos(\pi/2n)$$

and that $|T_n^{(k)}(y)|$ is increasing on $[a_k, 1]$. Thus, using (5), we have

(28) $\quad |\hat{p}^{(k)}(y)| \leqq |T_n^{(k)}(y)| \leqq |T_n^{(k)}(\cos \pi/2n)| \text{ if } y \in (a_k, \cos(\pi/2n)).$

Combining (27) and (28) and setting $y = \cos(\pi/2n)x$, we obtain the second inequality in b).

Proof of Theorem 2. We first consider the case $k = 1$. Let the function $f(\theta)$ be defined, for $\theta \in \mathbf{R}$, by

$$f(\theta) = \begin{cases} \cos 2\theta & \text{if } (\lambda, \mu) = (1, 0) \\ \cos \theta & \text{if } (\lambda, \mu) = (1, 1) \\ \left(\cos \dfrac{\pi}{4(n+1)}\right)^{-2}\left(\cos \theta + \sin^2 \dfrac{\pi}{4(n+1)}\right) & \text{if } (\lambda, \mu) = (2, 0) \\ \left(\cos \dfrac{\pi}{2(n+2)}\right)^{-1} \cos \theta & \text{if } (\lambda, \mu) = (2, 2) \end{cases}$$

and let

$$g(\theta) = (1 - f(\theta))^{\lambda}(1 + f(\theta))^{\mu}.$$

If $p \in \pi_n$, the function $g(\theta)p(f(\theta))$ will be a trigonometric polynomial of order $m = 2n + 1$ if $(\lambda, \mu) = (1, 0)$, of order $m = (n + 1)$ if $(\lambda, \mu) = (1, 1)$

or (2, 0) and of order $m = n + 2$ if $(\lambda, \mu) = (2, 2)$. Moreover, using table 1, we can easily check that, in those cases,

(29) $g(\theta)(t_{\lambda,\mu,n}(f(\theta)) + is_{\lambda,\mu,n}(f(\theta))) = e^{im\theta}$ for all $\theta \in \mathbf{R}$.

Now, using Theorem A as in the proof of Theorem 1, we see that the inequality

$$|(1 - x)^{\lambda}(1 + x)^{\mu}p(x)| \leqq 1$$

is valid on the interval $[-1, +1]$ if $(\lambda, \mu) = (1, 0)$ or $(1, 1)$; it is valid on the interval

$$\left[-1, \left(\cos \frac{\pi}{2(n + 1)}\right)^{-2}\left(1 + \sin^2 \frac{\pi}{2(n + 1)}\right)\right]$$

if $(\lambda, \mu) = (2, 0)$ and on the interval

$$\left[-\left(\cos \frac{\pi}{2(n + 2)}\right)^{-1}, \left(\cos \frac{\pi}{2(n + 2)}\right)^{-1}\right]$$

if $(\lambda, \mu) = (2, 2)$. This, together with (29), implies that, for $p \in \pi_{\lambda,\mu,n}$ the inequality

$$|g(\theta)p(f(\theta))| \leqq |g(\theta)(t_{\lambda,\mu,n}(f(\theta)) + is_{\lambda,\mu,n}(f(\theta)))|$$

is valid for all θ. On the other hand, using (29) again we see that the hypotheses of Lemma 4 are satisfied for any a and b if $c = 0$. Let $x = f(\theta_0)$ be fixed. Applying Lemma 4 with $a = -g(\theta_0)$, $b = g'(\theta_0)$ and $c = 0$, we get, for $\theta = \theta_0$,

$$|g^2(\theta_0)f'(\theta_0)p'(f(\theta_0))| \leqq |g^2(\theta_0)f'(\theta_0)(t'_{\lambda,\mu,n}(f(\theta_0)) + is_{\lambda,\mu,n}(f(\theta_0)))|.$$

Since

$$g^2(\theta_0)f'(\theta_0) \neq 0 \quad \text{for } x \in (-1, +1),$$

this inequality is equivalent to (12) when $k = 1$.

To proceed to the general case, we verify that Lemma 2 applies here. Indeed, as already noticed, the functions $t_{\lambda,\mu,n}(x)$ and $s_{\lambda,\mu,n}(x)$ are linearly independent solutions of a differential equation of the form (20). The verification of conditions (i) to (iii) can readily be made with the help of table 1 whereas Lemma 3 states that condition (iv) is also fulfilled. Finally we just proved that (21) is true for any $p \in \pi_{\lambda,\mu,n}$.

Let us now suppose that $k > 1$ is fixed. If, for any real α we set

$$R_{\alpha}(x) = \cos \alpha t_{\lambda,\mu,n}(x) + \sin \alpha s_{\lambda,\mu,n}(x),$$

we have

(30) $|R_\alpha^{(k)}(x)| \leqq |t_{\lambda,\mu,n}^{(k)}(x) + is_{\lambda,\mu,n}^{(k)}(x)|$ for $x \in (-1, +1)$.

Suppose that, for a given $p \in \pi_{\lambda,\mu,n}$ and a fixed $x_0 \in (-1, +1)$, the inequality

$$|p^{(k)}(x_0)| > |t_{\lambda,\mu,n}^{(k)}(x_0) + is_{\lambda,\mu,n}^{(k)}(x_0)|$$

is true. In view of (30) we can choose $\gamma \in [0, 1)$ such that

(31) $R_\alpha^{(k)}(x_0) + \gamma p^{(k)}(x_0) = 0.$

We now choose α such that x_0 is a local maximum of

$$|R_\alpha^{(k)}(x)/p^{(k)}(x)|$$

i.e., such that

(32) $R_\alpha^{(k+1)}(x_0)p^{(k)}(x_0) - R_\alpha^{(k)}(x_0)p^{(k+1)}(x_0) = 0.$

This is always possible since the last equation is of the form

$$a \cos \alpha + b \sin \alpha = 0.$$

Using (31), (32) then becomes

$$p^{(k)}(x_0)\,[R_\alpha^{(k+1)}(x_0) + \gamma p^{(k+1)}(x_0)] = 0$$

which implies that x_0 is a double zero of $R_\alpha^{(k)}(x) + \gamma p^{(k)}(x)$. Since $\gamma p \in \pi_{n,\lambda,\mu}$, this contradicts the conclusion of Lemma 2, hence (12) is satisfied for any k.

Proof of Theorem 3. The functions $t_{1,1,n}$ and $s_{1,1,n}$ satisfy the conditions (i) and (ii) of Lemma 2 as well as the equation

(33) $(1 - x^2)y'' - 3xy' + n(n + 2)y = 0.$

Differentiating both sides of (33), k times with respect to x, we get

(34) $$(1 - x^2)y^{(k+2)} - (2k + 3)xy^{(k+1)} + (n - k)(n + k + 2)y^{(k)} = 0.$$

In view of Rolle's Theorem, $t_{1,1,n}^{(k)}(x)$ has exactly $(n - k)$ roots in $(-1, +1)$ while $s_{1,1,n}^{(k)}(x)$ has at least $(n - k + 1)$ roots there. Using the Sturm separation Theorem, we see that the latter has, in fact, precisely $(n - k + 1)$ roots in $(-1, +1)$ which separate those of $t_{1,1,n}^{(k)}$. Let us denote them by $a_1, \dots, a_{n-k+1}$.

Let $p \in \pi_{1,1,n}$, using (12) we see that

$$|p^{(k)}(a_i)| \leqq |t_{1,1,n}^{(k)}(a_i)| \quad \text{for } i = 1, \dots, n - k + 1$$

whereas

$$\operatorname{sgn}(t_{1,1,n}^{(k)}(a_i)) = -\operatorname{sgn}(t_{1,1,n}^{(k)}(a_{i+1})), \quad i = 1, \dots, n - k.$$

Hence, according to Theorem 1 of [6],

(35) $|p^{(k)}(x)| \leqq |t^{(k)}_{1,1,n}(x)|$ for $x \notin [a_1, a_{n-k+1}]$.

Set

$$N_k(x) = (t^{(k)}_{1,1,n}(x))^2 + (s^{(k)}_{1,1,n}(x))^2 = \sum_0^{\infty} \alpha_{k,j} x^{2j},$$

where the development is valid in $(-1, +1)$. We use induction to show that, for each fixed k, $\alpha_{k,j} \geqq 0$ for each j.

Using (2), (3) and table 1 we get

$$N_1(x) = \frac{(n+1)^2}{(1-x^2)^2} + \frac{x^2}{(1-x^2)^3}$$

which implies that $\alpha_{1,j} \geqq 0$ for $j \geqq 0$. Let us suppose that the same is true for a fixed $k > 1$. Going back to (34), we obtain the following relation

$$\frac{d}{dx}\{N_{k+1}(x)(1-x^2) + (n-k)(n+k+2)N_k(x)\}$$
$$= 4(k+1)xN_{k+1}(x)$$

from which we deduce that, for each positive j,

$$2(j+1)[\alpha_{k+1,j+1} + (n-k)(n+k+2)\alpha_{k,j+1}]$$
$$= [4(k+1) + 2(j+1)]\alpha_{k+1,j}.$$

Thus, as soon as $\alpha_{k+1,j} > 0$ for some j, the same is true for $\alpha_{k+1,i}$ if $i \leqq j$. On the other hand

$$\lim_{x \to 1} N_{k+1}(x) = +\infty$$

and $N_{k+1}(x)$ is always positive. This implies that there exist arbitrarily large values of j for which $\alpha_{k+1,j} > 0$; whence this is true for all j. This completes the induction.

Now, if $x \in [a_1, a_{n-k+1}]$, we infer from (12) and the last paragraph, that the inequality

(36) $|p^{(k)}(x)|^2 \leqq N_k(x) \leqq N_k(a_1) = N_k(a_{n-k+1})$,

is valid. Since

$$N_k(a_i) = (t^{(k)}_{1,1,n}(a_i))^2,$$

(36) together with (35), lead to the estimate

$$|p^{(k)}(x)| \leqq \|t^{(k)}_{1,1,n}(x)\|_\infty \quad \text{for } x \in [-1, +1],$$

which is the desired result.

Proof of Theorem 4. The functions $t_{\lambda,\mu,n}(x)$ and $s_{\lambda,\mu,n}(x)$ satisfy the hypotheses of Lemma 2. If we repeat the reasoning of the first paragraph of the proof of that lemma, we will obtain that $s'_{\lambda,\mu,n}(x)$ has n roots in $(-1, +1)$ which separate those of $t'_{\lambda,\mu,n}(x)$. Let us denote them by $a_1, \dots, a_n$.

Let $p \in \pi_{\lambda,\mu,n}$, using (12) and Theorem 1 of **[6]** as in the proof of Theorem 3, we get

$$(37) \quad |p'(x)| \leqq |t'_{\lambda,\mu,n}(x)| \quad \text{for } x \notin [a_1, a_n].$$

Set

$$N(x) = (t'_{\lambda,\mu,n}(x))^2 + (s'_{\lambda,\mu,n}(x))^2.$$

In view of (12) and (37), it is enough, to complete the proof, to show that

$$\max_{[a_1,a_n]} N(x) = \max\{N(a_1), N(a_n)\}.$$

We consider the three cases separately.

a) $(\lambda, \mu) = (2, 2)$. Using table 1, we get

$$N(x) = \frac{1}{(1 - x^2)^2}\left\{\frac{(n + 2)^2\alpha^2}{(1 - \alpha^2x^2)} + \frac{4x^2}{(1 - x^2)^2}\right\}$$

where $\alpha = \cos(\pi/2(n + 2))$. That function is clearly even and increasing on $(0, 1)$, hence

$$\max_{[a_1,a_n]} N(x) = N(a_n) = N(a_1).$$

b) $(\lambda, \mu) = (1, 0)$. The function N can be expressed as

$$N(x) = \frac{1}{4(1 - x)^2}\left[\frac{(2n + 1)^2}{(1 + x)} + \frac{1}{(1 - x)}\right].$$

If we differentiate, we will be lead to $N'(x) = R(x)/S(x)$, where $S(x) > 0$ for $x \in (-1, +1)$ while $R(x)$ is a polynomial of degree 2. This shows that $N(x)$ has at most two local extrema in $(-1, +1)$. But $N(x)$ is always positive and

$$\lim_{|x|\to 1} N(x) = +\infty,$$

hence the only possibility is that $N(x)$ has one local minimum. It follows from there that

$$\max_{[a_1,a_n]} N(x) = \max\{N(a_1), N(a_n)\}.$$

c) $(\lambda, \mu) = (2, 0)$. Setting

$$A = \sec^2 \frac{\pi}{2(n+1)} + tg^2 \frac{\pi}{2(n+1)},$$

the function $N(x)$ can be expressed as

$$N(x) = \frac{1}{(1-x)^4} + \frac{(n+1)^2}{(1+x)(1-x)^2(A-x)}.$$

Differentiating we get

$$N'(x) = \frac{4(1+x)^2(A-x)^2 - (n+1)^2(1-x)^2}{(1-x)^5(1+x)^2} \times \frac{[4x^2 - (3A-1)x - (A+1)]}{(A-x)^2}.$$

Using the reasoning of part b) we see that the proof will be complete if we show that $N'(x)$ has exactly one root in $(-1, +1)$. For this set

$$f_1(x) = 4(1+x)^2(A-x)^2,$$

$$f_2(x) = (n+1)^2(1-x)^2(4x^2 - (3A-1)x - (A+1)).$$

Using the relations $f_2(-1) = 0, f_1'(-1) = 0, f_1(A) = 0, f_1'(A) = 0$ and the relations

$$f_2(-1) = 8(n+1)^2(A+1), f_2(0) = -(n+1)^2(A+1),$$

$$f_2(+1) = 0, f_2'(+1) = 0,$$

$$f_2(A) = (n+1)^2(1-A)^2(A^2+1),$$

we can, by properly locating the zeros of $f_1(x), f_1'(x), f_2(x), f_2'(x)$, plot the graphs of these functions as below:

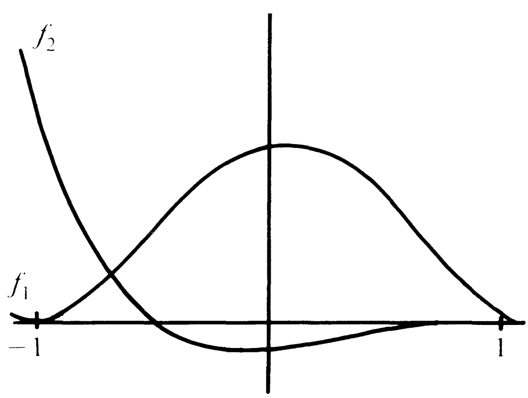

Figure 2

https://doi.org/10.4153/CJM-1986-044-1 Published online by Cambridge University Press

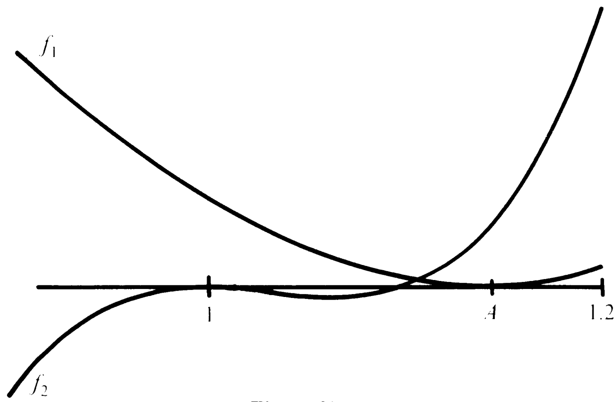

Figure 2′

This shows that the equation $N'(x) = 0$ has exactly one root in $(-1, +1)$.

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