OBITUARY

GORDON MARSHALL PETERSEN

Gordon Marshall Petersen, a member of the London Mathematical Society from 1958, was born on 25 November 1921 in San Francisco, and died on 9 November 1996 in Christchurch, New Zealand.

His father was Danish, born in 1883 in Esbønderup, a village on the island of Zealand. While still only 15, he suddenly announced to his family that, having secretly saved up his fare, he was emigrating to America the next day. He had little formal education, but was able to support his family comfortably. One day during the Depression, father and son, in their shabbiest clothes, visited a Cadillac showroom and were about to be ejected when the father produced a paper bag stuffed with enough banknotes to buy the car they had been inspecting. No wonder Gordon, an only child, adored his father!

Gordon's mother Catherine was born in 1883 in America to Richard and Ann Marshall, who had not long before emigrated from Nottingham in England. She had three older siblings and also a younger brother, Gordon, who operated several garages and built his own aeroplanes. He survived the First World War in the US Army Air Force, but was killed in an air crash in 1920, at the age of 32.

After obtaining his BA degree from Stanford University in 1943, Gordon taught for a while at a boys' boarding school in Deep Springs, a remote Californian settlement not far from Death Valley; he also worked at Moffett Field, a Government aeronautical establishment in the San Francisco area. He subsequently returned to Stanford, taking his MA degree in 1947, and from then until 1949 he taught at the University of British Columbia. His next move was to the University of Toronto, to work for a PhD under the supervision of G. G. Lorentz, and this he achieved within two years, in 1951. There followed short-term posts at the Universities of Manitoba (1951/2), Arizona (1952/3) and Oklahoma (1953/5). Next, Gordon secured a two-year Senior Research Fellowship at the University College of Swansea (1955/7), which he—and the College—found enormously stimulating. On its expiry, Gordon returned to America as an Associate Professor at the University of New Mexico. However, two years later, in 1959, a Lectureship fell vacant at Swansea and he was appointed, having eagerly applied in spite of the considerable financial sacrifice that success entailed. Gordon was promoted to Senior Lecturer in 1961 and to Reader in 1964. He was awarded the degree of DSc (Wales) in 1963. His last move came in 1965, when he was elected to the Chair of Pure Mathematics in the University of Canterbury in Christchurch, New Zealand. In 1967 he became head of the Department of Mathematics, and in 1973 he was elected a Fellow of the Royal Society of New Zealand. Sadly, after a stroke, he had to take early retirement in 1984. Further strokes, from 1990 onwards, necessitated continuous nursing care. The devoted attention accorded to him in the St John of God Hospital in Christchurch, and his powerful constitution, enabled him to survive for six more years, the last few of which were distressing for himself and his friends.

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Gordon was well over six feet tall and unusually broad. His exceptional bulk went with a great booming voice, which gave him a formidable presence. He himself recorded with amusement an incident when he asked a widow and her young son to tea. In response to their knock, Gordon flung open the door and roared a welcome. Thereupon the boy, who had wondered what the initials G. M. stood for, asked his mother 'do you think it's Great Monster Petersen?'. Gordon could, in fact, be intimidating, but for every person who found him so, there were many who enjoyed his stimulating company. A friend from his Swansea days called him a gentle giant. Of course, a giant has a commensurate appetite, and anyone who entertained Gordon came up against this problem. On the other hand, a satisfying meal produced a degree of appreciation that was ample reward. But Gordon also greatly enjoyed the role of host, in or out of doors, and in Swansea the picnics he organised are still fondly remembered. In addition, he was always happy to show the local sights to visitors or to new members of the Department and their families, but the pleasure he thus gave was not always unalloyed: though his driving record was actually admirable, this was sometimes hard to believe.

While still living in America, Gordon began to explore his roots in England and Denmark. He grew close to some of his English relatives, especially the younger generation, and on a number of occasions he took a party of the latter on long cycle trips in England and Denmark. They stayed mostly in youth hostels, but fed lavishly. However, what is mainly remembered by at least some of the then teenagers is Gordon's enthusiasm for all they saw and his ability to pass it on to his young companions. Years later, the sons of one of these relatives were also taken on trips by Gordon, but now by car or canal boat. Gordon always had a special affinity for the land of his father's birth, and he called the diary he kept of some of his wanderings there 'Three trips *home* to Denmark'. He even left instructions for his ashes to be interred in the family plot in Esbønderup. Another country that he particularly enjoyed visiting was China; in fact, travel was one of his great pleasures.

A characteristic that made Gordon such an interesting friend was his erudition. His knowledge of history may well have led to his delight in coin collecting; but he also avidly collected objects as diverse as chess sets, teapots, enamelled coins and oriental rugs. However, unlike many collectors, he freely made presents of items from his collections. Major gifts were his entire collection of coins, with the heads of all the Roman emperors, to the University of Canterbury, and another collection of Parthian and Roman Imperial coins to the University College of Swansea. Particulars of his generosity in financial affairs were, naturally, kept confidential.

Gordon had a great interest in the arts. In Swansea he painted abstract pictures and did some sculpting, but lack of time must have precluded these activities in Christchurch. On the other hand, as something of a real-life showman, he always made time for acting, which he loved. He also much enjoyed opera.

On taking up his Chair in Christchurch, Gordon gave first priority to the promotion of research. Overseas visitors provided an extra stimulus, and everything was done to encourage the Department's MSc and PhD programmes. On the wider national stage, Gordon was one of the prime agents in the establishment of the New Zealand Mathematical Colloquium, which first met a year after his arrival. Later, Gordon was the instigator and chief organiser of the first ever Australasian Mathematics Convention, appropriately held in the University of Canterbury in 1978.

Gordon took his Departmental and Faculty duties very seriously, and he enjoyed ceremonial occasions, resplendent in his scarlet DSc robes. That he cared about advanced teaching was very evident, but the book [**B1**] shows that he was also interested in the presentation of elementary material. During his first sojourn in Swansea (1955/7), he and Hazel Perfect, then a lecturer in the department, both noted the dearth of introductions in English to abstract algebra, and independently conceived the notion of translating the German edition of Alexandroff's classic text. In the event, they collaborated in the enterprise, providing also about 60 exercises. The fact that the book had seven printings proves that it very much filled a need. Incidentally, success did not bring riches: there were no royalties, only a modest once-for-all fee.

Pólya in Stanford, Lorentz in Toronto and Goffman in Oklahoma were regarded by Gordon as the principal moulders of his mathematical personality. He, in turn, must have left a lasting impression on his students and on his many collaborators who, though often not primarily interested in his field, were seduced by his bubbling enthusiasm. He was a powerful classical analyst whose *forte* was the solution of hard problems rather than the creation of new theories. In this context, his special gift for constructing examples to illustrate, prove and disprove propositions was particularly useful. Although his reputation to a great extent derives from his work in general summability theory, he also published extensively on a variety of other topics. An indication of the subjects tackled by him follows.

General analysis

A. *Almost convergence* [**26**, **30**]. The sequence (s_k) is said to be *almost convergent* to s if $p^{-1}\sum_{k=n+1}^{n+p} s_k \to s$ as $p \to \infty$, uniformly for $n \in \{1, 2, 3, \ldots\}$. The concept, which implies $(C, 1)$ limitability to the same value, was introduced by Lorentz in ⟨**9**⟩.

The principal result of [**26**] is that if the real function F is of bounded variation on $[0, 2\pi]$ and $c_n = (2\pi)^{-1}\int_0^{2\pi} e^{in\theta}dF(\theta)$ $(n = 0, 1, 2, \ldots)$, then $(|c_n|^2)$ is almost convergent to $(4\pi)^{-2}\sum \mu_m^2$, where the μ_m are the jumps of F at its discontinuities. This leads to an improvement of a theorem of Wiener ⟨**14**⟩ which states that F is continuous if and only if $(|c_n^2|)$ is $(C, 1)$ limitable to 0: for F to be continuous, it is necessary that $(|c_n^2|)$ is almost convergent to 0, and it is sufficient that $(|c_n^2|)$ is limitable to 0 by some method containing almost convergence.

B. *Uniformly and well-distributed sequences* [**12**, **23**, **38**, **39**, **45**, **52**, **56**, **57**, **60**, **73**, **76**]. For any interval $[a, b] \subseteq [0, 1]$, let $I_{[a,b]}(x)$ be the characteristic function of $[a, b]$. Then (see Weyl ⟨**13**⟩) the sequence (s_k) in $[0, 1]$ is *uniformly distributed* if for every $[a, b] \subseteq [0, 1]$, $p^{-1}\sum_{k=1}^{p} I_{[a,b]}(s_k) \to b-a$ as $p \to \infty$, that is, if $(I_{[a,b]}(s_k))$ is $(C, 1)$ limitable to $b-a$. In [**12**] Gordon strengthens this property: (s_k) in $[0, 1]$ is called *well-distributed* if for every $[a, b] \subseteq [0, 1]$, $(I_{[a,b]}(s_k))$ is almost convergent to $b-a$. Several well-known results on uniformly distributed sequences are shown to have analogues for well-distributed sequences. When α is irrational, $(\{k\alpha\})$ is well-distributed (where $\{\theta\}$ denotes the fractional part of θ). In addition, it is proved in [**23**] that to every irrational α, there corresponds a sequence $(n(k))$ of integers such that $n(k)/n(k-1) > \lambda > 1$ and $(\{n(k)\,\alpha\})$ is well-distributed.

In the next six papers of this group, the aim is largely to obtain results of a type opposite to the last one. The principal theorem of [**38**] is that if the sequence of integers $n(k)$ is such that $n(k)/n(k-1)$ increases to ∞, then, for almost all $\alpha \in (0, 1]$,

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($\{n(k)\,\alpha\}$) is not well-distributed. The condition on $n(k)$ is gradually relaxed until, in **[57]**, $(n(k))$ is required only to be a sequence of real numbers such that $n(k)/n(k-1) > \lambda > 1$.

If (s_n) is well (uniformly) distributed and (t_n) converges, then (s_n+t_n) is well (uniformly) distributed. These facts motivate the following definitions in **[60]**: (t_n) is called *admissible* (*u-admissible*) if, whenever (s_n) is well (uniformly) distributed, then so is (s_n+t_n). Among the results of this paper, two stand out. Theorem 5: if $|t_n-t_{n-1}| \leqslant \frac{1}{2}$ for all n, then (t_n) is admissible if and only if $(|t_n-t_{n-1}|)$ almost converges to 0. Theorem 6: if $|t_n-t_{n-1}| \leqslant \frac{1}{2}$ and $t_n = u_n+v_n$, where $|u_n-u_{n-1}| = O(1/n)$ and $(|v_n-v_{n-1}|)$ is $(C,1)$ limitable to 0, then (t_n) is u-admissible.

C. *An inequality of Hardy* **[36, 42, 62]**. The Hardy inequality which inspired these papers is

$$\sum_{n=1}^{\infty}\left(\frac{A_n}{n}\right)^p < \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p,$$

where $p > 1$, $A_n = a_1+\cdots+a_n$, and $a_n \geqslant 0$ for all n, but a_n is not 0 for all n. Moreover, the constant is best possible (see ⟨**7**, page 239⟩).

In **[36]** the arithmetic means are replaced by the means derived from Jurkat and Peyerimhoff's M-matrices $(a_{m\nu})$ (see ⟨**8**⟩) which are positive, triangular and satisfy some additional conditions. Also, $f(m)$ is a certain positive increasing function, and $u_\nu \geqslant 0$ for $\nu \geqslant 1$. Then

$$\sum_{m=1}^{\infty}\left(\sum_{\nu=1}^{m} a_{m\nu}u_\nu\right)^p \leqslant C\sum_{m=1}^{\infty}(u_m f(m)\,a_{mm})^p,$$

where $p = 2, 3, \ldots$ and C is a constant. The result is improved in **[42]**: p is now any real number > 1, while $(a_{m\nu})$ and f are less restricted than before. There is also an inequality in the opposite direction when $0 < p < 1$.

D. *Infinite linear systems* **[40, 41, 43, 44, 46]**. The (real) systems considered are of the form

$$\sum_{j=1}^{\infty} a_{ij}u_j = b_i \quad (i = 1, 2, \ldots), \tag{$*$}$$

where (a_{ij}) is a given matrix and (b_i) is an arbitrary sequence. Pólya ⟨**12**⟩ has shown that if (i) $a_{1j} \neq 0$ for infinitely many j, and (ii) $\liminf_{j\to\infty} A_{ij} = 0$ for every i, where $A_{ij} = (\sum_{k=1}^{i-1}|a_{kj}|)/|a_{ij}|$, then, for each sequence (b_i), $(*)$ has a solution (u_j) for which the left-hand side of $(*)$ converges absolutely.

In **[40]** a theorem cognate to Pólya's is proved: (i) is retained, while (ii) is replaced by more elaborate conditions on (a_{ij}), and also the left-hand side of $(*)$ need not converge absolutely. There is an example of a system to which the new theorem applies, but not Pólya's, and in which all left-hand sides converge conditionally.

[41] investigates the extent to which Pólya's sufficient conditions are also necessary. It is shown that if (i)′ $a_{1j} \neq 0$ for $j = 1, 2, \ldots$, and the conclusion of Pólya's theorem holds, then (ii)′ $\liminf_{j\to\infty} A_{ij} = 0$ for infinitely many i. This motivates the variant of Pólya's theorem proved in **[43]**: the conclusion of the theorem holds if (i)′, (ii)′ and three further conditions are satisfied.

E. *Metric density and Lusin's theorem* [**63**, **68**, **72**]. The basic Lusin–Menchoff theorem for Euclidean space E_n is that if E is a Borel set, X is a closed subset of E and $d_o(x, E)$, the ordinary metric density of E at x, equals 1 for all $x \in X$, then there exists a perfect set P with $X \subseteq P \subseteq E$ and $d_o(x, P) = 1$ for all $x \in X$. There is a proof in ⟨**5**⟩, where it is also shown that when $n \geqslant 2$, the metric density d_o may not be replaced by the strong kind d_s. On the other hand, in [**63**] it is proved that an appropriate concept is *uniformly regular* metric density $d(\lambda, x, E)$. Here λ is a family of rectangles such that for each $x \in E_n$, there is a sequence of members of λ which converges to x and is uniformly thick (in a precise sense). An example shows that the thickness condition must not be infringed even at a single point, and another, in [**68**], that thickness at each point, but without uniformity, is also inadequate.

F. *Miscellaneous topics* (*i*) [**1**, **13**, **29**]. Hardy ⟨**6**⟩ proved that if $f \in L^p$ $(p \geqslant 1)$ and $f(x) \sim \sum a_n \cos nx$, then there is a function in L_p with Fourier constants $A_n = n^{-1} \sum_{k=1}^{n} a_k$. It is shown in [**1**] that Hardy's L^p may be replaced by other Banach spaces, for example, by (see ⟨**10**⟩) $\Lambda(\alpha)$ $(0 < \alpha < 1)$, the space of measurable f on $(0, 1)$ with norm $\alpha \int_0^1 x^{\alpha-1} f^*(x)\, dx$, where f^* is the decreasing rearrangement of f. Also, other means can replace A_n.

In [**13**] the Riesz product $\prod_{n=1}^{\infty}(1+\cos 4^n \theta)$ is expanded as the formal series $1+\sum_{n=1}^{\infty} \alpha_n \cos p_n \theta$ with a set procedure for evaluating each α_n and p_n.

A Hardy–Littlewood theorem (for example, ⟨**15**, page 63⟩) is that the Fourier series of an integrable function $f \sim (a_n, b_n)$ converges at x to $f(x)$ if (i) $|f(x+h)-f(x)| = o((\log |h|^{-1})^{-1})$ and (ii) $a_n, b_n = O(n^{-\delta})$ for some $\delta > 0$. However, (i) alone is compatible with divergence of the Fourier series ⟨**15**, page 303⟩. Nevertheless, according to [**29**], (i) implies summability to $f(x)$ by triangular matrices of a certain kind.

Summability

G. *Special summation methods* [**2**, **3**, **32**, **74**]. The series $\sum_{\nu=0}^{\infty} u_\nu$ is said to be summable to s by Bernstein's method (B^h) if $\sum_{\nu=0}^{n} u_\nu \cos \frac{1}{2}\pi \left(\frac{\nu}{n+h}\right) \to s$ as $n \to \infty$. The first parts of both [**2**] and [**3**] deal with this method. It was known that $(B^h) \supseteq (C, 1)$ for all h, and the challenge was to make this relation more precise. In [**2**] it is shown that $(B^h) \equiv (C, 1)$ for $h > \frac{1}{2}$, and in [**3**] that $(B^h) \supset (C, 1)$ for $h < 0$. These results and others in ⟨**1**⟩ combine comprehensively: for $h = 0$ and $h > \frac{1}{2}$, $(B^h) \equiv (C, 1)$, and for all other h, $(B^h) \supset (C, 1)$.

Let $k > 0$ and $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$. In the study of trigonometric series, (i) Riesz's typical means and (ii) Riemann's method occur in a natural way. $\sum u_n$ is (i) summable (R, λ_n, k) to s if $\sum_{\lambda_n < \omega}(1-\lambda_n/\omega)^k u_n \to s$ as $\omega \to \infty$, and (ii) summable (R, k, λ_n) to s if $u_0 + \sum_{n=1}^{\infty}(\sin \lambda_n h/\lambda_n h)^k u_n \to s$ as $h \to 0$. It is proved in [**32**] that if $0 < p \leqslant \lambda_{n+1} - \lambda_n \leqslant q$ $(n = 1, 2, \ldots)$, then, when $k > 2$, $(R, k, \lambda_n) \supseteq (R, \lambda_n, 1)$.

Denote by E the set of sequences $s = (s_n)$ such that $s_n - s_{n-1} = O(1/n)$. According to a famous theorem of Littlewood, if s is Abel limitable and also $s \in E$, then s converges. Gordon shows in [**74**] that E may be replaced by $\bar{E}$, the closure of E in the topology generated by $\|s\| = \sup_n |s_n|$.

H. *Sets of subsequences* [**28**, **33**, **35**]. Each subsequence s^* of a given sequence s is formed by the deletion from s of all terms s_k with k in a certain set $K \subseteq \{1, 2, \ldots\}$;

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and s^* is paired with $x \in (0, 1]$ if the non-terminating binary expansion of x has 0 in the kth place when $k \in K$ and 1 when $k \notin K$. The resulting bijection between $(0, 1]$ and the set of all subsequences of s enables one to talk of 'almost all subsequences of s' or 'a set of subsequences of s of measure $\frac{1}{2}$', and so on.

[**28**] exhibits an intriguing pair (s_n), (t_n) of sequences; both are almost convergent (see Subsection A) though not convergent, while almost all subsequences of (s_n) are almost convergent, and almost no subsequences of (t_n) are almost convergent.

Let (p_n) be a real sequence, and put $P_m = p_1 + \cdots + p_m$. Then (s_n) is limited by the Riesz method (R, p_n) to s if $(p_1 s_1 + \cdots + p_m s_m)/P_m \to s$ as $m \to \infty$. An interesting result of [**33**] is that if (R, p_n) is regular (which means $P_m \to \infty$ as $m \to \infty$) and satisfies some additional conditions, then almost all subsequences of a bounded (R, p_n) limitable sequence are (R, p_n) limitable to the same value.

In [**35**] there is a theorem of opposite type. Conditions on a matrix A are found which ensure that if almost all subsequences of a sequence (s_n) are A-limited to s, then (s_n) itself is A-limited to s.

I. *Tauberian theorems* [**21**, **22**, **34**, **71**(ii), **77**]. R. C. Buck proved in ⟨**4**⟩ that a real sequence (s_n) converges if there exists a regular matrix limitation method which limits every subsequence of (s_n). In [**21**] it is shown that when the subsequences of (s_n) and the real numbers in $(0, 1]$ are paired as in Subsection H, this theorem can be strengthened by the replacement of 'every subsequence' by 'a set of subsequences of the second category'.

The main result of [**34**] is that $s_n - s_{n-1} = O(1/n)$ is a Tauberian condition for a class of matrices related to the M-matrices featuring in Subsection C; and in [**71**(ii)] a general technique involving M-matrices leads fairly rapidly to known Tauberian conditions for Riesz and Nørlund means.

Let $f(n)$ decrease and tend to 0 as $n \to \infty$, and let E be the set of sequences $s = (s_n)$ such that $s_n - s_{n-1} = O(f(n))$. The most attractive result in the interesting paper [**77**], Gordon's last, is that if A is a member of a wide class of limitation methods, and E is a Tauberian set for A, then so is the closure $\bar{E}$ of E generated by the norm $\|s\| = \sup_n |s_n|$ (compare Subsection G, [**74**]).

J. *Limitation of bounded and of unbounded sequences* [**7**, **8**, **10**, **11**, **15**, **16**, **24**, **25**, **27**, **37**]. Let A, B be limitation matrices. If the comparison $A \supseteq B$ refers to (a) all sequences, (b) all bounded sequences, then A is called (a) *a-stronger*, (b) *b-stronger* than B. If each of A, B is a-stronger than the other, then A, B are called *a-equivalent*; *b-equivalence* is similarly defined. Finally, A, B are said to be (a) *a-consistent*, (b) *b-consistent* if they limit to the same value (a) every sequence limited by both, (b) every bounded sequence limited by both.

[**25**] consists of concise proofs of four fundamental theorems of Brudno in ⟨**2**⟩. The earlier papers [**8**, **10**] contain slightly longer proofs of the first two. Let A, B be regular matrices: (i) if A is b-stronger than B, then A, B are b-consistent; (ii) if A is strictly b-stronger than B, written $A \supset B$, then there is a regular matrix C such that $A \supset C \supset B$. A generalisation of (i) is proved in [**27**].

In [**7**] there is a simple example of a set $\mathfrak{A}$ of regular matrices, b-consistent in pairs, such that every bounded sequence is limited by at least one member of $\mathfrak{A}$. Here $\mathfrak{A}$ is the set of all regular non-negative matrices with $\frac{1}{2}$ in each row and column.

One of the differences between the situations when unbounded sequences are and

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are not admitted is shown up in [**24**]: there are pairs of regular matrices which are a-equivalent, but not a-consistent (though they are necessarily b-consistent).

K. *Iterations and norms* [**9**, **17–20**, **31**, **49**]. The following notation and nomenclature relating to regular limitation matrices $A = (a_{mn})$, $B = (b_{mn})$ are used in this subsection.

(i) If $c_{mn} = \sum_{k=1}^{\infty} b_{mk}\, a_{kn}$, then $C = (c_{mn})$ is called the *product* $B \cdot A$ or the *iteration* of B with A.

(ii) The *limitation method* $\mathscr{A}$ is the (equivalence) class of regular matrices b-equivalent to A.

(iii) The *norm* $h(A)$ is $\sup_m \sum_{n=1}^{\infty} |a_{mn}|$, and the *norm* $\|\mathscr{A}\|$ is defined as $\inf h(A^*)$, where the infimum is taken over all $A^* \in \mathscr{A}$. Also, $N(A)$ is defined as $\sup |A\text{-}\lim s|$, where the supremum is taken over all sequences s in the unit sphere limited by A (see ⟨**2**, **3**⟩).

It is shown in [**20**] that if $h'(A) = \limsup_m \sum_{n=1}^{\infty} |a_{mn}|$, then $\|\mathscr{A}\|$ is also given by $\inf h'(A)$, where the infimum is taken over the same matrices as before. One of Brudno's striking theorems in ⟨**2**⟩ is that if the method $\mathscr{A}$ is b-stronger than $\mathscr{B}$, then $\|\mathscr{A}\| \geqslant \|\mathscr{B}\|$. In [**19**] Gordon proves the complementary result that $\mathscr{A}$ being strictly b-stronger than $\mathscr{B}$ does not imply that $\|\mathscr{A}\| > \|\mathscr{B}\|$. The theme of [**49**] is the attainment of norms. For instance, there is a new proof of Brudno's theorem in ⟨**3**⟩ to the effect that if A is regular, then there is a sequence s in the unit sphere such that $N_A = A\text{-}\lim s$.

A specimen theorem of [**9**] is that if A, B are regular, then there exists a regular C such that C and $B \cdot C$ are b-equivalent to A. The sole and surprising result of [**18**] is that if the regular matrices A_k are such that $\prod_{k=1}^{\infty} h(A_k) < \infty$, then there exists a matrix which is b-stronger than each of the products $A_k \cdot A_{k-1} \cdots A_1$ $(k = 1, 2, \ldots)$.

L. *Singularities* [**47**, **48**, **50**, **51**, **53**, **55**, **58**, **59**, **64**, **66**, **66a**, **70**, **75**]. Some of the notation introduced in Subsection K is now changed. Given the regular matrix A, in this section $\mathscr{A}$ denotes the limitation field of A, that is, the set of bounded sequences limited by A.

There are various types of singularity, most with highly technical definitions. However, very informally one may say that two or more regular matrices have a singularity if there are bounded sequences such that they and their transforms behave rather differently.

Singularities were first defined in ⟨**11**⟩. One of these, the type S_2, concerns a finite number of matrices, but an extension, the type S_3 for a countable set of matrices, is introduced in [**47**]. The significance of both types is explored in this paper. For instance, there is Theorem 8: if A, B are regular and $\mathscr{A} + \mathscr{B}$ spans all the bounded sequences, then A, B have a singularity S_2. In [**50**] it is deduced that under the previous conditions, A, B are not b-consistent.

The notion of b-consistency is in [**51**] generalised to more than two matrices, and the new concept leads to the corresponding generalisation of the theorem of [**50**]. Another extension appears in [**58**]: if the $\mathscr{A}^i$ $(i = 1, 2, \ldots)$ span the bounded sequences, then, for some p, $A_1, \ldots, A_p$ are not b-consistent. Yet another related result is Theorem 5 of [**66**]: if the $\mathscr{A}^i$ are such that $\mathscr{A}^i \subseteq \mathscr{A}^{i+1}$ $(i = 1, 2, \ldots)$, then they cannot span the space of bounded sequences densely.

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M. *Factor sequences* [**61**, **65**, **67**, **69**]. Let $A = (a_{mn})$ be a regular matrix. Changing earlier notation again, denote by $\mathfrak{A}$ the set of bounded sequences limited by A, and by $\mathfrak{A}_0$ the set of bounded sequences limited by A to 0.

The bounded sequence $\xi = (\xi_n)$ is now called a *factor sequence* for $\mathfrak{A}$ (or A) if $\xi x \in \mathfrak{A}_0$ whenever $x \in \mathfrak{A}_0$. The set of factor sequences for $\mathfrak{A}$ is denoted by $\mathfrak{A}^*$. (This is the definition as it appears in [**65**]; the form given in [**61**] is less compact.) In [**67**] it is shown that $\mathfrak{A}^*$, with the usual definitions of $x+y$, xy and $\|x\|$ for $x, y \in \mathfrak{A}^*$, is a Banach algebra.

A further definition (in [**61**]) is productive. If (n_k) is such that $\lim_{m\to\infty} \sum_k |a_{mn_k}| = 0$ and (t_n) is such that $t_n = 0$ when $n \neq n_k$, then (t_n) is called a *thin sequence* (with respect to A).

The theorems below give a flavour of the results in [**65**] and [**69**].

(i) If A is regular, then $\mathfrak{A}^*$ is the set of all bounded sequences if and only if $\mathfrak{A}$ consists of the sequences $r+t$, where r converges and t is thin.

(ii) If A is regular and $\mathfrak{A}^*$ is the set of bounded sequences, then A is b-equivalent to a positive regular matrix.

N. *Miscellaneous topics* (*ii*) [**4–6**, **14**, **54**, **71**(i)]. Much as in Subsection H, sets of submethods of a regular matrix method [**6**] and sets of subseries [**14**] can be associated with subsets of $(0,1]$. An appealing theorem in [**6**] is that the $(C,1)$ method is b-equivalent to almost all of its submethods.

The long review paper [**54**] appeared soon after the book [**B2**], but is very different in character. As Gordon says, the content is much influenced by his own preferences, and therefore leans towards areas in which he has been active. However, the last four of the 13 sections, on summability factors, Abel typical means, logarithmic means and Tauberian classes for functions, are entirely devoted to the work of others.

[**71**(i)] is a highly appreciative and affectionate biographical sketch of G. G. Lorentz which introduces a special volume of the *Journal of Approximation Theory* celebrating his 65th birthday. Gordon's mathematical tribute is [**71**(ii)], noted in Subsection I.

Regular matrix transformations [**B2**]. Although, in the preface, Gordon acknowledges his debt to G. H. Hardy's *Divergent series* (1949), R. G. Cooke's *Infinite matrices and sequence spaces* (1950) and K. Zeller's *Theorie der Limitierungsverfahren* (1958), there is very little overlap between these books and [**B2**]. After two classical chapters on matrix limitation methods and some well-known particular instances as well as Abel's method, [**B2**] proceeds to much newer material. Thus Chapter III introduces almost convergence and strongly regular matrices, and leads up to the Universal Tauberian Theorem of [**21**]. Chapter IV contains material on bounded convergence fields, with Brudno's fundamental theorems proved in the simple way set out in [**25**]. The discussion of matrix norms is continued in Chapter V, much of which is devoted to singularities and the inclusion of matrices. It is remarkable how this short book leads the reader from the very beginning of summability theory to what, in the late 1960s, was the frontier of research.

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