Canad. Math. Bull. Vol. **26** (1), 1983

UNIFORMLY CONTINUOUS PARTITIONS OF UNITY ON A METRIC SPACE

BY

STEWART M. ROBINSON AND ZACHARY ROBINSON

ABSTRACT. In this paper, we construct, for any open cover of a metric space, a partition of unity consisting of a family of uniformly continuous functions.

It is a well-known characterization of paracompact spaces that every open covering admits a partition of unity subordinate to it. In a series of lectures on "Non-Linear Analysis on Banach Spaces" given at both Dalhousie University and Cleveland State University, K. Sundaresan posed the following question: If (X, d) is a metric space, may one select this partition of unity to consist of uniformly continuous functions? In this note we provide an affirmative answer to this question.

THEOREM. *Let X be a paracompact space in which each open set is the support of a Lipschitz continuous function. Then there is a Lipschitz continuous partition of unity subordinate to any open cover of X.*

Proof. Let X be a space which satisfies the hypotheses and let $\mathcal{U}$ be an open cover of X. Since X is paracompact, $\mathcal{U}$ admits an open refinement that is both σ-discrete and locally finite; denote this refinement by $\bigcup_{n\in Z_+} \mathcal{U}_n$, where each $\mathcal{U}_n$ is a discrete family of open sets. Let $S_n = \bigcup_{A\in\mathcal{U}_n} A$. We may select a $g_n : X \to [0, 1]$ to be a Lipschitz continuous function with support S_n. For $j \in Z_+$, define $k_j : [0, 1] \to [0, 1]$ by

$$k_j(x) = \begin{cases} j \cdot x & \text{if } x \in [0, 1/j] \\ 1 & \text{otherwise,} \end{cases}$$

and define $g_{n,j} : X \to [0, 1]$ by

$$g_{n,j}(x) = k_j(g_n(x)).$$

Being the composition of Lipschitz continuous functions, $g_{n,j}$ is itself Lipschitz continuous.

Let $\tau : Z_+ \to Z_+ \times Z_+$ be a bijection, and for convenience, denote $\tau(n)$ by

Received by the editors, January 16, 1981.

AMS (MOS) Subject Classification (1980): Primary 54E35, 54C30, 54D99.

Key words and phrases: metric spaces, uniform continuity, partitions of unity.

(i_n, j_n). Define $h_n : X \to [0, 1]$ by

$$h_n(x) = \begin{cases} g_{i_n, j_n}(x) \prod_{k=1}^{n-1} (1 - g_{i_k j_k}(x)), & \text{if } n > 1 \\ g_{i_1, j_1}(x), & \text{if } n = 1. \end{cases}$$

Being the product of finitely many bounded Lipschitz continuous functions, h_n is Lipschitz continuous. For $A \in \mathcal{U}_n$, define $h_{n,A} : X \to [0, 1]$ by

$$h_{n,A}(x) = \begin{cases} h_n(x), & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

It is evident that $h_{n,A}$ is Lipschitz continuous.

CLAIM. $\mathcal{H} = \{h_{n,A} \mid n \in Z_+, A \in \mathcal{U}_n\}$ is the desired partition of unity.

Proof. (i) Support $(h_{n,A}) = A \in \mathcal{U}_n$, so $\mathcal{H}$ is subordinate to $\mathcal{U}$.

(ii) Since $\{\text{Support } (h_{n,A}) : n \in Z_+, A \in \mathcal{U}_n\} = \bigcup_{n \in Z_+} \mathcal{U}_n$ and since $\bigcup_{n \in Z_+} \mathcal{U}_n$ is locally finite, $\mathcal{H}$ is a locally finite family of functions.

(iii) The following identity may be proved by induction:

$$\sum_{k=1}^{n} h_k(x) = 1 - \prod_{k=1}^{n} (1 - g_{i_k, j_k}(x)).$$

Clearly, $\sum \{h_{n,A}(x) \mid n \in Z_+, A \in \mathcal{U}_n\} = \sum_{n \in Z_+} h_n(x)$; therefore it will suffice to show that for every $x \in X$, there is some $k \in Z_+$ such that $g_{i_k, j_k}(x) = 1$. Since $\bigcup_{n \in Z_+} \mathcal{U}_n$ is a cover of X, for some $n \in Z_+$ and $A \in \mathcal{U}_n$, $x \in A$. Thus, $g_n(x) > 0$ and for some $j \in Z_+$, $g_n(x) > 1/j$; i.e., $g_{n,j}(x) = 1$. But τ is a bijection, so for some $k \in Z_+$, $g_{i_k, j_k}(x) = 1$.

COROLLARY. *There is a uniformly continuous partition of unity subordinate to any open cover of a metric space.*

Proof. Let (X, d) be a metric space and U an open subset. For $x \in X$, define

$$f_U(x) = d(x, X \setminus U).$$

f_U is Lipschitz continuous and Support $(f_U) = U$. Furthermore, Lipschitz continuity implies uniform continuity.

REMARK. The condition that every open set be the support of a uniformly continuous functions is not a necessary condition for a space to satisfy the conclusion of the corollary. For example, any uncountable product of closed

https://doi.org/10.4153/CMB-1983-017-x Published online by Cambridge University Press

unit intervals does not satisfy this condition, since any singleton is a closed set that is not a G_δ; nevertheless as a compact space, it clearly satisfies the conclusion.

Cleveland State University
Cleveland, Ohio
MacGregor House, MIT
Cambridge, Mass.