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SKEW-PRODUCT DYNAMICAL SYSTEMS, ELLIS GROUPS AND TOPOLOGICAL CENTRE

A. JABBARI ✉ and H. R. E. VISHKI

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Abstract

In this paper, a general construction of a skew-product dynamical system, for which the skew-product dynamical system studied by Hahn is a special case, is given. Then the ergodic and topological properties (of a special type) of our newly defined systems (called Milnes-type systems) are investigated. It is shown that the Milnes-type systems are actually natural extensions of dynamical systems corresponding to some special distal functions. Finally, the topological centre of Ellis groups of any skew-product dynamical system is calculated.

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1. Introduction

In his well-known work [9], Hahn studied ergodic and topological properties of dynamical systems (X, T) in which $X = G^k$ and $T : X \to X$ is defined by

$$T(x_1, x_2, \ldots, x_k) = (\gamma_0 + x_1, x_1 + x_2, \ldots, x_{k-1} + x_k)$$

where $k = 1, 2, 3, \ldots$ or $k = \infty$ and G is a compact monothetic group with generator γ_0, that is, G is a compact group and the subgroup generated by γ_0 is dense in G (so G is Abelian). He called (X, T) a skew-product dynamical system. Among other things, Hahn [9] showed that if the dual group of G has no element of finite order, then (X, T) is a (distal) minimal dynamical system which is also uniquely ergodic. Such transformations on the torus were first studied by Anzai in [1], and the question of strict ergodicity of these transformations on the torus was studied by Furstenberg [7]. Also Milnes [11, Example 3] studied a different (kind of skew product) dynamical system on the 4-torus defined as

$$T(x, y, z, u) = (\lambda^4 x, \lambda^6 x^3 y, \lambda^4 x^3 y^2 z, \lambda xyzu)$$

where λ is an irrational (that is, nonroot of unity) element of the unit circle.

In this paper, a general construction of a skew-product dynamical system is given. It is shown that Hahn-type skew-product dynamical systems [9] and also the Milnes dynamical system on the 4-torus [11] are special cases of our definition of a skew-product system, (see Section 3 for the definition and results). Then, in Section 4, we examine the ergodic and topological properties of Milnes-type skew-product dynamical systems. These skew-product systems, on the finite-dimensional tori, are actually extensions of the dynamical systems corresponding to the functions $n \to \lambda^{n^k}$ on the integers (Theorem 5.3). Also Section 5 is devoted to a generalization of a result of Salehi [16] on the unique ergodicity of the functions $n \to \lambda^{n^k}$ ($k \in \mathbb{N}$, $\lambda \in \mathbb{T}$) on the additive group of integers by using the unique ergodicity of Milnes-type skew-product systems.

An interesting problem concerning any compact right-topological semigroup is characterizing its topological centre. Such semigroups in topological dynamics arise naturally in two examples. First, semigroup compactifications of any semitopological semigroup, and second, the enveloping semigroups of flows. In either case, there are rather few situations in which the corresponding topological centre is calculated explicitly. We refer the interested reader to [10, 14] or, in general, [3] for some interesting results on the first case. For the latter case, there is a nice result due to Namioka [13] which computes the topological centre of the Ellis group of a dynamical system on the 2-torus. Section 6 of this paper is devoted to the study of the Ellis groups of Hahn-type and Milnes-type skew-product dynamical systems and their topological centres. In particular, our results generalize Namioka's result.

2. Preliminaries

For the background materials and notations we follow Berglund *et al.* [2] as much as possible. For a semigroup S, the right translation ρ_t and the left translation λ_s on S are defined by $\rho_t(s) = st = \lambda_s(t)$, $(s, t \in S)$. A semigroup S, equipped with a topology, is said to be right topological if all of the right translations are continuous, semitopological if all of the left and right translations are continuous. If S is a right-topological semigroup then the set $\Lambda(S) = \{s \in S \mid \lambda_s \text{ is continuous}\}$ is called the topological centre of S.

By a flow (S, X) we mean a semitopological semigroup S and a compact Hausdorff space X in which there exists a mapping $S \times X \to X$, $(s, x) \mapsto sx$ such that for all $s, t \in S$ and $x \in X$, $(st)x = s(tx)$ and $ex = x$ if S has an identity e, and the mapping $\varepsilon_s : X \to X$, $x \mapsto sx$ is continuous for each $s \in S$. If (S, X) is a flow, then the closure of the set $\{\varepsilon_s : X \to X : s \in S\}$ in X^X, with the relativization of the product topology from X^X, is a compact right-topological semigroup under the composition operator. This closure is denoted by $\Sigma(S, X)$, or simply Σ if there is no risk of confusion, and is called the enveloping semigroup of the flow. It is obvious from the definition that $\Lambda(\Sigma(S, X))$ always contains the set $\{\varepsilon_s \mid s \in S\}$. The idea of enveloping semigroup was introduced by Ellis [6]. It was also proved by Ellis [5] that the enveloping semigroup of a flow (S, X) is a group if and only if (S, X) is distal, that is,

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$\lim_\alpha s_\alpha x = \lim_\alpha s_\alpha y$ for some net $\{s_\alpha\}$ in S and for some $x, y \in X$ implies that $x = y$. The enveloping semigroup of a distal flow is called the Ellis group of the flow. A flow (S, X) is called minimal if the orbit closure of each point is dense in X.

The next proposition is a crucial tool in the characterization of the topological centre of the Ellis group corresponding to any minimal flow, and is used in Section 6 to characterize the topological centre of the Ellis groups of skew-product dynamical systems.

PROPOSITION 2.1. *Let (S, X) be a minimal flow, then the set Σ_c of all continuous elements of Σ coincides with $\Lambda(\Sigma)$.*

PROOF. Obviously $\Sigma_c \subseteq \Lambda(\Sigma)$. Let $\sigma \in \Lambda(\Sigma)$, $x \in X$ and let $\{x_\alpha\}$ be a net in X such that $x_\alpha \to x$. We have to show that $\sigma(x_\alpha) \to \sigma(x)$. Since X is minimal $\Sigma x = X$, hence for each α there exists $\tau_\alpha \in \Sigma$ such that $\tau_\alpha x = x_\alpha$. By taking a convergent subnet of τ_α if necessary, we may assume that $\tau_\alpha \to \tau \in \Sigma$; it follows that $\tau_\alpha x \to \tau x$. On the other hand $\tau_\alpha x = x_\alpha \to x$, hence $\tau x = x$. Since $\sigma \in \Lambda(\Sigma)$, $\sigma x_\alpha = \sigma(\tau_\alpha x) = \sigma \circ \tau_\alpha(x) \to \sigma \circ \tau(x) = \sigma(\tau x) = \sigma x$, as required. □

A dynamical system is a pair (X, T) where X is a compact Hausdorff space and T is a homeomorphism from X onto X. Obviously, if (X, T) is a dynamical system, then $(\mathbb{Z}, X)$ is a flow with the action $(n, x) \mapsto T^n(x)$, for $n \in \mathbb{Z}$ and $x \in X$. Hence (X, T) is called minimal (distal) if the corresponding flow $(\mathbb{Z}, X)$ is minimal (distal).

A dynamical system (X, T) is called uniquely ergodic if there exists a unique T-invariant positive probability measure μ on X. A minimal uniquely ergodic dynamical system is called strictly ergodic. A sequence $\{x_n\}$ in X is said to be uniformly distributed if for all $x \in X$ and for each $f \in C(X)$, $\lim_{N\to\infty}(1/N)\sum_{n=0}^{N-1} f(T^n(x)) = \int_X f\, d\mu$.

The following two theorems due to Weyl [17] and Oxtoby [15] will be used in Section 4 to prove the minimality and unique ergodicity of the Milnes-type skew-product dynamical systems.

THEOREM 2.2 (Weyl). *Let $p(z) = a_0 + a_1 z + \cdots + a_{k-1}z^{k-1} + a_k z^k$ be a polynomial of degree k with real coefficients. If for some j, $1 \le j \le k$, a_j is irrational, then*

$$\sum_{n=0}^{N} \exp(p(n)) = \circ(N)$$

uniformly in $a_0, a_1, \ldots, a_{k-1}$.

For an interesting proof of the following theorem see [9].

THEOREM 2.3 (Oxtoby). *Let (X, T) be a minimal dynamical system. If for each $f \in C(X)$*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n(x))$$

exists uniformly in x, then (X, T) is strictly ergodic.

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3. The construction of skew-product dynamical systems

If $\{J_i \mid i \geq 0\}$ is a sequence of nonzero real numbers with $J_0 = 1$, define $J(i, 0) = 1$ and $J(i, 1) = J_{i-1}$ for $i \geq 1$ and

$$J(i, l) = J_{i-1} J_{i-2} \cdots J_{i-l} / (J_1 J_2 \cdots J_{l-1}) \quad \text{for } i \geq l \geq 2.$$

Let $Q_0 = 1$ and, for each $i \geq 1$, let $Q_i(n)$ be a polynomial which satisfies the following equation:

$$\sum_{j=l}^{i} J(i-l, j-l) Q_{i-j}(1) Q_{j-l}(n-1) = Q_{i-l}(n) \qquad \text{for all } i \geq l \geq 0 \text{ and for all } n \in \mathbb{N}. \tag{3.0.1}$$

For example if for each i, $J_i = 1$, then it is enough to take $Q_i(n) = (1/i!)n(n-1)\cdots(n-i+1)$ for $i \leq n$ and $Q_i(n) = 0$ elsewhere. Also if for each i, $J_i = i + 1$, then $Q_i(n) = n^i$ satisfies the cited equation. Now we are ready to make our main construction.

DEFINITION 3.1. Let $\{J_i \mid i \geq 0\}$ be a sequence of nonzero real numbers with $J_0 = 1$ and let $Q_i(n)$ be polynomials satisfying (3.0.1) such that for $0 \leq j \leq i \leq k$ one has $J(k-j, i-j)Q_{i-j}(1) \in \mathbb{Z}$. Let $k \in \mathbb{N}$, let G be a compact monothetic group with generator γ_0. Put $X = G^k$ and define $T : X \to X$ by

$$T(x_1, x_2, \ldots, x_k) = (y_1, y_2, \ldots, y_k)$$

where for each $i = 1, 2, \ldots, k$

$$y_i = J(k, i)Q_i(1)\gamma_0 + \sum_{j=1}^{i} J(k-j, i-j)Q_{i-j}(1)x_j. \tag{3.0.2}$$

Then (X, T) is called a skew-product dynamical system.

We can define two types of skew-product systems.

(i) For each $i \geq 0$ let $J_i = 1$. Let $Q_i(n) = (1/i!)n(n-1)\cdots(n-i+1)$ for $1 \leq i \leq n$ and $Q_i(n) = 0$ for $i > n$. Then it is easily seen that (in (3.0.2)) $y_1 = \gamma_0 + x_1$ and $y_i = x_{i-1} + x_i$ for $i \geq 2$. Hence we get the Hahn's skew-product transformation:

$$T(x_1, x_2, \ldots, x_k) = (\gamma_0 + x_1, x_1 + x_2, \ldots, x_{k-1} + x_k).$$

In this paper we refer to such systems as Hahn-type skew-product systems. In this case $k = \infty$ is also possible.

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(ii) For each $i \geq 0$ let $J_i = i + 1$ and let $Q_i(n) = n^i$. Some computation reveals that for each $l \leq i$, $J(i, l) = \binom{i}{l}$ and so if $T(x_1, x_2, \ldots, x_k) = (y_1, y_2, \ldots, y_k)$, then

$$\begin{aligned}
y_1 &= k\gamma_0 + x_1 \\
y_2 &= \binom{k}{2}\gamma_0 + \binom{k-1}{1}x_1 + x_2 \\
&\vdots \\
y_i &= \binom{k}{i}\gamma_0 + \sum_{j=1}^{i}\binom{k-j}{i-j}x_j \\
&\vdots \\
y_k &= \gamma_0 + \sum_{j=1}^{k} x_j.
\end{aligned}$$

In this case we call (X, T) a Milnes-type skew-product transformation, just because for $k = 4$ and $G = \mathbb{T}$ one has $T(x_1, x_2, x_3, x_4) = (\lambda^4 x_1, \lambda^6 x_1^3 x_2, \lambda^4 x_1^3 x_2^2 x_3, \lambda x_1 x_2 x_3 x_4)$, which is nothing but the Milnes dynamical system.

4. Milnes-type skew-product dynamical systems

The results and methods of this section are based on those of Hahn [9]. Unless otherwise stated, (X, T) denotes a Milnes-type skew-product dynamical system with the base group G, where G is a compact monothetic group with generator γ_0. The following theorem is the main result of this section.

THEOREM 4.1. *If the dual group of G has no element of finite order, then the following results hold.*

(i) *For each x in X the points $T^n(x)$, $n = 1, 2, 3, \ldots$, are uniformly distributed in X; in particular (X, T) is minimal.*

(ii) *(X, T) is strictly ergodic.*

To prove our main theorem we need to mention some preliminaries.

LEMMA 4.2. *Let $n \in \mathbb{Z}$. Then for $x = (x_1, x_2, \ldots, x_k)$ in X, $T^n(x_1, x_2, \ldots, x_k) = (y_{1,n}, y_{2,n}, \ldots, y_{k,n})$, where for each $i = 1, 2, \ldots, k$,*

$$y_{i,n} = \binom{k}{i} n^i \gamma_0 + \sum_{j=1}^{i}\binom{k-j}{i-j} n^{i-j} x_j.$$

PROOF. We prove the result for positive integers by induction, the general result will be then observed easily. Clearly the result holds for $n = 1$. Let $n \neq 1$ be

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a natural number and assume that for each $x = (x_1, x_2, \ldots, x_k)$ in X, $T^{n-1}(x) = (y_{1,n-1}, y_{2,n-1}, \ldots, y_{k,n-1})$, in which

$$y_{i,n-1} = \binom{k}{i}(n-1)^i\gamma_0 + \sum_{j=1}^{i}\binom{k-j}{i-j}(n-1)^{i-j}x_j$$

for each $i = 1, 2, \ldots, k$. Let $x = (x_1, x_2, \ldots, x_k)$, so $T^n(x) = T(T^{n-1}(x)) = T(y_{1,n-1}, y_{2,n-1}, \ldots, y_{k,n-1}) := (w_1, w_2, \ldots, w_k)$, and hence for each $i = 1, 2, \ldots, k$

$$\begin{aligned}
w_i &= \binom{k}{i}\gamma_0 + \sum_{j=1}^{i}\binom{k-j}{i-j}y_{j,n-1} \\
&= \binom{k}{i}\gamma_0 + \sum_{j=1}^{i}\binom{k-j}{i-j}\left[\binom{k}{j}(n-1)^j\gamma_0 + \sum_{l=1}^{j}\binom{k-l}{j-l}(n-1)^{j-l}x_l\right] \\
&= \binom{k}{i}\gamma_0 + \sum_{j=1}^{i}\binom{k-j}{i-j}\binom{k}{j}(n-1)^j\gamma_0 \\
&\quad + \sum_{j=1}^{i}\sum_{l=1}^{j}\binom{k-j}{i-j}\binom{k-l}{j-l}(n-1)^{j-l}x_l \\
&= \left[\sum_{j=0}^{i}\binom{k-j}{i-j}\binom{k}{j}(n-1)^j\right]\gamma_0 + \sum_{l=1}^{i}\sum_{j=l}^{i}\binom{k-j}{i-j}\binom{k-l}{j-l}(n-1)^{j-l}x_l \\
&= \left[\sum_{j=0}^{i}\binom{k}{i}\binom{i}{j}(n-1)^j\right]\gamma_0 + \sum_{l=1}^{i}\sum_{j=l}^{i}\binom{k-l}{i-l}\binom{i-l}{j-l}(n-1)^{j-l}x_l, \\
&\qquad \left(\binom{k-j}{i-j}\binom{k-l}{j-l} = \binom{k-l}{i-l}\binom{i-l}{j-l}\right) \\
&= \binom{k}{i}\left[\sum_{j=0}^{i}\binom{i}{j}(n-1)^j\right]\gamma_0 + \sum_{l=1}^{i}\binom{k-l}{i-l}\left[\sum_{j=l}^{i}\binom{i-l}{j-l}(n-1)^{j-l}\right]x_l \\
&= \binom{k}{i}n^i\gamma_0 + \sum_{l=1}^{i}\binom{k-l}{i-l}n^{i-l}x_l, \quad \left(\sum_{j=l}^{i}\binom{i-l}{j-l}(n-1)^{j-l} = n^{i-l}\right).
\end{aligned}$$

The lemma is now established. □

REMARK 4.3. If (X, T) is any skew-product dynamical system, then with the same method as in the above lemma, and the notations of Section 3, one may show that for $n \in \mathbb{Z}$ and $x = (x_1, x_2, \ldots, x_k)$ in X, $T^n(x_1, x_2, \ldots, x_k) = (y_{1,n}, y_{2,n}, \ldots, y_{k,n})$,

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where for each $i = 1, 2, \ldots, k$,

$$y_{i,n} = J(k, i)Q_i(n)\gamma_0 + \sum_{j=1}^{i} J(k-j, i-j)Q_{i-j}(n)x_j.$$

COROLLARY 4.4. *(X, T) is distal.*

PROOF. Let $(x_1, x_2, \ldots, x_k)$ and $(x'_1, x'_2, \ldots, x'_k)$ be members of X such that

$$\lim_\alpha T^{m_\alpha}(x_1, x_2, \ldots, x_k) = \lim_\alpha T^{m_\alpha}(x'_1, x'_2, \ldots, x'_k)$$

for some net $\{m_\alpha\}$ in $\mathbb{Z}$. Then from Lemma 4.2 for each $i = 1, 2, \ldots, k$ we obtain

$$\lim_\alpha \left[\binom{k}{i} m_\alpha^i \gamma_0 + \sum_{j=1}^{i} \binom{k-j}{i-j} m_\alpha^{i-j} x_j\right] = \lim_\alpha \left[\binom{k}{i} m_\alpha^i \gamma_0 + \sum_{j=1}^{i} \binom{k-j}{i-j} m_\alpha^{i-j} x'_j\right].$$

Let $i = 1$, then $\lim_\alpha k m_\alpha \gamma_0 + x_1 = \lim_\alpha k m_\alpha \gamma_0 + x'_1$, hence $x_1 = x'_1$. Now let $1 \le l < k$ and assume that we have already shown that $x_j = x'_j$ for $j = 1, 2, \ldots, l$. We shall show that $x_{l+1} = x'_{l+1}$. Taking $i = l + 1$ in the above equation, we obtain

$$\lim_\alpha \binom{k}{l+1} m_\alpha^{l+1} \gamma_0 + x_{l+1} = \lim_\alpha \binom{k}{l+1} m_\alpha^{l+1} \gamma_0 + x'_{l+1}.$$

Hence $x_{l+1} = x'_{l+1}$. It follows by induction that $(x_1, x_2, \ldots, x_k) = (x'_1, x'_2, \ldots, x'_k)$, that is, (X, T) is distal. □

For each x in X, $T(x) = \gamma + U(x)$ where $\gamma = \left(k\gamma_0, \binom{k}{2}\gamma_0, \binom{k}{3}\gamma_0 \ldots, \gamma_0\right)$ and U is the homomorphism on X defined by $U = T - \gamma$. Hence $T^n = \left(\binom{k}{1} n\gamma_0, \binom{k}{2} n^2\gamma_0, \ldots, n^k\gamma_0\right) + U^n$, for all $n \in \mathbb{Z}$. Let $\widehat{G}$ and $\widehat{X}$ denote the dual groups of G and X, respectively. And let $\widehat{T}$ and $\widehat{U}$ denote the corresponding dual transformations of T and U, respectively.

LEMMA 4.5. *For each $\eta = (\eta_1, \eta_2, \ldots, \eta_k) \in \widehat{X}$ and $n \in \mathbb{Z}$:*

(i) $\widehat{U}^n(\eta) = (\zeta_{1,n}, \zeta_{2,n}, \ldots, \zeta_{k,n})$ *where*

$$\zeta_{j,n} = \prod_{i=j}^{k} \eta_i^{\binom{k-j}{i-j} n^{i-j}} \quad \text{for } j = 1, 2, \ldots, k;$$

(ii) $\widehat{T}^n(\eta) = \eta\left(\binom{k}{1} n\gamma_0, \binom{k}{2} n^2\gamma_0, \ldots, n^k\gamma_0\right)\widehat{U}^n(\eta)$.

PROOF. (i) Let $x \in X$ and let $U(x) = (z_1, z_2, \ldots, z_k)$, then $z_i = \sum_{j=1}^{i} \binom{k-j}{i-j} x_j$. Now we have

$$\begin{aligned}
\widehat{U}(\eta)(x) &= \eta(Ux) = \eta_1(z_1)\eta_2(z_2)\cdots\eta_k(z_k) \\
&= \eta_1(x_1)\eta_2\left(\binom{k-1}{1} x_1 + x_2\right)\cdots\eta_k(x_1 + x_2 + \cdots + x_k) \\
&= \zeta_1(x_1)\zeta_2(x_2)\cdots\zeta_k(x_k)
\end{aligned}$$

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where $\zeta_j = \prod_{i=j}^{k} \eta_i^{\binom{k-j}{i-j}}$ for $j = 1, 2, \ldots, k$. Hence the result holds for $n = 1$. The general result holds by induction. (ii) follows from (i) and the formulas preceding the lemma. □

Let μ be the Haar measure on X. To prove our result on the unique ergodicity of (X, T), we first show that (X, T) is ergodic provided $\widehat{G}$ has no (nontrivial) element of finite order.

THEOREM 4.6. *If the character group of G has no element of finite order, then (X, μ, T) is ergodic.*

PROOF. Assume that $\widehat{G}$ has no element of finite order, and that (X, μ, T) is not ergodic. Hence by [8, Corollary 3] $\widehat{U}^n\eta = \eta$ for some positive integer n and for some nontrivial character $\eta \in \widehat{X}$. Let $\eta = (\eta_1, \eta_2, \ldots, \eta_k)$ in which $\eta_i \in \widehat{G}$, then for each $x = (x_1, x_2, \ldots, x_k)$ in X, $\eta(x) = \eta_1(x_1)\eta_2(x_2)\cdots\eta_k(x_k)$. By Lemma 4.5(i) the equation $\widehat{U}^n\eta = \eta$ implies that $\eta_{k-1}\eta_k^n = \eta_{k-1}$. Since $\widehat{G}$ has no element of finite order, $\eta_k \equiv 1$. Now by downward induction assume that for some i with $2 < i \leq k$, $\eta_i \equiv \eta_{i+1} \equiv \cdots \equiv \eta_k \equiv 1$. From $\widehat{U}^n\eta = \eta$ we conclude that $\eta_{i-2}\eta_{i-1}^{(k-i+1)n} \equiv \eta_{i-2}$. Since $\widehat{G}$ has no element of finite order we get $\eta_{i-1} \equiv 1$, so that by induction $\eta_2 \equiv \eta_3 \equiv \cdots \equiv \eta_k \equiv 1$. It remains to show that $\eta_1 \equiv 1$. By [8, Theorem 4] it follows that for at least one η for which $\widehat{U}^n\eta = \eta$ we have $\widehat{T}^n\eta = \eta$. Hence by Lemma 4.5 (ii) we have $\eta = \widehat{T}^n\eta = \eta_1^{kn}(\gamma_0)\eta$. Now using the facts that the subgroup generated by γ_0 is dense in G and that $\widehat{G}$ has no element of finite order we derive that $\eta_1 \equiv 1$. Hence $\eta \equiv 1$ which is a contradiction to the fact that η is nontrivial. So (X, μ, T) is ergodic. □

LEMMA 4.7. *If $\widehat{G}$ has no element of finite order, then for each $f \in C(X)$*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n(x))$$

exists uniformly in x.

PROOF. Let $f \in C(X)$. Since $\widehat{X}$ is dense in $C(X)$, assume, without loss of generality, that $f \in \widehat{X}$. Then $f = (f_1, f_2, \ldots, f_k)$ where $f_i \in \widehat{G}$ for $i = 1, 2, \ldots, k$. That is, for each $x = (x_1, x_2, \ldots, x_k)$ in X, $f(x) = f_1(x_1)f_2(x_2)\cdots f_k(x_k)$. To examine the values $f(T^n(x)) = f(y_{1,n}, y_{2,n}, \ldots, y_{k,n})$, where $y_{i,n} = \binom{k}{i} n^i\gamma_0 + \sum_{j=1}^{i} \binom{k-j}{i-j} n^{i-j}x_j$ for $i = 1, 2, \ldots, k$, let

$$f_i(x_j) = \exp(\phi_i(x_j))$$
$$f_i(\gamma_0) = \exp(\alpha_i)$$

in which $0 \leq \phi_i(x_j) < 1$ and $0 \leq \alpha_i < 1$ for $i, j = 1, 2, \ldots, k$. Then

$$f(T^n(x)) = f_1(y_{1,n})f_2(y_{2,n})\cdots f_k(y_{k,n}).$$

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Since

$$f_i(y_{i,n}) = f_i\left(\binom{k}{i} n^i \gamma_0 + \sum_{j=1}^{i} \binom{k-j}{i-j} n^{i-j} x_j\right)$$
$$= \exp\left(\binom{k}{i} n^i \alpha_i + \sum_{j=1}^{i} \binom{k-j}{i-j} n^{i-j} \phi_i(x_j)\right)$$

we have

$$f(T^n(x)) = \exp\left(\sum_{i=1}^{k} \binom{k}{i} n^i \gamma_0 + \sum_{i=1}^{k}\sum_{j=1}^{i} \binom{k-j}{i-j} n^{i-j} \phi_i(x_j)\right)$$
$$= \exp(q(n))$$

where

$$q(n) = \sum_{i=1}^{k} \binom{k}{i} n^i \gamma_0 + \sum_{i=1}^{k}\sum_{j=1}^{i} \binom{k-j}{i-j} n^{i-j} \phi_i(x_j).$$

Suppose that $f \neq 1$ and choose j such that $1 \leq j \leq k$, $f_j \neq 1$ and $f_{j+1} = f_{j+2} = \cdots = f_k = 1$. Then $q(n)$ is of order j and the leading coefficient in $q(n)$ equals to $\binom{k}{j} \alpha_j$. Since f_j is not of finite order and since the subgroup generated by γ_0 is dense in G, $\exp(\alpha_j)$ is not of finite order, that is, α_j is irrational, and so is $\binom{k}{j} \alpha_j$. Let $q(n) = a_0 + a_1 n + \cdots + a_{j-1} n^{j-1} + \binom{k}{j} \alpha_j n^j$. Then, by Theorem 2.2,

$$\sum_{n=0}^{N-1} f(T^n(x)) = \sum_{n=0}^{N-1} \exp(q(n)) = \circ(N)$$

uniformly in $a_0, a_1, \ldots, a_{j-1}$. Since these a_i are combinations of $\phi_l(x_i)$, $\sum_{n=0}^{N-1} f(T^n(x)) = \circ(N)$ uniformly in x. This proves the lemma. □

Now we are ready to prove the main result of this section.

PROOF OF THEOREM 4.1.

(i) Since (X, μ, T) is ergodic (Theorem 4.6), by the ergodic Theorem for each $f \in C(X)$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n(x)) = \int_X f\, d\mu$$

for almost all x in X. But it follows from Lemma 4.7 that for each $f \in C(X)$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n(x)) := f^*(x)$$

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exists uniformly in x. So f^* is continuous. On the other hand by the above equations f^* is constant almost everywhere, hence the continuity of f^* implies that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n(x)) = \int_X f\, d\mu$$

for all x in X. Hence for each x in X the points $T^n(x)$ $(n = 1, 2, 3, \ldots)$ are uniformly distributed (and so dense) in X, that is, (X, T) is minimal.

(ii) Since (X, T) is minimal and $\widehat{G}$ has no element of finite order, the result follows from Lemma 4.7 and Theorem 2.3. □

REMARK 4.8. As a matter of fact, all the results of this section could be generalized to arbitrary skew-product dynamical systems. We avoided doing this just because of the huge calculations needed.

5. Some consequences

In this section, among other things, we deduce a simple proof for the unique ergodicity of a class of functions on integers by using the unique ergodicity of Milnes-type skew-product systems (Theorem 4.1(ii)). Also we show that Milnes-type systems are natural extensions of dynamical systems corresponding to some special functions. To this end, consider the homeomorphism $V : l^\infty(\mathbb{Z}) \to l^\infty(\mathbb{Z})$ defined by $V(f)(n) = f(n+1)$ (for $f \in l^\infty(\mathbb{Z})$), called the shift operator. For each $f \in l^\infty(\mathbb{Z})$, let X_f be the pointwise closure of the set $\{V^n(f) \mid n \in \mathbb{Z}\}$ in $l^\infty(\mathbb{Z})$. Since on a bounded subset of $l^\infty(\mathbb{Z})$ the weak*-topology coincides with the topology of pointwise convergence on $\mathbb{Z}$, X_f is compact with the pointwise topology. If we denote the restriction of V to X_f by V again, then the pair (X_f, V) defines a dynamical system. A function $f \in l^\infty(\mathbb{Z})$ is called uniquely ergodic if its corresponding dynamical system (X_f, V) is uniquely ergodic.

The following proposition was first proved in [16], but for the sake of completeness, we give the proof here.

PROPOSITION 5.1. *Let (X, T) be a strictly ergodic dynamical system. Define $\Delta : C(X) \times X \to l^\infty(\mathbb{Z})$ by $\Delta(F, x)(n) = F(T^n(x))$. Then any element in the range of Δ is uniquely ergodic.*

PROOF. Let $x_0 \in X$ and $F \in C(X)$, and let $f = \Delta(F, x_0)$. Define $\varphi : X \to l^\infty(\mathbb{Z})$ by $\varphi(x)(n) = F(T^n x)$. Then $\varphi(x_0) = f$, and $\varphi(Tx)(n) = F(T^n(Tx)) = F(T^{n+1}x) = \varphi(x)(n+1) = V(\varphi(x))(n)$, that is, $\varphi T = V\varphi$. Hence φ is a flow homomorphism. Since (X, T) is minimal, $\varphi(X) = X_f$. It follows that f is uniquely ergodic, as claimed. □

Using Proposition 5.1 we derive a simple proof for the next result which is due to Salehi [16] when $G = \mathbb{T}$.

COROLLARY 5.2. *Let G be a compact monothetic group with generator γ_0 such that $\widehat{G}$ has no element of finite order, then for every continuous character χ on G and for each $k \in \mathbb{N}$ the function $f(n) = \chi(n^k \gamma_0)$ is uniquely ergodic. In particular for each m in $\mathbb{Z}$ and for each irrational element λ in $\mathbb{T}$, the function $f(n) = \lambda^{mn^k}$ is uniquely ergodic.*

PROOF. Let $X = G^k$ and let (X, T) be a Milnes-type skew-product dynamical system with the base group G, then by Theorem 4.1 (X, T) is strictly ergodic. Now $T^n(0) = (kn\gamma_0, \binom{k}{2} n^2\gamma_0, \ldots, n^k\gamma_0)$. Let $\chi \in \widehat{G}$ and use Proposition 5.1 with $x_0 = 0$ and the function F defined by $F(x_1, x_2, \ldots, x_k) = \chi(x_k)$. Then $\Delta(F, 0) = f$, where $f(n) = \chi(n^k\gamma_0)$. For the special case, let $G = \mathbb{T}$, $\chi = m$ and $\gamma_0 = \lambda$, and use the first part. □

Let $X = \mathbb{T}^k$, let $\lambda \in \mathbb{T}$ and consider the mapping $f : n \mapsto \lambda^{n^k}$ on $\mathbb{Z}$. Let the mapping $\Gamma : X \to \mathbb{T}^{\mathbb{Z}}$ be defined by

$$\Gamma(x_1, x_2, \ldots, x_k)(n) = f(n) x_1^{n^{k-1}} x_2^{n^{k-2}} \cdots x_{k-1}^{n} x_k$$

where $(x_1, x_2, \ldots, x_k) \in X$ and $n \in \mathbb{Z}$. With these hypotheses we have the next result.

THEOREM 5.3. *Let (X, T) be a Milnes-type skew-product dynamical system with the base group $G = \mathbb{T}$. If $\lambda \in \mathbb{T}$ is irrational, then* $\mathrm{Range}(\Gamma) = X_f$*, and $\Gamma : (X, T) \to (X_f, V)$ is a homomorphism, that is $V\Gamma = \Gamma T$ and so X_f is a factor of $\mathbb{T}^k$.*

PROOF. Since λ is irrational, by Theorem 4.1 (X, T) is minimal and hence the points $(\lambda^{kn}, \lambda^{\binom{k}{2} n^2}, \ldots, \lambda^{n^k}) = T^n(1, 1, \ldots, 1)$ $(n = 1, 2, 3, \ldots)$ are dense in X. Now some computation reveals that $\mathrm{Range}(\Gamma) = X_f$ and also $V\Gamma = \Gamma T$. □

REMARKS 5.4.

(a) The mapping $\Gamma : (X, T) \to (X_f, V)$ in Theorem 5.3 is not necessarily one-to-one; for example, if $k = 3$ then $\Gamma(-1, -1, 1) = \Gamma(1, 1, 1)$ because for each n, $n^2 + n$ is divisible by 2.

(b) Since any factor of any distal dynamical system is again distal, it follows from Theorem 5.3 and Corollary 4.4 that for each $k \in \mathbb{N}$ and for each $\lambda \in \mathbb{T}$ the function $f : n \mapsto \lambda^{n^k}$ is a distal function (note that if λ is not irrational, then f is periodic and so distal). This result is due to Namioka [12].

6. Ellis groups of skew-product systems

In this section, unless otherwise stated, (X, T) denotes a Milnes-type skew-product dynamical system with the base group G and generator γ_0 such that $\widehat{G}$ has no element of finite order, $1 \neq k \in \mathbb{N}$, Σ is the Ellis group of (X, T) and $E = E(G)$ (with its usual topology) is the (compact) set of all (not necessarily continuous) endomorphisms of G.

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If $\tau = \lim_\alpha T^{m_\alpha} \in \Sigma$, then (passing to a subnet, if necessary) one may assume that for each t in G and for $i = 1, 2, \ldots, k-1$

$$\lim_\alpha T^{m^i_\alpha}(t) = \theta_i(t)$$

and

$$\lim_\alpha T^{m^k_\alpha}\gamma_0 = u.$$

If $x = (x_1, x_2, \ldots, x_k) \in X$ and $\tau x = (y_1, y_2, \ldots, y_k)$, then by Lemma 4.2 (with $\theta_0 = \mathrm{id}_G$ in mind) for $i = 1, 2, \ldots, k-1$

$$y_i = \theta_i\left(\binom{k}{i}\gamma_0\right)\prod_{j=1}^{i}\theta_{i-j}\left(\binom{k-j}{i-j}x_j\right)$$

and

$$y_k = u\prod_{j=1}^{k}\theta_{k-j}(x_j).$$

Hence each τ in Σ corresponds to a k-fold $(\theta_1, \theta_2, \ldots, \theta_{k-1}, u)$ in $E^{k-1} \times G$. In the following lemma it is shown that this correspondence is in fact an embedding isomorphism, in which the product in $E^{k-1} \times G$ is as follows: if $\tau \cong (\theta_1, \ldots, \theta_{k-1}, u)$ and $\tau' \cong (\theta'_1, \ldots, \theta'_{k-1}, u')$, then $\tau'\tau \cong (\phi_1, \ldots, \phi_{k-1}, z)$, where for each $i = 1, 2, \ldots, k-1$,

$$\phi_i = \prod_{j=0}^{i}\binom{i}{j}(\theta'_{i-j} o \theta_j), \quad \text{and} \quad z = u'\prod_{j=1}^{k-1}\binom{k}{j}\theta'_{k-j}\circ\theta_j(\gamma_0)u.$$

For more information on these products see [11, 13].

The main theme of this section is to prove the following theorem which characterizes the topological centre of Σ. Our method is partly similar to [11, Example 3]. In what follows, for each n in $\mathbb{Z}$, by $n(\,)$ we mean the endomorphism $x \mapsto nx$ in $E(G)$.

THEOREM 6.1. *If G is a connected monothetic Lie group, then under the identification given above:*

$$\Lambda(\Sigma) = \{(n(\,), n^2(\,), \ldots, n^{k-1}(\,), u) \in \Sigma \mid n \in \mathbb{Z} \text{ and } u \in G\} \subset E^{k-1} \times G.$$

To prove this theorem, we need to mention some preliminaries and lemmas.

LEMMA 6.2. *Define $\Theta : \Sigma \to E^{k-1} \times G$ by $\Theta(\tau) = (\theta_1, \theta_2, \ldots, \theta_{k-1}, u)$ where $\tau \in \Sigma$ and $\theta_1, \theta_2, \ldots, \theta_{k-1}, u$ are as above. Then Θ is an embedding isomorphism of Σ into $E^{k-1} \times G$.*

PROOF. Θ is well defined. In fact, if for nets $\{m_\alpha\}$ and $\{m'_\beta\}$, $\tau = \lim_\alpha T^{m_\alpha}$, $\tau' = \lim_\beta T^{m'_\beta}$ are members of Σ and $\tau = \tau'$, we have to show that the associated elements $\Theta(\tau) = (\theta_1, \dots, \theta_{k-1}, u)$ and $\Theta(\tau') = (\theta'_1, \dots, \theta'_{k-1}, u')$ coincide in $E^{k-1} \times G$. If $x = (x_1, x_2, \dots, x_k) \in X$, then $\tau(x) = \tau'(x)$, hence

$$u \prod_{j=1}^{k} \theta_{k-j}(x_j) = u' \prod_{j=1}^{k} \theta'_{k-j}(x_j).$$

By taking $x = 0$, we obtain $u = u'$. Now let $1 \le l \le k-1$ and let $(x_1, x_2, \dots, x_k) \in X$ such that x_l is an arbitrary element of G and for each $i \ne l$, $x_i = 0$, the identity of G. Then it follows that $u\theta_{k-l}(x_l) = u'\theta'_{k-l}(x_l)$. But $u = u'$ hence $\theta_{k-l} = \theta'_{k-l}$ and Θ is well defined. That Θ is a group isomorphism is clear from the formula. To prove that Θ is continuous, let $\tau_\beta \to \tau$ in Σ. Let $(x_1, x_2, \dots, x_k) \in X$ and let $\tau_\beta(x_1, x_2, \dots, x_k) = (y_{1,\beta}, y_{2,\beta}, \dots, y_{k,\beta})$ and $\tau(x_1, x_2, \dots, x_k) = (y_1, y_2, \dots, y_k)$. If for each β, $\Theta(\tau_\beta) = (\theta_{1,\beta}, \dots, \theta_{(k-1),\beta}, u_\beta)$ and $\Theta(\tau) = (\theta_1, \dots, \theta_{k-1}, u)$, then $y_{k,\beta} = u_\beta \prod_{j=1}^{k} \theta_{k-j,\beta}(x_j)$, for all β, and $y_k = u \prod_{j=1}^{k} \theta_{k-j}(x_j)$. If we take $x_j = 0$, for all $j = 1, 2, \dots, k$, then since $y_{k,\beta} \to y_k$, we get $u_\beta \to u$. Again let $1 \le l \le k-1$ and let $(x_1, x_2, \dots, x_k) \in X$ such that x_l is an arbitrary element of G and, for each $i \ne l$, $x_i = 0$; then since $y_{k,\beta} \to y_k$ and $u_\beta \to u$ we have $\theta_{k-l,\beta} \to \theta_{k-l}$. Therefore Θ is an embedding isomorphism, as claimed. $\square$

LEMMA 6.3. *Let $\tau \in \Lambda(\Sigma)$ and let $\Theta(\tau) = (\theta_1, \dots, \theta_{k-1}, u)$, where Θ, $\theta_1, \theta_2, \dots, \theta_{k-1}$ and u are as in Lemma 6.2; then for each $i = 1, 2, \dots, k-1$, θ_i is continuous.*

PROOF. Let $\{t_\alpha\}$ be a net in G and let $t_\alpha \to t \in G$. Fix $1 \le i < k$. For each α let x_α (resp. x) be the element of X in which all of its coordinates are equal to 0 except its ith coordinate which is equal to t_α (resp. t). Then $x_\alpha \to x$ in X. Let $\tau = \lim_\beta T^{m_\beta} \in \Lambda(\Sigma)$ and let $\Theta(\tau) = (\theta_1, \dots, \theta_{k-1}, u)$. Assume that for each α, $\tau(x_\alpha) = (z_{1,\alpha}, z_{2,\alpha}, \dots, z_{k,\alpha})$ and $\tau(x) = (z_1, z_2, \dots, z_k)$. Since (X, T) is minimal (Theorem 4.1(i)) by Proposition 2.1 τ is continuous. Hence for each $j = 1, 2, \dots, k$, $z_{j,\alpha} \to z_j$, in particular, $z_{k,\alpha} \to z_k$. But $z_{k,\alpha} = (\lim_\beta T^{m^k_\beta}\gamma_0)\theta_{k-i}(t_\alpha)$ and $z_k = (\lim_\beta T^{m^k_\beta}\gamma_0)\theta_{k-i}(t)$, hence $\theta_{k-i}(t_\alpha) \to \theta_{k-i}(t)$, that is, θ_{k-i} is continuous and the lemma is proved. $\square$

PROOF OF THEOREM 6.1. Since G is a connected monothetic Lie group it is isomorphic (as a Lie group) to a torus $\mathbb{T}^m$ for some m (see [4]). Hence $\widehat{G}$ has no elements of finite order. Therefore by Theorem 4.1(i) (X, T) is minimal. Now it follows from Proposition 2.1 that $\Lambda(\Sigma) = \Sigma_c$. To prove the theorem let $\tau = \lim_\alpha T^{m_\alpha}$ be in $\Lambda(\Sigma)$ and let the mapping $\Theta : \Sigma \to E^{k-1} \times G$ be defined as in Lemma 6.2 and let $\Theta(\tau) = (\theta_1, \dots, \theta_{k-1}, u)$. Hence by Lemma 6.3 and the above results, θ_j is continuous for each $j = 1, 2, \dots, k-1$. Consider θ_j as an endomorphism on $\mathbb{T}^m$. Then $\theta_j(\eta_1, \eta_2, \dots, \eta_m) = (f_j(\eta_1), f_j(\eta_2), \dots, f_j(\eta_m))$

where $f_j(\eta) = \lim_\alpha \eta^{m_\alpha^j}$ for all η in $\mathbb{T}$. The continuity of θ_j implies that $f_j \in E(\mathbb{T})$ is continuous, hence $f_j = (\ )^{n_j}$ for some n_j in $\mathbb{Z}$, and so $\theta_j = n_j(\ )$ on G. We shall show that for every $j = 2, \ldots, k-1$, $n_j(\ ) = n_1^j(\ )$ on G. Fix p as a nonzero integer. Then for $\eta = \exp(1/p)$, $f_1(\eta) = \lim_\alpha (\eta)^{m_\alpha} = (\eta)^{n_1}$, hence $m_\alpha = n_1$ (mod. p), eventually. Thus for each $j = 2, \ldots, k-1$, $m_\alpha^j = n_1^j$ (mod. p), eventually. Therefore $\lim_\alpha \exp((1/p)m_\alpha^j) = \exp((1/p)n_1^j)$. It follows that for each integer q with $0 < q < p$ one has $\lim_\alpha \exp((q/p)m_\alpha^j) = \exp((q/p)n_1^j)$. On the other hand $\lim_\alpha \exp((q/p)m_\alpha^j) = \theta_j(\exp(q/p))$. Thus $\exp(q/p)^{n_1^j} = \exp(q/p)^{n_j}$. Since $(\ )^{n_1^j}$ and $(\ )^{n_j}$ are both continuous functions on $\mathbb{T}$ and the set $\{\exp(q/p) \mid p, q$ are positive integers, $0 < q < p\}$ is dense in $\mathbb{T}$ we have $(\ )^{n_1^j} = (\ )^{n_j}$. Thus $f_j = (\ )^{n_1^j}$ and so $\theta_j = n_1^j(\ )$. The latter means that $\tau = (n_1(\ ), n_1^2(\ ), \ldots, n_1^{k-1}, u)$. To prove the converse inclusion, assume that $\tau = (n(\ ), n^2(\ ), \ldots, n^{k-1}(\ ), u) \in \Sigma$ for some $n \in \mathbb{Z}$ and $u \in G$, then the continuity of $n^j(\ )$ for $j = 1, 2, \ldots, k-1$ implies that τ is continuous, hence (by Proposition 2.1) $\tau \in \Lambda(\Sigma)$, and the theorem is proved. □

REMARKS 6.4.

(a) Let (X, T) denote a Hahn-type skew-product system. It would be interesting to investigate the Ellis group Σ and $\Lambda(\Sigma)$ of the finite (that is, $k < \infty$) or infinite (that is, $k = \infty$) skew-product system (X, T). Hahn [9] proved that if the base group G has no element of finite order, then (X, T) is a (distal) minimal dynamical system which is also uniquely ergodic. It is readily proved by induction that for each n in $\mathbb{Z}$ and for each $(x_1, x_2, \ldots, x_k) \in X$

$$T^n(x_1, x_2, \ldots, x_k) = (y_{1,n}, y_{2,n}, \ldots, y_{k,n})$$

where for each $j = 1, 2, \ldots k$

$$y_{j,n} = p_j(n)\gamma_0 + p_{j-1}(n)x_1 + p_{j-2}(n)x_2 + \cdots + p_1(n)x_{j-1} + x_j$$

in which $p_j(n) = (1/j!)n(n-1)\cdots(n-j+1)$. The same formulas work for the case in which $k = \infty$. Let Σ be the Ellis group corresponding to (X, T). If $\tau \in \Sigma$ and for some net $\{m_\alpha\}$ in $\mathbb{Z}$, $\tau = \lim_\alpha T^{m_\alpha}$ then, by what we already discussed, if $k < \infty$ then τ corresponds to an element $(\theta_1, \theta_2, \ldots, \theta_{k-1}, u)$ in $E^{k-1} \times G$, and if $k = \infty$ then τ corresponds to $(\theta_1, \theta_2, \theta_3, \ldots) \in \prod_{j=1}^{\infty} E_j$ in which for all j, $E_j = E(G)$ and for all $t \in G$, $\theta_j(t) = \lim_\alpha p_j(m_\alpha)t$ (with $\theta_0 = \mathrm{id}_G$ in mind), and $u = \lim_\alpha p_k(m_\alpha)\gamma_0$. It is not hard to verify that under this correspondence if $\tau \cong (\theta_1, \ldots, \theta_{k-1}, u)$ and $\tau' \cong (\theta_1', \ldots, \theta_{k-1}', u')$, then $\tau'\tau \cong (\phi_1, \ldots, \phi_{k-1}, z)$, where for each $i = 1, 2, \ldots, k-1$, $\phi_i = \prod_{j=0}^{i} \theta_{i-j}' o \theta_j$ and $z = u' \prod_{j=1}^{k-1} \theta_{k-j}' o \theta_j(\gamma_0)u$. Similarly for the case $k = \infty$ if $\tau \cong (\theta_1, \theta_2, \theta_3, \ldots)$ and $\tau' \cong (\theta_1', \theta_2', \theta_3', \ldots)$ then $\tau'\tau \cong (\phi_1, \phi_2, \phi_3, \ldots)$, where for each i, $\phi_i = \prod_{j=0}^{i} \theta_{i-j}' o \theta_j$. With the same methods as in Lemmas 6.2 and 6.3 one may easily verify that if $k < \infty$, then the mapping $\Theta : \Sigma \to E^{k-1} \times G$ defined by $\Theta'(\tau) = (\theta_1, \theta_2, \ldots, \theta_{k-1}, u)$ where $\theta_1, \theta_2, \ldots, \theta_{k-1}$ and u are as above, is an embedding isomorphism, and if $k = \infty$, then the mapping $\Theta : \Sigma \to \prod_{j=1}^{\infty} E$, $\tau \mapsto (\theta_1, \theta_2, \theta_3, \ldots)$ is an embedding isomorphism. Also if $\tau \in \Lambda(\Sigma)$, then θ_i is continuous for each $i = 1, 2, \ldots, k-1$, (and for all i, if $k = \infty$).

Now with the method of Theorem 6.1 one may observe that if (X, T) is a finite or infinite Hahn-type skew-product dynamical system with the base group G which is a connected monothetic Lie group, then under the identifications given above the following statements hold.

(i) If $k < \infty$, then

$$\Lambda(\Sigma) = \{(p_1(n)(\,), p_2(n)(\,), \ldots, p_{k-1}(n)(\,), u) \in \Sigma \mid n \in \mathbb{Z} \text{ and } u \in G\}.$$

(ii) If $k = \infty$, then $\Lambda(\Sigma) \equiv \mathbb{Z}$.

(b) In part (a) let $k = 2$. Let $G = \mathbb{T}$ and let $\gamma_0 \in \mathbb{T}$ be irrational. Namioka [13] proved that for this case $\Sigma \cong E(\mathbb{T}) \times \mathbb{T}$ and characterized the topological centre of Σ explicitly. In fact, he showed that $\Lambda(\Sigma) \cong \mathbb{Z} \times \mathbb{T} = \{((\,)^n, u) : n \in \mathbb{Z} \text{ and } u \in \mathbb{T}\}$. So our results generalize Namioka's example. Also, as discussed in [13] for the case $k = 3$ in part (a), Σ is not isomorphic to the whole space $E(\mathbb{T}) \times E(\mathbb{T}) \times \mathbb{T}$. So it would be interesting to determine which part of $E^{k-1} \times G$ is covered by Σ (and also by Σ in Theorem 6.1). There is a conjecture due to Milnes [11] concerning this problem for the case when $G = \mathbb{T}$ and $k = 4$ in Theorem 6.1.

(c) We do not know whether for an arbitrary $k \in \mathbb{N}$,

$$\Lambda(\Sigma(X, T_M)) \cong \{(n(\,), n^2(\,), \ldots, n^{k-1}(\,), u) \mid n \in \mathbb{Z}, u \in G\}$$

and

$$\Lambda(\Sigma(X, T_H)) \cong \{(p_1(n)(\,), p_2(n)(\,), \ldots, p_{k-1}(n)(\,), u) \mid n \in \mathbb{Z}, u \in G\}$$

where (X, T_M) and (X, T_H) denote any Milnes-type and Hahn-type skew-product dynamical system, respectively. To decide this using Theorem 6.1 and part (a), we have to characterize for which pairs (n, u) in $\mathbb{Z} \times G$, one has $(n(\,), n^2(\,), \ldots, n^{k-1}(\,), u) \in \Sigma(X, T_M)$ and $(p_1(n)(\,), p_2(n)(\,), \ldots, p_{k-1}(n)(\,), u) \in \Sigma(X, T_H)$, respectively. Of course (by part (b) and a similar method for $\Sigma(X, T_M)$) if $k = 2$ and $G = \mathbb{T}$, then the answer to both questions is affirmative. Hence another question arises. Is it true that if $k < \infty$, then $\Lambda(\Sigma(X, T_M)) \neq \mathbb{Z} \neq \Lambda(\Sigma(X, T_H))$? Note however that, as stated in part (a), for the case $k = \infty$ we have $\Lambda(\Sigma(X, T_H)) \equiv \mathbb{Z}$.

(d) Following similar methods for the Ellis groups of Hahn-type and Milnes-type systems, one may show that if (X, T) is any skew-product dynamical system with the base group G which is a connected monothetic Lie group, then, with our notations of Section 3,

$$\Lambda(\Sigma(X, T)) \cong \{(Q_1(n)(\,), Q_2(n)(\,), \ldots, Q_{k-1}(n)(\,), u) \in \Sigma(X, T) \mid n \in \mathbb{Z}, u \in G\}.$$

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Also the product in $\Sigma(X, T)$ is given as follows. If $\tau \cong (\theta_1, \ldots, \theta_{k-1}, u)$ and $\tau' \cong (\theta'_1, \ldots, \theta'_{k-1}, u')$, then $\tau'\tau \cong (\phi_1, \ldots, \phi_{k-1}, z)$, where for each $i = 1, 2, \ldots, k-1$,

$$\phi_i = \prod_{j=0}^{i} J(i, j)\theta'_{i-j} o \theta_j \quad \text{and} \quad z = u' \prod_{j=1}^{k-1} J(k, j)\theta'_{k-j} o \theta_j(\gamma_0)u.$$

(See also [11, Theorem 1].)

(e) The mapping $\Gamma : (X, T) \to (X_f, V)$ in Theorem 5.3 induces a continuous homomorphism from Σ onto the Ellis group $\Sigma(\mathbb{Z}, X_f)$ of (X_f, V) (which is not necessarily an isomorphism). We believe that it is worth characterizing the topological centre of $\Sigma(\mathbb{Z}, X_f)$.

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A. JABBARI, Department of Mathematics, Ferdowsi University, PO Box 1159, Mashhad 91775, Iran
e-mail: shahzadeh@math.um.ac.ir

H. R. E. VISHKI, Department of Mathematics, Ferdowsi University, PO Box 1159, Mashhad 91775, Iran
and
Centre of Excellence in Analysis on Algebraic Structures (CEAAS), Ferdowsi University of Mashhad, Iran
e-mail: vishki@ferdowsi.um.ac.ir