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NORMED LIE ALGEBRAS

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0. Introduction. In this paper we attempt to define the notion of normed Lie algebra by endowing the corresponding algebraic concept with topological-metric properties.† More precisely, we define normed Lie algebras as being normed spaces possessing a Lie product, the latter satisfying a compatibility relation. It turns out that any normed algebra, in particular the algebra of continuous linear operators on a normed space, is a normed Lie algebra in the sense defined below, with the usual Lie product given by the additive commutator.

It seems that some algebraic features have within the framework of normed Lie algebras the natural topological extensions. We mention, for instance, the convergence of the well-known Campbell-Hausdorff formula. Let us also mention the occurence of a variant of the Kleinecke-Sirokov theorem, obtained via universal enveloping algebra, which might be unknown in this context.

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1. Definitions. In this section we give a collection of definitions, most of them being borrowed from the algebraic theory of Lie algebras, adapted to our topological conditions (for algebraic references, see [**1**; **3**; **5**; **6**].)

In what follows, $(\mathfrak{L}, |*|)$ will be a complex normed vector space (most, but by no means all, of the results are true on a real vector space). Suppose that $\mathfrak{L}$ is endowed with a Lie product

$$(1) \qquad (x, y) \to [x, y];$$

i.e, $[x, y]$ is a bilinear form with values in $\mathfrak{L}$, having the properites:

$$(1') \qquad [x, x] = 0 \text{ for every } x \in \mathfrak{L};$$

$$(1'') \qquad [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \text{ for any } x, y, z \in \mathfrak{L}.$$

From $(1')$ follows easily the property of anticommutativity of the Lie product:

$$(1''') \qquad [x, y] = -[y, x] \text{ for any } x, y \in \mathfrak{L}.$$

We recall that the relation $(1'')$ is known as the *Jacobi formula.*

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†After the paper had been written, the author discovered that such a notion had been considered in some older articles (for instantce E. B. Dynkin, *Normed Lie algebras and analytic groups*, Uspehi Mat. Nauk. (N.S.) *5* (1950), 135–186; available in Amer. Math. Soc. Transl. (No. 97)) in order to obtain the convergence of the Campbell-Hausdorff formula. However this problem is treated here in a somewhat different way.

We say that $\mathfrak{L}$ is a *normed Lie algebra* (NL-algebra) if the Lie product on $\mathfrak{L}$ has the property

$$|[x, y]| \leqq 2|x|\,|y| \text{ for any } x, y \in \mathfrak{L}. \tag{2}$$

From (2) it follows that the Lie product is a bicontinuous form on $\mathfrak{L}$. If $\mathfrak{L}$ is actually a Banach space and satisfies (2), we call it a *Banach-Lie algebra* (BL-algebra). From the corresponding result in normed spaces (N-spaces) it follows easily that any NL-algebra can be extended to a BL-algebra, uniquely determined up to an isomorphism, in which it is dense.

If $\mathfrak{U}$ is a normed algebra (N-algebra), then it becomes a NL-algebra with the usual Lie product

$$[x, y] = xy - yx.$$

We denote by $\mathfrak{U}_L$ the NL-algebra canonically associated with any N-algebra $\mathfrak{U}$.

Evidently, the relation (2) reflects this example and the constant 2 cannot be replaced with a smaller number, actually in the finite-dimensional case. We may consider instead of (2) the more general inequality

$$|[x, y]| \leqq 2M|x|\,|y|, \tag{2'}$$

where $M > 0$ is a constant. But taking eventually an equivalent norm on $\mathfrak{L}$, we can always assume that (2) holds.

It is possible to construct simple examples of NL-algebras which satisfy (2′) and are not necessarily generated by a pre-existing multiplicative structure. Take an N-space $\mathfrak{A}$, an N-algebra $\mathfrak{B}$, and an isomorphism ϕ of $\mathfrak{A}$ onto the N-space $\mathfrak{B}$. Then the relation

$$[a, b]_\phi = \phi^{-1}(\phi(a)\phi(b) - \phi(b)\phi(a)) \ (a, b \in \mathfrak{A})$$

defines a Lie product on $\mathfrak{A}$, satisfying (2′), with $M = |\phi|\,|\phi^{-1}|$.

However, for the sake of simplicity, we always suppose that (2) holds for NL-algebras.

If $\mathfrak{L}, \mathfrak{L}'$ are two NL-algebras, then a (NL-) *homomorphism* $\mathfrak{c}$ of $\mathfrak{L}$ into $\mathfrak{L}'$ will be a continuous linear operator $\mathfrak{c}: \mathfrak{L} \to \mathfrak{L}'$ with the property

$$\mathfrak{c}[x, y] = [\mathfrak{c}(x), \mathfrak{c}(y)] \text{ for any } x, y \in \mathfrak{L}.$$

We say that $\mathfrak{L}, \mathfrak{L}'$ are *isomorphic* NL-algebras if there is an invertible homomorphism from $\mathfrak{L}$ onto $\mathfrak{L}'$.

A *derivation* D of $\mathfrak{L}$ is a linear continuous operator on $\mathfrak{L}$ with the property

$$D[x, y] = [Dx, y] + [x, Dy] \text{ for any } x, y \in \mathfrak{L}. \tag{3}$$

From (3) it is easy to obtain the usual Leibniz formula

$$D^n[x, y] = \sum_{k=0}^{n} \binom{n}{k} [D^k x, D^{n-k} y]. \tag{4}$$

For any $x \in \mathfrak{L}$ we put

(5) $$D_x(y) = [x, y]$$

and on account of the Jacobi formula (1″), it follows that D_x is a derivation of $\mathfrak{L}$ and $|D_x| \leqq 2|x|$. Such a derivation D_x will be called an *inner derivation* of $\mathfrak{L}$. Denote by $\mathfrak{D}_0(\mathfrak{L})$ the set of all inner derivations of $\mathfrak{L}$, by $\mathfrak{D}(\mathfrak{L})$ the set of all derivations of $\mathfrak{L}$, and by $\mathfrak{B}(\mathfrak{L})$ the set of linear continuous operators on $\mathfrak{L}$. We have obviously

$$\mathfrak{D}_0(\mathfrak{L}) \subset \mathfrak{D}(\mathfrak{L}) \subset \mathfrak{B}(\mathfrak{L}),$$

and they are NL-algebras under the Lie product

$$[T, S] = TS - ST.$$

When $\mathfrak{L}$ is a BL-algebra, $\mathfrak{D}(\mathfrak{L})$ and $(\mathfrak{B}(\mathfrak{L}))_L$ are also BL-algebras.

Let us notice that the mapping $x \to D_x$ of $\mathfrak{L}$ onto $\mathfrak{D}_0(\mathfrak{L})$ is an homomorphism, since it is easily seen that

$$[D_x, D_y]\ (z) = D_{[x,y]}(z),$$

when x, y, z are arbitrary.

A sub-NL-algebra is a vector subspace closed under the Lie product.

An *ideal* $\mathfrak{J}$ of $\mathfrak{L}$ is a vector subspace $\mathfrak{J} \subset \mathfrak{L}$ such that $[\mathfrak{J}, \mathfrak{L}] \subset \mathfrak{J}$ (i.e., $[x, y] \in \mathfrak{J}$ for any $x, y \in \mathfrak{L}$).

If $\mathfrak{J}$ is an ideal of $\mathfrak{L}$, then we can construct the quotient Lie algebra $\mathfrak{L}' = \mathfrak{L}/\mathfrak{J}$. When $\mathfrak{L}$ is a BL-algebra and $\mathfrak{L}$ is a closed ideal in $\mathfrak{L}$, $\mathfrak{L}' = \mathfrak{L}/\mathfrak{J}$ is again a BL-algebra.

Denote by $\mathfrak{C}$ the set

(6) $$\{y \in \mathfrak{L} | [x, y] = 0 \text{ for every } x \in \mathfrak{L}\}.$$

We call $\mathfrak{C}$ the *center* of $\mathfrak{L}$; $\mathfrak{C}$ is a closed ideal in $\mathfrak{L}$.

Let $\mathfrak{L}$ be an NL-algebra and V an N-space. We say that θ is a *representation* of $\mathfrak{L}$ in V if θ is a homomorphism from $\mathfrak{L}$ into $(\mathfrak{B}(V))_L$.In particular, $x \to D_x$ is a representation of $\mathfrak{L}$ in $\mathfrak{L}$ called the *adjoint representation of* $\mathfrak{L}$. Obviously, the kernel in $\mathfrak{L}$ of the adjoint representation is just the center $\mathfrak{C}$ of $\mathfrak{L}$.

An element $x \in \mathfrak{L}$ is *nilpotent* (*quasi-nilpotent*) if there is an $n \geqq 1$ such that

$$D_x^n = 0\ (\lim_{n\to\infty} |D_x^n(y))|^{1/n} = 0 \text{ for all } y \in \mathfrak{L}).$$

If $\mathfrak{L}$ is a BL-algebra, then x is a quasi-nilpotent if and only if D_x is quasi-nilpotent as an element of $\mathfrak{B}(\mathfrak{L})$ (see [**2**, Problem 7] for details).

Finally, let us mention the following result, connected with the adjoint representation of a special Lie algebra:

Let H be a Hilbert space, $\mathfrak{B}(H)$ the B-algebra of linear continuous operators on H, $\mathfrak{C}$ the center of $\mathfrak{B}(H)$ (which coincides with the set of scalar multiples of identity), and $\mathfrak{L} = (\mathfrak{B}(H))_L/\mathfrak{C}$. Then $\mathfrak{L}$ is isometrically isomorphic with $\mathfrak{D}_0(\mathfrak{L})$ (an equivalent assertion and similar topics can be found in [**7**]).

This result and some others [**8**; **9**] seem to justify the introduction of normed Lie algebras and they might become a useful tool in the theory of linear operators.

2. Universal enveloping algebra. We are going to define a topologic equivalent of the notion of universal enveloping algebra.

Let $\mathfrak{L}$ be a fixed BL-algebra. Define

$$\mathfrak{L}^{(n)} = \begin{cases} \mathbf{C} & \text{if } n = 0 \\ \mathfrak{L} & \text{if } n = 1 \\ \underbrace{\mathfrak{L} \hat{\otimes} \ldots \hat{\otimes} \mathfrak{L}}_{n\text{-times}} & \text{if } n > 1, \end{cases} \tag{7}$$

where $\mathbf{C}$ is the complex field and $\mathfrak{L} \hat{\otimes} \ldots \hat{\otimes} \mathfrak{L}$ is the tensor product of $\mathfrak{L}$ with itself, endowed with the projective norm and complete for this topology.

Let $\mathfrak{T}_0$ be the collection of all formal series $\sum_{j=0}^{\infty} t_j$ with $t_j \in \mathfrak{L}^{(j)}$, and for $\rho > 0$ arbitrary, set

$$\mathfrak{T}_\rho = \left\{ t = \sum_{j=0}^{\infty} t_j \,\middle|\, t \in \mathfrak{T}_0, \sum_{j=0}^{\infty} \rho^j |t_j| < +\infty \right\}. \tag{8}$$

We can define in $\mathfrak{T}_\rho$ a product by extending the natural embedding

$$\mathfrak{L}^{(m)} \otimes \mathfrak{L}^{(n)} \subset \mathfrak{L}^{(m+n)}.$$

2.1. Proposition. *For each* $\rho > 0$, $\mathfrak{L}_\rho$ *is a unital* B*-algebra.*

Proof. Indeed, $\mathfrak{T}_\rho$ is a B-space as a direct sum of B-spaces, where we put for $t = \sum_{j \geqq 0} t_j$,

$$|t|_\rho = \sum_{j \geqq 0} \rho^j |t_j|. \tag{9}$$

Moreover, since the projective norm has the property

$$|t' \otimes t''| \leqq |t'|\,|t''| \quad (t', t'' \in \mathfrak{L}),$$

we can easily prove that $\mathfrak{T}_\rho$ is actually a B-algebra, by using the same argument as above to show that the product of two absolutely convergent series is an absolutely convergent series.

The algebra given by (8) will be called the *ρ-tensor algebra* associated with $\mathfrak{L}$. Evidently, the construction of $\mathfrak{T}_\rho$ uses only the fact that $\mathfrak{L}$ is a B-space.

Denote by $\mathfrak{R}_\rho$ the two-sided closed ideal generated in $\mathfrak{T}_\rho$ by the elements of the form

$$[x, y] - x \otimes y + y \otimes x \quad (x, y \in \mathfrak{L}). \tag{10}$$

The quotient $\mathfrak{U}_\rho = \mathfrak{T}_\rho / \mathfrak{R}_\rho$ is again a unital B-algebra. Since there is a natural embedding of $\mathfrak{L}$ into $\mathfrak{T}_\rho$, we can construct a canonical mapping $\mathfrak{c}_\rho$ from $\mathfrak{L}$ into $\mathfrak{U}_\rho$, which is obviously continuous. Furthermore, $\mathfrak{c}_\rho$ is a BL-homomorphism of $\mathfrak{L}$ into $(\mathfrak{U}_\rho)_L$.

By definition, the pair $(\mathfrak{U}_\rho, \mathfrak{c}_\rho)$ will be called the *ρ-universal enveloping algebra* of $\mathfrak{L}$. This definition may be justified by the next results.

2.2 PROPOSITION. *Let $\mathfrak{B}$ be a unital B-algebra and θ_0 a BL-homomorphism of $\mathfrak{L}$ into $\mathfrak{B}_L$. Then there is a unique B-homomorphism θ of $\mathfrak{U}_\rho$ into $\mathfrak{B}$ such that $\theta_0 = \theta\mathfrak{c}_\rho$ for every $\rho \geqq |\theta_0|$.*

Proof. Define θ' from $\mathfrak{T}_\rho$ into $\mathfrak{B}$ by putting

$$\theta'(1) = 1, \theta'(x_1 \otimes \ldots \otimes x_n) = \theta_0(x_1) \ldots \theta_0(x_n).$$

Since $|\theta_0| \leqq \rho$, it follows that $|\theta'(t_n)| \leqq \rho^n|t_n|$ for $t_n \in \mathfrak{L}^{(n)}$; hence, θ' can be prolonged continuously on $\mathfrak{T}_\rho$. Moreover, as θ_0 is a BL-homomorphism,

$$\theta'([x, y] - x \otimes y + y \otimes x) = 0.$$

hence, θ' induces an $\mathfrak{U}_\rho$ a B-homomorphism θ of $\mathfrak{U}_\rho$ into $\mathfrak{B}$. The relation $\theta_0 = \theta\mathfrak{c}_\rho$ and the uniqueness of θ follow from the fact that the algebra generated in $\mathfrak{U}_\rho$ by 1 and $\mathfrak{c}_\rho(\mathfrak{L})$ is dense.

2.3. THEOREM. *Let $\mathfrak{B}$ be a unital B-algebra and ϕ a BL-homomorphism of $\mathfrak{L}$ into $\mathfrak{B}_L$. If $(\mathfrak{B}, \phi)$ has the property that for any B-algebra $\mathfrak{B}'$ and for any BL-homomorphism σ of $\mathfrak{L}$ into $\mathfrak{B}_L'$ with $|\sigma| \leqq |\phi|$ there is a unique B-homomorphism $\bar{\sigma}$ of $\mathfrak{B}$ into $\mathfrak{B}'$ such that $\sigma = \bar{\sigma}\phi$, then $\mathfrak{B}$ is isomorphic with $\mathfrak{U}_\rho$ where $\rho = |\phi|$.*

Proof. On account of Proposition 2.2, we can write $\phi = \bar{\phi}\mathfrak{c}_\rho$, where $\bar{\phi}$ is a B-homomorphism of $\mathfrak{U}_\rho$ into $\mathfrak{B}$ and where $\rho = |\phi|$. On the other hand, since $|\mathfrak{c}_\rho| \leqq \rho = |\phi|$, by our assumption we can write $\bar{\mathfrak{c}}_\rho = \bar{\mathfrak{c}}_\rho\phi$, where $\bar{\mathfrak{c}}_\rho$ is a B-homomorphism of $\mathfrak{B}$ into $\mathfrak{U}_\rho$. Therefore

$$\phi = \bar{\phi}\bar{\mathfrak{c}}_\rho\phi = 1_{\mathfrak{B}}\phi$$

and

$$\mathfrak{c}_\rho = \bar{\mathfrak{c}}_\rho\phi\mathfrak{c}_\rho = 1_{\mathfrak{U}_\rho}\mathfrak{c}_\rho$$

and the uniqueness involves

$$\bar{\phi}\bar{\mathfrak{c}}_\rho = 1 \quad \text{and} \quad \bar{\mathfrak{c}}_\rho\bar{\phi} = 1 \quad ;$$

consequently, $\mathfrak{B}$ and $\mathfrak{U}_\rho$ are isomorphic.

We recall that if $\mathfrak{B}$ is a B-algebra and δ is a linear continuous operator on $\mathfrak{B}$ then δ is a *derivation* of $\mathfrak{B}$ if for any $a, b \in \mathfrak{B}$,

$$\delta(ab) = (\delta a)b + a(\delta b). \tag{11}$$

If we put $\delta_a(b) = ab - ba$, then δ_a is an *inner derivation of* $\mathfrak{B}$.

2.4 PROPOSITION. *If D is a derivation of a BL-algebra $\mathfrak{L}$, then for any ρ-universal enveloping algebra $(\mathfrak{U}_\rho, \mathfrak{c}_\rho)$ with $\rho > 1$, there is a unique derivation δ of $\mathfrak{U}_\rho$ with the property $\delta\mathfrak{c}_\rho = \mathfrak{c}_\rho D$. If $D = D_x$, then $\delta = \delta_{\mathfrak{c}_\rho(x)}$.*

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Proof. Let us define on $\mathfrak{L}^{(n)} \subset \mathfrak{T}_\rho$ the operator

$$\delta'(x_1 \otimes \ldots \otimes x_n) = Dx_1 \otimes x_2 \otimes \ldots \otimes x_n + \ldots + x_1 \otimes x_2 \otimes \ldots \otimes x_{n-1} \otimes Dx_n,$$

and prolong it on $\mathfrak{T}_\rho$ by setting $\delta' = 0$ on $\mathbf{C}$. Then, if $t \in \mathfrak{T}_\rho$, $t = \sum_{j \geqq 0} t_j$, we may write

$$|\delta'(t)| \leqq \sum_{j \geqq 0} |\delta'(t_j)| \leqq |D| \sum_{j \geqq 0} j|t_j| \leqq M|D| \sum_{j \geqq 1} \rho^j |t_j| = M'|t|,$$

where $M > 0$ depends only on $\rho > 1$.

It is not difficult to verify that δ' is a derivation on $\mathfrak{T}_\rho$. Let us also remark

$$\delta'([x, y] - x \otimes y + y \otimes x) = ([Dx, y] - Dx \otimes y + y \otimes Dx) + ([x, Dy] - x \otimes Dy + Dy \otimes x) \in \mathfrak{N}_\rho;$$

hence, δ' induces on $\mathfrak{U}_\rho$ an operator δ which is again a derivation. The equality $\delta \mathfrak{c}_\rho = \mathfrak{c}_\rho D$ is immediate and it implies the uniqueness of δ, because 1 and $\mathfrak{c}_\rho(\mathfrak{L})$ generate a dense algebra of $\mathfrak{U}_\rho$. If $D = D_x$, then for any $y \in \mathfrak{L}$ we have

$$\begin{aligned}\delta \mathfrak{c}_\rho(y) &= \mathfrak{c}_\rho D_x(y) = \mathfrak{c}_\rho([x, y]) \\ &= \mathfrak{c}_\rho(x)\mathfrak{c}_\rho(y) - \mathfrak{c}_\rho(y)\mathfrak{c}_\rho(x) \\ &= \delta_{\mathfrak{c}_\rho(x)}(\mathfrak{c}_\rho(y));\end{aligned}$$

hence, $\delta = \delta_{\mathfrak{c}_\rho(x)}$.

The next result is an application of the existence and the properties of ρ-universal enveloping algebras and can be considered as a variant of the Kleinecke-Sirokov theorem [**4**] for BL-algebras.

2.5. THEOREM. *Let D be a derivation on $\mathfrak{L}$ with the property that $D^2x = 0$ for a certain $x \in \mathfrak{L}$. Then the element Dx is quasi-nilpotent in $\mathfrak{L}$.*

Proof. Let us take a $\rho \geqq 2$ and denote by δ the derivation induced by D on $\mathfrak{U}_\rho$, on account of the previous proposition. Notice that $\phi(x) = D_x (x \in \mathfrak{L})$ is a BL-homomorphism of $\mathfrak{L}$ into $(\mathfrak{B}(\mathfrak{L}))_L$ and $|\phi| \leqq 2$; hence, by Proposition 2.2 there is a B-homomorphism θ of $\mathfrak{U}_\rho$ into $\mathfrak{B}(\mathfrak{L})$ such that

$$\phi(x) = \theta(\mathfrak{c}_\rho(x)) \quad (x \in \mathfrak{L}).$$

On the other hand, $\delta^2 \mathfrak{c}_\rho(x) = \mathfrak{c}_\rho D^2 x = 0$; hence, by the Kleinecke-Sirokov theorem for B-algebras [**4**], the element $\delta \mathfrak{c}_\rho(x)$ is quasi-nilpotent in $\mathfrak{U}\rho$. Consequently, we may write

$$\begin{aligned}\lim_n |(\phi(Dx))^n|^{1/n} &= \lim_n |(\theta \mathfrak{c}_\rho(Dx))^n|^{1/n} \\ &= \lim_n |\theta((\mathfrak{c}_\rho(Dx))^n)|^{1/n} \\ &\leqq \lim_n |(\mathfrak{c}_\rho(Dx))^n|^{1/n} \\ &= \lim_n |(\delta \mathfrak{c}_\rho(x))^n|^{1/n} = 0;\end{aligned}$$

hence, $|D_{Dx}{}^n|^{1/n} \to 0$ and this means that Dx is quasi-nilpotent in $\mathfrak{L}$.

Our notion of universal enveloping algebra seems, in general, not to enjoy a structure as described by the Poincare-Birchoff-Witt theorem, in the main, since the basis problem is a painful one in Banach spaces. In the following we give a much weaker result, corresponding to this well-known theorem.

We say that $\mathfrak{A} = \{x_j | j \in \mathfrak{J}\} \subset \mathfrak{L}$ is a *total* set in $\mathfrak{L}$ if the span of $\{x_j | j \in \mathfrak{J}\}$ is a dense subspace of $\mathfrak{L}$. Let us suppose that the set of indices $\mathfrak{J}$ is ordered.

A monomial of the form

$$x_{j1} \otimes \ldots \otimes x_{jn}$$

where $j_1 \leqq j_2 \leqq \ldots \leqq j_n$, $x_{jk} \in \mathfrak{A}$, and $\mathfrak{A}$ is total in $\mathfrak{L}$, will be called *standard*.

2.6 Lemma. *Every element $t \in \mathfrak{T}_\rho$ of the form $t = \sum_{j=1}^m t_j$ where*

$$t_j = \sum \lambda_{j1} \ldots {}_{jn} x_{j1} \otimes \ldots \otimes x_{jn}, \ \lambda_{j1} \ldots {}_{jn} \in \mathbf{C}, x_{jk} \in \mathfrak{L},$$

is congruent modulo $\mathfrak{R}_\rho$ to a linear combination of 1 *and standard monomials.*

Proof. Indeed, the formula

$$x_{j1} \otimes \ldots \otimes x_{jn} \equiv x_{j1} \otimes \ldots \otimes x_{jk+1} \otimes x_{jk} \otimes \ldots \otimes x_{jn} - x_{j1} \otimes \ldots \otimes [x_{jk}, x_{jk+1}] \otimes \ldots \otimes x_{jn} \pmod{\mathfrak{R}_\rho},$$

involves the result via an induction hypothesis on the degree of the monomials.

2.7. Lemma. *The elements $t \in \mathfrak{T}_\rho$, described in the previous lemma, are dense in $\mathfrak{T}_\rho$.*

Proof. The assertion follows from the fact that $\mathfrak{A}$ is a total set in $\mathfrak{L}$.

2.8. Theorem. *Let $\mathfrak{A}$ be a total set in the BL-algebra $\mathfrak{L}$. Then the cosets of* 1 *and the standard monomials with elements of $\mathfrak{A}$ form a dense set in $\mathfrak{U}_\rho$ for any $\rho > 0$.*

Proof. The result follows from Lemmas 2.6 and 2.7.

3. The Campbell-Hausdorf formula. In this section we show that the Campbell-Hausdorff formula is convergent in BL-algebras in a neighbourhood of the origin.

Let $\mathfrak{L}$ be a BL-algebra and $\mathfrak{T}_\rho$ its ρ-tensor algebra, $\rho > 0$ arbitrary. It will be convenient to denote the product of two elements $t', t'' \in \mathfrak{T}_\rho$ simply by $t't''$. There is a canonical mapping i_ρ of $\mathfrak{L}$ into $\mathfrak{T}_\rho$; we denote by $\mathfrak{L}_\rho$ the range of i_ρ. Consider now the BL-algebra $(\mathfrak{T}_\rho)_L$ and let $\mathfrak{L}\hat{}_\rho$ be the BL-algebra generated in $(\mathfrak{T}_\rho)_L$ by $\mathfrak{L}_\rho$. Obviously, the Lie product of two elements $t', t'' \in (\mathfrak{T}_\rho)_L$ is given by

$$[t', t''] = t't'' - t''t'.$$

As a matter of fact, we can construct in a similar way the BL-algebra $\mathfrak{L}\hat{}_\rho$ generated in $(\mathfrak{T}_\rho)_L$ by any B-space, which is not necessarily a BL-algebra.

Denote by $\mathfrak{x}_\rho$ the restriction of the identity to $\mathfrak{L}\hat{}_\rho$.

3.1. Proposition. *The pair $((\mathfrak{T}_\rho)_L, \mathfrak{x}_\rho)$ is a* 1*-universal enveloping algebra of $\mathfrak{L}\hat{}_\rho$.*

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Proof. Remark that $|\mathfrak{x}_\rho| = 1$ and take a B-algebra $\mathfrak{B}$ and $\sigma: \mathfrak{L}^\wedge{}_\rho \to \mathfrak{B}_L$ a homomorphism, $|\sigma| \leqq 1$. As $\mathfrak{L}^\wedge{}_\rho \supset \mathfrak{L}_\rho$, we have that there is a unique homomorphism $\bar{\sigma}$ of $\mathfrak{T}_\rho$ into $\mathfrak{B}$ such that $\sigma = \bar{\sigma}\mathfrak{x}_\rho$ (see the proof of Proposition 2.2). By Theorem 2.3, $(\mathfrak{T}_\rho)_L$ is isomorphic with the 1-universal enveloping algebra of $\mathfrak{L}^\wedge{}_\rho$.

As $\mathfrak{T}_\rho$ is a Banach algebra, we can define the functions

$$\exp(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}, \tag{12}$$

where the series is always convergent, and

$$\log(1 + t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^n}{n}, \tag{13}$$

whenever the right side is convergent (for example, if $|t|_\rho < 1$). Moreover,

$$\exp(\log(t)) = \log(\exp(t)) = t,$$

if the function "log" is defined in t.

3.2. Lemma. *There is a neighbourhood of the origin $\mathfrak{N}_\rho$ in $\mathfrak{T}_\rho$ such that for any pair $t, s \in \mathfrak{N}_\rho$ the element $v = \log(\exp(t)\exp(s))$ exists in $\mathfrak{T}_\rho$.*

Proof. Indeed,

$$\exp(t)\exp(s) = 1 + \sum_{p+q\geqq 1} \frac{t^p s^q}{p!q!};$$

whence formally,

$$v = \log(\exp(t)\exp(s)) = \sum_{n\geqq 1} \sum \frac{(-1)^{m+1}}{m} \frac{t^{p_1}s^{q_1} \ldots t^{p_m}s^{q_m}}{p_1!q_1! \ldots p_m!q_m!}, \tag{14}$$

where the inner sum is over

$$p_1 + \ldots + p_m + q_1 + \ldots + q_m = n.$$

Take $\mathfrak{N}_\rho = \{t \in \mathfrak{L}_\rho \mid |t|_\rho < 1/2\}$ and $t, s \in \mathfrak{N}_\rho$. Then we can write

$$\begin{aligned} \sum \frac{|t|^{p_1+\ldots+p_m}|s|^{q_1+\ldots+q_m}}{mp_1!q_1! \ldots p_m!q_m!} &\leqq \sum_{p+q=n} |t|^p|s|^q \\ &\leqq \sum_{p=0}^{n} \binom{n}{p} |t|^p|s|^{n-p} \\ &\leqq (|t| + |s|)^n, \end{aligned}$$

where the first sum is over

$$p_1 + \ldots + p_m + q_1 + \ldots + q_m = n;$$

therefore, (14) is a convergent series and $v \in \mathfrak{T}_\rho$.

3.3. LEMMA. *If $t, s \in \mathfrak{L}^\wedge_\rho$, then we have the following formula*:

$$(15)\quad \sum \frac{(-1)^{m+1}}{m} \frac{t^{p_1}s^{q_1}\dots t^{p_m}s^{q_m}}{p_1!q_1!\dots p_m!q_m!}$$

$$= \frac{1}{n}\sum \frac{(-1)^{m+1}}{m} \frac{[\dots[\overbrace{[t,t],\dots,t]}^{p_1},\overbrace{s],\dots,s]}^{q_1},\dots \overbrace{s],\dots,s]}^{q_m}}{p_1!q_1!\dots p_m!q_m!}$$

where both sums are over

$$p_1 + \dots + p_m + q_1 + \dots + q_m = n.$$

Proof. This assertion is well known in the algebraic theory of Lie algebras. Take two elements $t, s \in \mathfrak{L}$ and consider the algebra $\mathfrak{T}_0$ of formal series; in $\mathfrak{T}_0$ take the Lie algebra generated by t and s, say $\mathfrak{L}(t, s)$; then the element $v = \log(\exp(t)\exp(s)) \in \mathfrak{L}(s, t)$ by the Campbell-Hausdorff theorem [**3**]. In particular, the homogeneous terms of (14) are in $\mathfrak{L}(t, s)$ and they have the form (15) (for details, see [**3**; **6**]).

3.4. PROPOSITION. *There is a neighbourhood $\mathfrak{N}^\wedge_\rho$ of the origin in $\mathfrak{L}^\wedge_\rho$ such that for any pair $t, s \in \mathfrak{N}^\wedge_\rho$ the element $v = \log(\exp(t)\exp(s))$ is also in $\mathfrak{L}_\rho^\wedge$.*

Proof. If we take $\mathfrak{N}^\wedge_\rho = \mathfrak{N}_\rho \cap \mathfrak{L}^\wedge_\rho$, where $\mathfrak{B}_\rho$ is given by Lemma 3.2, then for any $t, s \in \mathfrak{N}^\wedge_\rho$ we obtain that $v \in \mathfrak{L}^\wedge_\rho$ on account of Lemma 3.3.

3.5. THEOREM. *Let $\mathfrak{L}$ be a BL-algebra, $\mathfrak{B}$ a B-algebra, and ϕ a homomorphism of $\mathfrak{L}$ into $\mathfrak{B}_L$. There is a neighbourhood of the origin $\mathfrak{N}$ in $\mathfrak{L}$ such that for any $x, y \in \mathfrak{N}$ there is an element $z \in \mathfrak{L}$ satisfying the equality*

$$\exp \phi(x) = \exp \phi(x) \exp \phi(y).$$

Proof. First, let us show that it suffices to solve the problem for a ρ-universal enveloping algebra $(\mathfrak{U}_\rho, \mathfrak{c}_\rho)$ of $\mathfrak{L}$, with $\rho \geqq |\phi|$. Suppose that there is an element $z \in \mathfrak{L}$ such that

$$\exp \mathfrak{c}_\rho(z) = \exp \mathfrak{c}_\rho(x) \exp \mathfrak{c}_\rho(y).$$

According to Proposition 2.2, there is a B-homomorphism $\bar{\phi}$ of $\mathfrak{U}_\rho$ into $\mathfrak{B}$ such that $\phi = \bar{\phi}\mathfrak{c}_\rho$; therefore,

$$\begin{aligned}\bar{\phi} \exp \mathfrak{c}_\rho(z) &= \exp \bar{\phi}(\mathfrak{c}_\rho(z)) = \exp \phi(z)\\ &= \bar{\phi}(\exp \mathfrak{c}_\rho(x)\exp \mathfrak{c}_\rho(y))\\ &= \exp \phi(x) \exp \phi(y).\end{aligned}$$

Denote now by $\mathfrak{N}'$ the set $\{x \in \mathfrak{L} \mid |x| < 1/4\}$ and take $x, y \in \mathfrak{N}'$ arbitrary. Define the element $z = \sum_{n=1}^\infty z_n$, where

$$(16)\quad z_n = \frac{1}{n}\sum \frac{(-1)^{m+1}}{m} \frac{[\dots[\overbrace{[x,x],\dots,x]}^{p_1},\overbrace{y],\dots,y]}^{q_1},\dots,\overbrace{y],\dots,y]}^{q_m}}{p_1!q_1!\dots p_m!q_m!},$$

and the sum is over

$$p_1 + \ldots + p_m + q_1 + \ldots + q_m = n.$$

Remark that z is well defined as an element of $\mathfrak{L}$. Indeed, since $\mathfrak{L}$ is a BL-algebra, for arbitrary $x_1, \ldots, x_n \in \mathfrak{L}$,

$$|[x_1, [x_2, \ldots [x_{n-1}, x_n], \ldots]| \leqq 2^n |x_1| \ldots |x_n|;$$

therefore,

$$\begin{aligned} |z_n| &\leqq \sum \frac{(2|x|)^{p_1+\ldots+p_m}(2|y|)^{q_1+\ldots+q_m}}{mp_1!q_1!\ldots p_m!q_m!} \\ &\leqq \frac{1}{n} \sum_{p+q=n} (2|x|)^p(2|y|)^q \\ &\leqq \frac{1}{n} \sum_{p=0}^{n} \binom{n}{p} 2(|x|)^p (2|y|)^{n-p} \\ &= \frac{1}{n} 2^n(|x| + |y|)^n \end{aligned}$$

where the first sum is over

$$p_1 + \ldots p_m + q_1 + \ldots + q_m = n.$$

As $2(|x| + |y|) < 1$, the series defining z is convergent in $\mathfrak{L}$. If i_ρ is the canonical mapping of $\mathfrak{L}$ into $\mathfrak{T}_\rho$, then

$$i_\rho([a, b]) \equiv [i_\rho(a), i_\rho(b)] \pmod{\mathfrak{N}_\rho}, \tag{17}$$

for any $a, b \in \mathfrak{L}$ on account of the definition of $\mathfrak{N}_\rho$. Take now $\mathfrak{N} = \mathfrak{N}' \cap i_\rho^{-1}(\mathfrak{N}\hat{}_\rho)$, where $\mathfrak{N}\hat{}_\rho$ is given by Proposition 3.4, and $x, y \in \mathfrak{N}$. Denote by $t = i_\rho(x)$, $s = i_\rho(y)$, and let v be the element $\log(\exp(t)\exp(s)) \in \mathfrak{L}\hat{}_\rho$. If $\bar{\mathfrak{c}}_\rho$ is the canonical mapping of $\mathfrak{T}_\rho$ into $\mathfrak{U}_\rho$, then by (14), (15), (16), and (17) we can write

$$\begin{aligned} \mathfrak{c}_\rho(z) &= \bar{\mathfrak{c}}_\rho(i_\rho(z)) = \bar{\mathfrak{c}}_\rho(v) \\ &= \bar{\mathfrak{c}}_\rho(\log(\exp i_\rho(x) \exp i_\rho(y)) \\ &= \log(\exp \mathfrak{c}_\rho(x) \exp \mathfrak{c}_\rho(y)), \end{aligned}$$

and the proof is finished.

This result is a variant of the Campbell-Hausdorff theorem, stated in our conditions, and the relation given by (16) is just the Campbell-Hausdorff formula.

4. Solvable *NL*-algebras. In the following we deal with solvable NL-algebras and give structure theorems for them.

We say that an NL-algebra $\mathfrak{L}$ is N-solvable [**8**], where $N \leqq \aleph_0$ is a cardinal, if there is a system of ideals $\{\mathfrak{J}_n\}_{0 \leqq n < N}$ with the properties:

(1) $0 = \mathfrak{J}_0 \subset \mathfrak{J}_1 \subset \ldots . \subset \mathfrak{J}_n \subset \ldots \subset \mathfrak{L}$;
(2) $\dim \mathfrak{J}_n = n$ $(0 \leqq n < N)$;
(3) $\mathfrak{L} = \bigcup_{0 \leqq n < N} \mathfrak{J}_n$.

Let us remark that an N-solvable infinite-dimensional NL-algebra is never complete. Indeed, if $\mathfrak{L}$ were complete, since the $\mathfrak{J}_n$ being finite-dimensional spaces are closed, there would result from (3), on account of a Baire theorem, that at least one $\mathfrak{J}_n$ has a non-void interior; therefore, $\mathfrak{L}$ would be finite-dimensional.

We say that two elements $x, y \in \mathfrak{L}$ are *commuting* if $[x, y] = 0$. It is easy to see that if $x, y \in \mathfrak{L}$ are commuting then the corresponding derivations D_x, D_y are commuting as linear operators.

Let us remark that Theorem 2.5 is valid for any not-necessarily-complete NL-algebra $\mathfrak{L}$, taking in the proof instead of $\mathfrak{L}$ its completion $\mathfrak{L}^\wedge$ and the conclusion follows also in $\mathfrak{L}^\wedge$.

4.1. PROPOSITION. *Let $\mathfrak{L}$ be an N-solvable NL-algebra. Then for any $n > 0$, $n < N$, one of the following properties holds*:

(α) *For any $x \in \mathfrak{J}_n$ one has $D_x = 0$.*

(β) *There is a non-null ideal $\mathfrak{Q}$ of $\mathfrak{L}$, $\mathfrak{Q} \subset \mathfrak{J}_n$ such that the elements of $\mathfrak{Q}$ are commuting quasi-nilpotent.*

Proof. We get this result by induction with respect to n. When $n = 1$, we have $\mathfrak{J}_1 = \{\lambda x_1 | \lambda \in \mathbf{C}\}$; hence, each $x \in \mathfrak{L}$ has the property

$$[x_1, x] = \lambda_x x_1.$$

If $\lambda_x = 0$ for any $x \in \mathfrak{L}$, then $\mathfrak{J}_1$ satisfies (α). When $\lambda_{x_0} \neq 0$ for a certain $x_0 \in \mathfrak{L}$, then $x_1 = \lambda_{x_0}^{-1} [x_1, x_0]$, whence $D^2_{x_1}(x_0) = 0$. By Theorem 2.5 we have that $[x_1, x_0]$ is quasi-nilpotent in $\mathfrak{L}^\wedge$; thus, $x_1 = \lambda_{x_0}^{-1} [x_1, x_0]$ is quasi-nilpotent in $\mathfrak{L}^\wedge$. Hence (β) holds. Assume now that the assertion is true for $n = k$. If (β) is valid for $n = k$, then (β) is also valid for $n = k + 1$; hence, we may consider only the case when (α) is fulfilled for $n = k$. As $\mathfrak{J}_{k+1} \supset \mathfrak{J}_k$, we can take a basis $x_1, \ldots, x_k, x_{k+1}$ in $\mathfrak{J}_{k+1}$ such that $x_j \in \mathfrak{J}_k$ $(1 \leqq j \leqq k)$. For any $y \in \mathfrak{L}$ we have

$$[x_{k+1}, y] = \eta_1 x_1 + \ldots + \eta_k x_k + \eta_{k+1} x_{k+1}.$$

Since $\mathfrak{J}_k$ has the property (α), it follows that $[x_{k+1}, [x_{k+1}, y]] = 0$; hence, $q_y = [x_{k+1}, y] = D_{x_{k+1}}(y)$ is quasi-nilpotent in $\mathfrak{L}^\wedge$ (Theorem 2.5) and $[q_y, \mathfrak{J}_{k+1}] = 0$. If $q_y = 0$ for any $y \in \mathfrak{L}$, then $D_{x_{k+1}} = 0$ and, as $x_k, \ldots, x_k, x_{k+1}$ is a basis in $\mathfrak{J}_{k+1}$ and $D_{x_j} = 0$ $(j = 1, \ldots . , k + 1)$, it follows that $D_x = 0$ for every $x \in \mathfrak{J}_{k+1}$. Hence (α) holds. If $q_y \neq 0$ for at least one $y \in \mathfrak{L}$, then the set $\mathfrak{Q}$ of quasi-nilpotent elements of $\mathfrak{J}_{k+1}$ is non-null. Since $\mathfrak{J}_k$ satisfies (α) we have $[\mathfrak{J}_{k+1}, \mathfrak{J}_{k+1}] = 0$. In particular, $\mathfrak{Q}$ contains only commuting elements. If $q_1, \ldots, q_m \in \mathfrak{Q}$, then $q = \alpha_1 q_1 + \ldots + \alpha_m q_m$ is again a quasi-nilpotent element of $\mathfrak{J}_{k+1}$ because $D_q = \alpha_1 D_{q_1} + \ldots + \alpha_m D_{q_m}$ and the derivations $D_{q_1}, \ldots, D_{q_m}$ are commuting quasi-nilpotent operators in $\mathfrak{B}(\mathfrak{L}^\wedge)$. We have also $[\mathfrak{L}, \mathfrak{Q}] \subset [\mathfrak{L}, \mathfrak{J}_{k+1}] \subset \mathfrak{Q}$; therefore, $\mathfrak{Q}$ is an ideal of $\mathfrak{L}$ and (β) is fulfilled.

This result can be slightly improved if we suppose that $\mathfrak{L}$ is a sub-NL-algebra of $\mathfrak{B}_L$, where $\mathfrak{B}$ is a unital B-algebra. In case of linear operators on B-spaces, a first variant is implicitly contained in [**8**]. A more explicit formulation can be found in [**9**].

4.2. PROPOSITION. *Let $\mathfrak{L}$ be a sub-NL-algebra of $\mathfrak{B}_L$, where $\mathfrak{B}$ is a unital B-algebra. If $\mathfrak{L}$ is N-solvable, then for any $n > 0$, $n < N$, one of the following properties holds*:

(i) *Any $x \in \mathfrak{J}_n$ has the form $x = \lambda + q$, where λ is scalar, q quasi-nilpotent in $\mathfrak{B}$, commuting with $\mathfrak{L}$.*

(ii) *There is an $x \in \mathfrak{J}_n$ commuting with $\mathfrak{L}$ such that the spectrum of x in $\mathfrak{B}$ has at least two points.*

(iii) *There is a non-null ideal $\mathfrak{Q}$ of $\mathfrak{L}$, $\mathfrak{Q} \subset \mathfrak{J}_n$, containing commuting quasi-nilpotent (in $\mathfrak{B}$) elements.*

Proof. We apply Proposition 4.1 which in this case can be formulated a little more strongly. Namely, we can use in the proof of Proposition 4.1 actually the Kleinecke-Sirokov theorem [**4**] for B-algebras and we get quasi-nilpotent elements in $\mathfrak{B}$. Then, as $\mathfrak{L}$ is N-solvable, the condition (β) in Proposition 4.1 is just the condition (iii). If (α) is fulfilled for $\mathfrak{J}_n$, we have two cases:

(1) The spectrum of each element $x \in \mathfrak{J}_n$ contains only one point; hence $x = \lambda + q$ where q is quasi-nilpotent. Since $D_x = 0$ for any x it follows that $qa - aq = 0$ for every $a \in \mathfrak{J}$; therefore, we have (i).

(2) There is an element $x \in \mathfrak{J}_n$ such that its spectrum has at least two points. As $D_x = 0$, (ii) holds.

Such results can be used in order to obtain infinite-dimensional versions of a finite-dimensional theorem of Lie [**8**; **9**].

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