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Introduction – from type theory and homotopy theory to univalent foundations

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We give an overview of the main ideas involved in the development of homotopy type theory and the univalent foundations of Mathematics programme. This serves as a background for the research papers published in the special issue.

1. Introduction

This special issue is devoted to a new area of research, generally known as homotopy type theory, which connects type theory and homotopy theory, and to the univalent foundations of Mathematics programme, formulated by Vladimir Voevodsky, which seeks to develop a new, computational foundation of Mathematics on the basis of type theories which include axioms (such as the univalence axiom) motivated by homotopy theory.

Type theory originated within mathematical logic in the work of Russell, who used the notion of a type in order to resolve the paradoxes that arise from the unrestricted formation of collections. Since then, thanks to the work of several logicians and theoretical computer scientists (including Church, de Bruijn, Curry, Howard, Scott and Martin-Löf), type theory has evolved into a rich, independent discipline and has found significant real-world applications. In particular, type theories have been implemented in computer systems (such as Coq and Agda) which have been used for the formalization of large mathematical proofs (Gonthier *et al.* 2013) and the verification of the correctness of computer programs (Leroy 2009). The fundamental feature that distinguishes type theories from set theories is that, while in set theories all mathematical objects are treated indiscriminately as sets, within type theories they are classified using the primitive notion of a type, in a way that is analogous to that in which expressions are classified into

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data-types in programming languages. Indeed, a type theory can also be seen as a programming language with a rich typing mechanism, capable of expressing sophisticated software specifications, and the development of some modern programming languages has been guided by exploiting this very fact.

Homotopy theory, on the other hand, is a branch of algebraic topology which is generally concerned with the problem of classifying topological spaces up to a suitable notion of equivalence (e.g. weak homotopy equivalence), making precise the idea that one is a continuous deformation of the other. The subject has a long history, which is far too rich and complex to be summarized here. Suffice it to say that, over the years, many notions and techniques that arose originally in homotopy theory have found important applications in many other areas of Mathematics. For example, ideas from homotopy theory have been fundamental in the recent creation of derived algebraic geometry (Lurie 2004; Toën and Vezzosi 2005) and higher-dimensional category theory (Joyal 2008; Lurie 2009; Rezk 2001). A crucial role in this cross-fertilization has been played by seminal work of Quillen (1967), which developed an axiomatic approach to homotopy theory based on structures now generally called Quillen model categories. Roughly speaking, a Quillen model category is a category equipped with additional structure (e.g. a distinguished class of maps playing the role of weak homotopy equivalences) which allows one to reproduce in a general context some of the classical development of homotopy theory. This general development can then be instantiated in any of the numerous examples of Quillen model categories that exist in Mathematics.

The first indication of a connection between type theory and homotopy theory was the discovery by Hofmann and Streicher that Martin-Löf type theory admits an interpretation in the category of groupoids (Hofmann and Streicher 1998). This model was later generalized in two different, although related, ways. On the one hand, Awodey and Warren discovered that Martin-Löf's type-theoretic rules for identity types can be interpreted in any Quillen model category (Awodey and Warren 2009) (subject to subtle conditions which were investigated further by Warren (2008) and by van den Berg and Garner (2012)). The discovery of this new class of models arose from the basic observation that, if we think of types as spaces and of elements of types as points, then it is natural to think of proofs of equalities between two elements of a type (as given by elements of identity types) as paths connecting the two points. Accordingly, families of proofs of equalities can be regarded as homotopies (which are suitable families of paths). At around the same time, Voevodsky discovered that the type theory underpinning the Coq proof assistant has a model in the category of simplicial sets, in which types are interpreted as Kan complexes (Voevodsky 2009). Simplicial sets is one of the most fundamental examples of a Quillen model category, and the specific interpretation given to identity types by Voevodsky agrees with the general one given independently by Awodey and Warren. The discovery of these homotopical models is important because it gives us a clear, precise, topological intuition for working with constructive type theories, for which no natural models were previously available.

On the basis of this work, several other researchers began to explore the topic in depth. In particular, it was soon shown by Gambino and Garner that the syntactic category associated to Martin-Löf type theories can be equipped with a weak factorization system

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(a structure closely related to that of a Quillen model category) (Gambino and Garner 2008) and by van den Berg and Garner and, independently, by Lumsdaine, that every type can be equipped with the structure of a weak ω-groupoid (Lumsdaine 2010; van den Berg and Garner 2011). Around the same time, Voevodsky introduced the univalence axiom and started to develop a new approach to the formalization of Mathematics in type theory, using the proof assistant Coq. A paper describing the current state of this ground breaking and influential library, along with the associated Coq files themselves, is part of this special issue. Further impetus for the development of the subject was recently given by a special thematic program on univalent foundations, organized at the Institute for Advanced Study in Princeton in the academic year 2012/13 by Awodey, Coquand, and Voevodsky. Several of the papers in this special issue were written during that year by participants in the program, recording some of the advances made during that time.

2. The special issue

The papers in this special issue can be divided roughly into three thematic groups. The first group, comprising the papers by Ahrens, Kapulkin and Shulman; by Pelayo, Voevodsky and Warren; by Rijke and Spitters; and by Voevodsky, is most closely concerned with the development of the univalent foundations programme. In particular, the paper by Voevodsky gives an overview of the fundamental definitions in this approach, including that of equivalence, and of several fundamental constructions. Readers can use this paper and the accompanying Coq code as a good introduction to the subject. The other papers in this group deal with the development of more specialized topics, namely, the development of various topics in category theory, set theory and algebra under the new, univalent approach.

A second group of papers, which includes the papers by Barras, Coquand and Huber; by van den Berg and Moerdijk; by van Oosten; and by Shulman, is concerned with the investigation of models of type theories. Very briefly, the contents of these contributions are as follows: Barras, Coquand and Huber undertake a constructivisation of the simplicial model of Voevodsky; this work serves as an important step toward a closely related, cubical approach under active current investigation (Bezem *et al.* 2014). In their paper, van den Berg and Moerdijk investigate Martin-Löf's types of well-founded trees, or W-types, in the simplicial model of homotopy type theory. The homotopical aspects of the effective topos, a topos built using ideas from realizability, are the topic of van Oosten's contribution. Shulman produces the first models of the univalence axiom in settings other than Voevodsky's original one of simplicial sets, via a general procedure from homotopy theory.

Finally, a third group consisting of the papers by Avigad, Kapulkin and Lumsdaine and by Herbelin, focuses on the definition within homotopy type theory of structures inspired by classical homotopy theory, namely homotopy limits and semi-simplicial sets, respectively.

Many of the papers in this special issue have accompanying files consisting of formalizations of the main results in the Coq proof assistant. These Coq files are published on the journal webpage alongside the papers themselves. In particular, Vladimir Voevodsky's library, on which some of the others are based, is at the DOI http://dx.doi.org/10.1017/S0960129514000577.

We believe that these papers, in combination with the book HoTT (2013), present a good overall view of the current state of the art of the subject.

3. Open problems

We conclude these introductory remarks with a few comments on the main, current outstanding issues in the field, as an invitation to prospective researchers, and for the sake of establishing some benchmarks for the record. A leading conjecture of Voevodsky concerns the constructive character of the univalence axiom. Roughly speaking, it states that the addition of that axiom to the system of type theory does not spoil the good computational properties of the latter, in that the normalization algorithm can be modified to yield a routine of 'normalization up to homotopy'. Some partial results, including ones by Harper and Licata (2012) and by Shulman (see the paper in this issue), have been promising, but a full proof is still outstanding. The truth of this conjecture, if established, is expected to have applications in the theory of computation as well as in the design of future proof assistants better suited to the implementation of univalent foundations.

A second direction for further research concerns higher inductive types, which have been introduced and used with great success in the book HoTT (2013) for everything from the smooth introduction of quotients in type theory, thus giving an alternative to setoids, to the calculations of some higher homotopy groups of spheres. But the computational character of these constructions, too, remains largely unknown. Moreover, although a good stock of examples have now been produced, and experts have developed precise techniques for dealing with them, a general theory of the kind available for conventional inductive types is still lacking.

Much current work is also devoted to the semantics of homotopy type theory, especially in connection with the possibility of modelling type theory using cubical sets, following the seminal preprint (Bezem *et al.* 2014). Preliminary results toward a constructive interpretation, which could lead to a solution of Voevodsky's conjecture and other results, appear to be quite promising.

Finally, toward strengthening the connection to contemporary homotopy theory and higher category theory, the precise relation between homotopy type theory and the notion of an ∞-topos (e.g. in the sense of Lurie (2009)) remains to be spelled out; for example, the apparent equivalence of the univalence axiom in the former with the notion of object classifier in the latter is tantalizing. Again, experts in the field have a good, working understanding, but a general theory of the kind available for extensional type theory and ordinary toposes is still lacking.

Of course, there are many other fascinating topics and problems under current investigation. Some of them will undoubtedly turn out to be very difficult; but in a subject as young and, apparently, deep as this one, there are doubtless also still some delectable low-hanging fruits. Moreover, the next advance may well come, not from the solution of one of these 'open problems', but from an entirely unexpected and surprising direction, just as did the subject itself.

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References

Awodey, S. and Warren, M. A. (2009) Homotopy-theoretic models of identity types. *Mathematical Proceedings of the Cambridge Philosophical Society* **146** (1) 45–55.

Bezem, M., Coquand, T. and Huber, S. (2014) *A model of type theory in cubical sets*. Preprint available from the second-named author's web page http://www.cse.chalmers.se/∼coquand/mod1.pdf

Gambino, N. and Garner, R. (2008) The identity type weak factorisation system. *Theoretical Computer Science* **409** (1) 94–109.

Gonthier, G. *et al.* (2013) A machine-checked proof of the odd order theorem. In: Interactive Theorem Proving. *Springer Lecture Notes in Mathematics* **7998** 163–179.

Harper, R. and Licata, D. R. (2012) Canonicity for 2-dimensional type theory. In: *Proceedings of the 39th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages*, 337–348.

Hofmann, M. and Streicher, T. (1998) The groupoid interpretation of type theory. *Twenty-Five Years of Constructive Type Theory*, Oxford University Press 83–111.

Joyal, A. (2008) *The Theory of Quasi-Categories and its Applications*. Quaderns **45** (II) Centre de Recerca Matemàtica.

LeFanu Lumsdaine, P. (2010) Weak ω-categories from intensional type theory. *Logical Methods in Computer Science* **6** (3:24) 1–19.

Leroy, X. (2009) Formal verification of a realistic compiler. *Communications of the ACM* **52** (7) 107–115.

Lurie, J. (2004) *Derived Algebraic Geometry*, Ph.D. thesis, M. I. T.

Lurie, J. (2009) *Higher Topos Theory*, Princeton University Press.

Quillen, D. G. (1967) *Homotopical Algebra*, Lecture Notes in Mathematics volume 43, Springer.

Rezk, C. (2001) A model for the homotopy theory of homotopy theories. *Transactions of the American Mathematical Society* **353** (3) 973–1007.

The Univalent Foundations Program (2013) *Homotopy Type Theory: Univalent Foundations of Mathematics*, Institute for Advanced Study.

Toën, B. and Vezzosi, G. (2005) Homotopical algebraic geometry. I. Topos theory. *Advances in Mathematics* **193** (2) 257–372.

van den Berg, B. and Garner, R. (2011) Types are weak ω-groupoids. *Proceedings of the London Mathematical Society* **102** (2) 370–394.

van den Berg, B. and Garner, R. (2012) Topological and simplicial models of identity types. *Transactions of the ACM on Computational Logic* **13** (1) 3:1–3:44.

Voevodsky, V. (2009) *Notes on Type Systems*. Preprint. Available from the author's web page http://www.math.ias.edu/∼vladimir/Site3/Univalent_Foundations.html

Warren, M. A. (2008) *Homotopy Theoretic Aspects of Constructive Type Theory*, Ph.D. thesis, Carnegie Mellon University.