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A LOGARITHMIC PROPERTY FOR EXPONENTS OF PARTIALLY ORDERED SETS

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1. Introduction. In an effort to unify the arithmetic of cardinal and ordinal numbers, Garrett Birkhoff [**2**; **3**; **4**; **5**] (cf. [**6**]) defined several operations on partially ordered sets of which at least one, (cardinal) exponentiation, is of considerable independent interest: for partially ordered sets P and Q let P^Q denote the set of all order-preserving maps of Q to P partially ordered by $f \leqq g$ if and only if $f(x) \leqq g(x)$ for each $x \in Q$. If P is a lattice we call P^Q a *function lattice.*

One important application of exponentiation lies in Birkhoff's fundamental representation theorem for finite distributive lattices: if D is a finite distributive lattice then $D \cong \mathbf{2}^Q$ where $\mathbf{2}$ denotes the two-element chain and Q^d, the dual of Q, is isomorphic to $\mathbf{J}(D)$, the partially ordered set of join irreducible elements of D [**3**]. Our main result is inspired by this elegant representation theorem (see Figure 1).

THEOREM. *Let L be a finite lattice and let Q be a finite partially ordered set. Then*

$$\mathbf{J}(L^Q) \cong \mathbf{J}(L) \times Q^d.$$

This *logarithmic* property provides us with a technique to reduce a problem concerning exponents of partially ordered sets to one concerning direct products of partially ordered sets. Indeed, it is this technique that enables to establish several important instances of two long-standing conjectures concerning exponents of partially ordered sets.

For combinatorialists, *cancellation law* problems have long inspired fascination; the best known cancellation results for partially ordered sets are concerned with factorizations. G. Birkhoff [**2**] was the first to show that *in a partially ordered set with least and greatest elements any two factorizations have a common refinement.* An induction on the length yields: *in a partially ordered set P of finite length with least and greatest elements $P \cong XY \cong XZ$ implies $Y \cong Z$.* J. Hashimoto [**8**; **9**] succeeded in generalizing these results by replacing the assumption of universal bounds by connectivity. Finally, L. Lovász [**10**] (cf. [**11**]) disclosed the complete story with an ingenious proof of a cancellation law for relational systems; in particular, *for finite partially ordered sets P, Q and R, $PQ \cong PR$ implies $Q \cong R$, moreover, for any positive integer n, $P^n \cong Q^n$ implies $P \cong Q$.*

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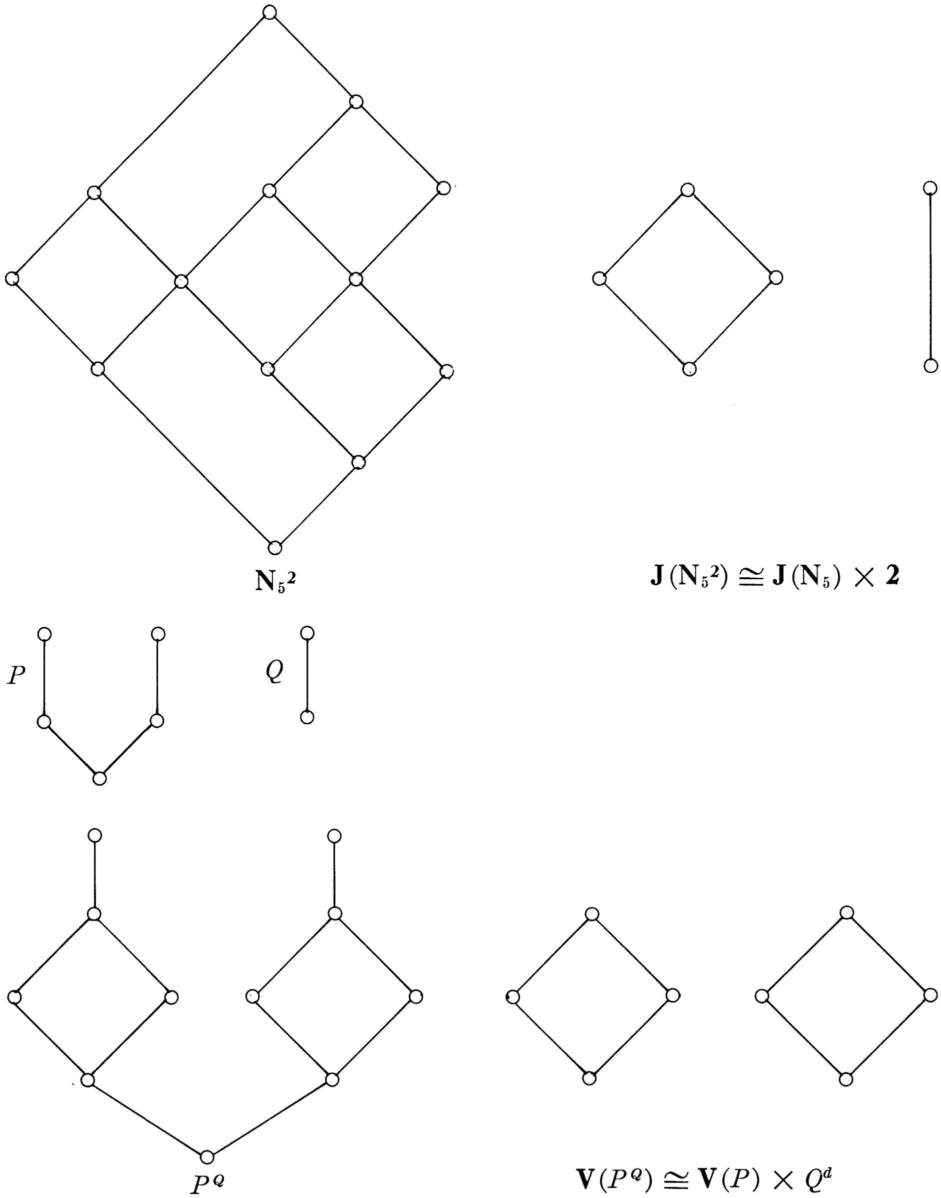

FIGURE 1.

The status of cancellation laws for exponents of finite partially ordered sets is much less satisfactory than that for factorizations. Indeed, although G. Birkhoff had already discussed the problem in [**4**] very little is known.

Recall, a partially ordered set is *bounded* if it has a least and a greatest element.

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THEOREM. *Let L and K be finite lattices and let Q be a finite, bounded partially ordered set. If $L^Q \cong K^Q$, then $L \cong K$.*

Our second application of the logarithmic property was obtained jointly with R. McKenzie.

THEOREM. *Let P be a finite partially ordered set that is not unordered and let Q and R be finite partially ordered sets. If $P^Q \cong P^R$, then $Q \cong R$.*

If P is unordered then $Q \cong R$ need not follow from $P^Q \cong P^R$. (For instance, let P be the two-element antichain 2, let Q be the two-element chain **2** and let $R \cong \mathbf{2}^2$.) E. Fuchs [**7**] has shown that if P is unordered and $|P| > 1$ then $P^Q \cong P^R$ if and only if Q and R have the same number of connected components.

2. Preliminaries. Let A and B be disjoint partially ordered sets. The (*disjoint*) *sum* $A + B$ is the set $A \cup B$ with partial ordering that induced by the partial orderings on A and B. A partially ordered set P is *connected* if $P = A + B$ implies $A = \emptyset$ or $B = \emptyset$. $P = P_1 + P_2 + \ldots + P_m$ is a *non-decomposable sum representation* of P if P_i is connected for each i; call P_i a *component* of P. Obviously, every finite partially ordered set has a non-decomposable sum representation that is unique (up to the order of the components in the sum) [**2**]. For partially ordered sets $P_i (i = 1, 2, \ldots, n)$ let $P_1P_2 \ldots P_n$ (or, $P_1 \times P_2 \times \ldots \times P_n$) denote the usual direct product of partially ordered sets.

Many laws of arithmetic can be generalized to arbitrary partially ordered sets. For instance (cf. [**2**]),

$$A(B + C) \cong AB + AC,$$
$$(AB)^C \cong A^C B^C \quad \text{and}$$
$$A^{B+C} \cong A^B A^C;$$

moreover, if C is connected then also

$$(A + B)^C \cong A^C + B^C.$$

We now dispense with preliminary remarks concerning *covers* in exponents of partially ordered sets. Recall, for elements $a > b$ in a partially ordered set $a \succ b$ (a *covers* b) if $a \geqq c > b$ implies $a = c$.

Let P and Q be finite partially ordered sets. Let $x \in Q$ and choose elements f and g of P^Q such that $f(y) = g(y)$ for all $y \neq x$ in Q. If $f(x) \succ g(x)$ in P then $f \succ g$ in P^Q. The converse of the statement also holds. Let $f \succ g$ in P^Q and let us suppose that f and g have different values at more than one element of Q. We choose $y \in Q$ minimal with respect to the property $f(y) > g(y)$ and choose $x \in Q - \{y\}$ minimal with respect to the property $f(x) > g(x)$. Clearly, either $x > y$ or x is noncomparable with y. Define a function of Q to P by

$$h(z) = \begin{cases} f(z) & \text{if } z \neq y \\ g(z) & \text{if } z = y. \end{cases}$$

To see that h is order-preserving let $u < v$ in Q. If $v = y$ then $h(u) = f(u) = g(u) \leqq g(y) = h(y)$. If $u = y$ then $h(y) = g(y) < f(y) \leqq f(v) = h(v)$. Since $f(y) > h(y)$ and $h(x) > g(x)$, we have $f > h > g$. Hence, $f \succ g$ in P^Q implies $f(x) > g(x)$ for precisely one element $x \in P$ and otherwise $f = g$. A similar argument shows that $f \succ g$ in P^Q implies $f(x) \succ g(x)$ in P.

PROPOSITION 2.1. *Let P and Q be finite partially ordered sets and let $f, g \in P^Q$. Then f covers g in P^Q if and only if there is $x \in Q$ such that $f|Q - \{x\} = g|Q-\{x\}$ and $f(x)$ covers $g(x)$ in P.*

Let P be a partially ordered set. The *length* of a chain C of P is defined by

$$l(C) = |C| - 1$$

and the *length* of P is defined by

$$l(P) = \sup\ (l(C)|C \text{ is a chain in } P).$$

COROLLARY 2.2 (G. Birkhoff [**5**]). *Let P and Q be finite partially ordered sets. Then $l(P^Q) = l(P)\cdot|Q|$.*

Proof. In view of Proposition 2.1 the length of P^Q is at most $l(P)\cdot|Q|$.

Let $P = \{a_1, a_2, \ldots, a_n\}$ and let $a_1 < a_2 < \ldots < a_k$ be a chain of maximum length in P. Now let $Q = \{b_1, b_2, \ldots, b_m\}$ and let $b_1 > b_2 > \ldots > b_m$ be a linear extension of Q. Define a map $f_{i,j}$ of Q to P by

$$f_{i,j}(b_l) = \begin{cases} a_{i+1} & \text{if } l < j \\ a_i & \text{if } l \geqq j \end{cases}$$

for $i = 1, 2, \ldots, k - 1$ and $j = 1, 2, \ldots, m$. Then $f_{i,j}$ is an order-preserving map from a linear extension of Q to P; hence, $f_{i,j} \in P^Q$. Also, it is clear that $f_{i,j+1} > f_{i,j}$ and $f_{i+1,1} > f_{i,m}$; whence, upon adjoining the function which maps Q to a_k, we have a chain of length $l(P)\cdot|Q|$ in P^Q.

Actually, Proposition 2.1 and an appropriate version of Corollary 2.2 hold even if it is assumed only that P and Q contain no infinite chains.

Similar arguments may be applied to establish further covering properties for function lattices. For instance, *if Q and L are finite partially orcdred sets then L^Q is a semimodular lattice if and only if L is a semimodular lattice.*

3. The logarithmic property. The importance of join irreducible elements of a finite lattice for combinatorial investigations in lattice theory is well known. The aim of this section is to describe the partially ordered set of all join irreducible elements of a finite function lattice L^P in terms of L and P (cf. Figure 1).

Actually we shall attack a somewhat more general question. For a partially ordered set P let $\mathbf{V}(P)$ denote the partially ordered subset of all elements of

P with precisely one lower cover. (Note that $\mathbf{J}(P) = \mathbf{V}(P)$ if P is a finite lattice.)

THEOREM 3.1. *Let P and Q be finite partially ordered sets. If P has a least element, then* $\mathbf{V}(P^Q) \cong \mathbf{V}(P) \times Q^d$.

Proof. If P has a least element O_P then O_{P^Q} exists and $\mathbf{V}(P^Q) \neq \emptyset \neq \mathbf{V}(P)$. Let L denote the set of all functions f in P^Q for which there exists $x \in Q$ and $a \in \mathbf{V}(P)$ such that

$$f(y) = \begin{cases} a & \text{if } y \geqq x \\ O_P & \text{if } y \ngeqq x. \end{cases}$$

Such a function we denote by $f_{(a,x)}$. The map which assigns (a, x) to $f_{(a,x)}$ is an isomorphism of L onto $V(P) \times Q^d$. We shall show that $L = \mathbf{V}(P^Q)$.

Let $f_{(a,x)} \in L$. If $g \in P^Q$ and $g < f_{(a,x)}$ then there exists $y \geqq x$ such that $g(y) < f(y) = a$. Since a has a unique lower cover a_* in P, $g(y) \leqq a_*$. Therefore, $g \leqq f_*$ where

$$f_*(z) = \begin{cases} f(z) & \text{if } z \neq x \\ a_* & \text{if } z = x. \end{cases}$$

Clearly, $f_* \in P^Q$ and $f \succ f_*$. It follows that $f \in V(P^Q)$ so that $L \subseteq \mathbf{V}(P^Q)$.

Conversely, let $f \in \mathbf{V}(P^Q)$ and let us suppose that $f(Q)$ contains at least two distinct elements of P distinct from O_P. Let a and b be minimal elements of $f(Q)$, different from O_P; then a and b are noncomparable. Let $a \succ a_*$ and $b \succ b_*$. Choose x_0 to be a minimal element of $f^{-1}(a)$ and y_0 to be a minimal element of $f^{-1}(b)$. Let g and h be maps of Q to P satisfying $g(z) = f(z)$ for all $z \neq x_0$, $h(z) = f(z)$ for all $z \neq y_0$, $g(x_0) = a_*$ and $h(y_0) = b_*$. To show that $g \in P^Q$ we need only show that $z < x_0$ implies $g(z) \leqq a_*$. Since x_0 is a minimal element of $f^{-1}(a)$ and a is a minimal nonzero element in $f(Q)$, $z < x_0$ implies $f(z) = 0$. Therefore, $g(z) = f(z) = 0 \leqq a_*$. Similarly, $h \in P^Q$. It is clear that $f \succ g$, $f \succ h$ and $g \neq h$. We may now assume that $f(Q)$ contains a least element, say a, different from O_P. Choose $b \in f(Q)$ such that $b \succ a$ in $f(Q)$. Again, let $a \succ a_*$ and $b \succ b_* \geqq a$ in P. Let g, h be the functions defined as above. We show that $h \in P^Q$; whence, contrary to assumption, $f \notin \mathbf{V}(P^Q)$. Let $z < y_0$ in Q. Since y_0 is a minimal element of $f^{-1}(b)$, $f(z) < b$. On the other hand, $b \succ a$ in $f(Q)$ and a is the minimum element in $f(Q)$ different than O_P. Therefore, $h(z) = f(z) \leqq a \leqq b_* = h(y_0)$. Therefore, if $f \in \mathbf{V}(P^Q)$ then $f(Q)$ contains exactly one element of P other than O_P.

Let a be the unique nonzero member of $f(Q)$. Let $a \succ b$ and $a \succ c$ in P. Choose a minimal element x_0 in $f^{-1}(a)$ and let g and h be maps of Q to P satisfying $g(z) = h(z) = f(z)$ for all $z \neq x_0$, $g(x_0) = b$ and $h(x_0) = c$. Then $g, h \in P^Q$, $f \succ g$, and $f \succ h$. We conclude that $f \in \mathbf{V}(P^Q)$ implies $b = c$; that is, $a \in \mathbf{V}(P)$.

Finally, we show that $f^{-1}(a)$ has a minimum element (in other words, $f^{-1}(a) = \{y \in Q | y \geqq x\}$ for some $x \in Q$). Let x_1 and x_2 be minimal elements of $f^{-1}(a)$ and let f_1 and f_2 be maps of Q to P defined by $f_i(z) = f(z)$ for all $z \neq x_i$

and $f_i(x_i) = a_*$, for $i = 1, 2$, where a_* is the unique lower cover of a in P. Again, it is clear that $f_i \in P^Q$ $(i = 1, 2)$. It must be the case that $f_1 = f_2$; that is, $x_1 = x_2$. (Note that $f \in \mathbf{V}(P^Q)$ may be a constant function. In fact this happens exactly when Q has a least element and $f(z) = a$ for all $z \in Q$ and $a \in \mathbf{V}(P)$.)

COROLLARY 3.2. *Let L be a finite lattice and let Q be a finite partially ordered set. Then*

$$\mathbf{J}(L^Q) \cong \mathbf{J}(L) \times Q^d.$$

Again, Theorem 3.1 holds under the less restrictive condition that P and Q contain no infinite chains. Moreover, Corollary 3.2 holds provided L satisfies some atomicity condition (for example, if L^d is an algebraic lattice).

As an easy illustration of the utility of Theorem 3.1, we can derive a special case of one of the cancellation laws for exponents first established by M. Novotný [**12**] (cf. [**7**]): *if P, Q and R are finite partially ordered sets such that P has a least element and $|P| > 1$, then $P^Q \cong P^R$ implies $Q \cong R$.* By Theorem 3.1, $\mathbf{V}(P^Q) \cong \mathbf{V}(P) \times Q^d$. On the other hand, $\mathbf{V}(P^Q) \cong \mathbf{V}(P^R)$ implies that $\mathbf{V}(P) \times Q^d \cong \mathbf{V}(P) \times R^d$. It follows that $Q \cong R$.

4. Applications: cancellation laws. We first establish the following result.

THEOREM 4.1. *Let L and K be finite lattices and let Q be a finite, bounded partially ordered set. If $L^Q \cong K^Q$, then $L \cong K$.*

In the interests of brevity and clarity of the proof we shall adopt several notational devices. Each of the partially ordered sets L, K, and Q of Theorem 4.1 is bounded. We shall let 0, respectively 1, stand for the least element, respectively greatest element, for each of L, K, and Q. As the particular partially ordered set under consideration will always be clear from the context there should be no confusion. Let $\mathbf{M}(L)$ denote the partially ordered set of all meet irreducible elements of L. For $a \in L$ and $x \in Q$ let a^x and ${}^x a$ be the elements of L^Q defined by

$$a^x(y) = \begin{cases} a & \text{if } y \geqq x \\ 0 & \text{if } y \ngeqq x \end{cases}$$

and

$${}^x a(y) = \begin{cases} a & \text{if } y \leqq x \\ 1 & \text{if } y \nleqq x. \end{cases}$$

Observe that $a^x \in \mathbf{J}(L^Q)$ for every $x \in Q$, provided that $a \in \mathbf{J}(L)$; similarly, if $a \in \mathbf{M}(L)$, then ${}^x a \in \mathbf{M}(L^Q)$ for every $x \in Q$.

Before proceeding with the proof, a few remarks concerning the idea behind it are in order. Let $\mathbf{P}(L) = \mathbf{J}(L) \cup \mathbf{M}(L)$ and let $\mathbf{L}(P)$ be the *normal completion* (Dedekind-MacNeille completion) of a partially ordered set P. Then $\mathbf{L}(\mathbf{P}(L)) \cong L$ (cf. [**1**; **13**]). Now, let

$$\mathbf{\Delta}(L^Q) = \{a^0 | a \in L\}$$

and let φ be an isomorphism of L^Q onto K^Q. As $\mathbf{P}(\mathbf{\Delta}(L^Q)) \cong \mathbf{P}(L)$, the conclusion of Theorem 4.1 follows if

$$\mathbf{P}(\mathbf{\Delta}(L^Q)) \cong \mathbf{P}(\mathbf{\Delta}(K^Q)).$$

We shall show that, while $\varphi(a^0)$, $a \in \mathbf{J}(L)$, need not be a member of $\mathbf{\Delta}(K^Q)$, it can, nevertheless, be associated, in a natural way, with a member of $\mathbf{\Delta}(K^Q)$.

LEMMA 4.2. *Let L and K be finite lattices and let Q be a finite, bounded, direct product nondecomposable partially ordered set. Let φ be an isomorphism of L^Q onto K^Q and let $a \in \mathbf{J}(L)$. Then there exist $b \in \mathbf{J}(L)$, α, $\beta \in \mathbf{J}(K)$, and a sequence $x_1, x_2, \ldots, x_n$ of nonzero elements of Q satisfying the following conditions:*

(*i*) $\varphi(a^0) = \alpha^{x_1}$, $\varphi(b^0) = \alpha^0$;
(*ii*) $\varphi(b^{x_1}) = \alpha^{x_2}$, $\varphi(b^{x_2}) = \alpha^{x_3}$, $\ldots$, $\varphi(b^{x_{n-1}}) = \alpha^{x_n}$;
(*iii*) $\varphi(b^{x_n}) = \beta^0$.

Proof. Observe that $\varphi^{-1}(\alpha^0) \in \mathbf{J}(L^Q)$ and $\varphi^{-1}(\alpha^0) > a^0$; therefore, $\varphi^{-1}(\alpha^0) = b^0$ for some $b \in \mathbf{J}(L)$. Let $\varphi(a^0) = \alpha^x$ and $\varphi(b^y) = \beta^z$. We claim that either $\beta = \alpha$ or $z = 0$. Once this is established, a simple induction completes the proof of the lemma.

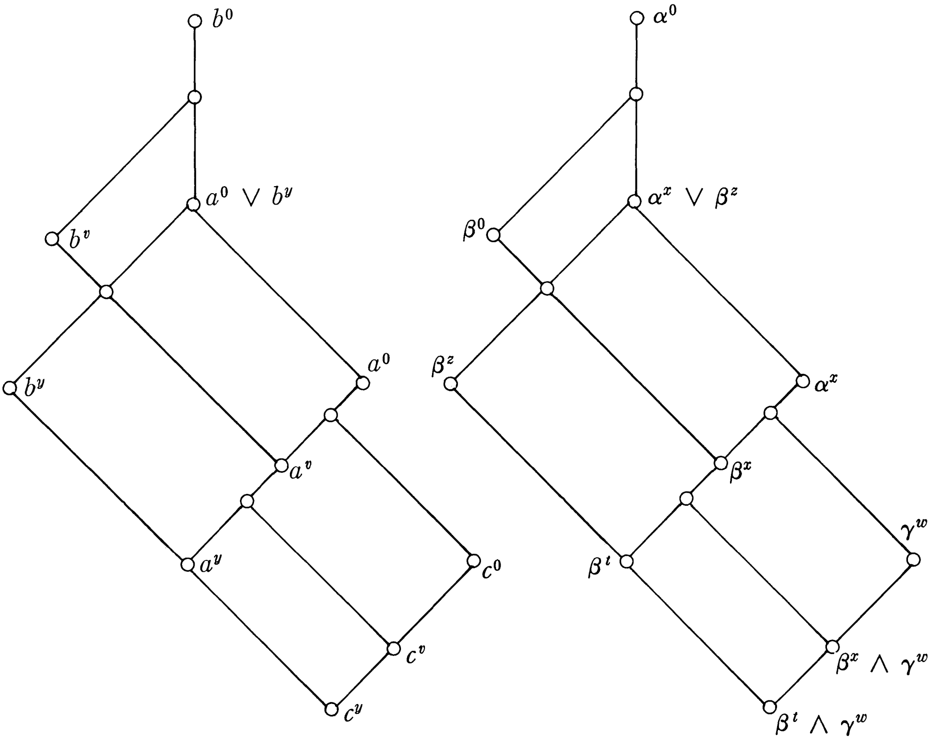

FIGURE 2.

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Let us suppose that $z > 0$ and $\beta \neq \alpha$. As $b^0 = \varphi^{-1}(\alpha^0) > \varphi^{-1}(\alpha^x) = a^0$ it follows that $b > a$ and $\alpha > \beta$ (see Figure 2). Observe that $a \in \mathbf{J}(L)$ implies that $a^u \in \mathbf{J}(L^Q)$ and $\varphi(a^u) \in \mathbf{J}(K^Q)$, for every $u \in Q$; hence, also α^u, $\beta^u \in \mathbf{J}(K^Q)$ and $b^u \in \mathbf{J}(L^Q)$, for every $u \in Q$. Since $a^0 \wedge b^y = a^y \in \mathbf{J}(L^Q)$ it follows that $\varphi(a^y) = \alpha^x \wedge \beta^z \in \mathbf{J}(K^Q)$ so that $x \vee z$ exists in Q, say $t = x \vee z$, and $\varphi(a^y) = \beta^t$. Moreover, $\beta^z > \beta^t$ implies that $z \ngeqq x$ and $a^0 \ngeqq b^y$ implies that $x \ngeqq z$. Let $u \in Q$ such that $u \leqq x$ and $u \leqq z$. Then $a^0 \vee b^y \leqq \varphi^{-1}(\alpha^u) \leqq b^0$. Now, $\varphi^{-1}(\alpha^u) \in \mathbf{J}(L^Q)$ so that $u = 0$. In particular, $x \wedge z$ exists in Q and $x \wedge z = 0$.

Let us consider β^0 in K^Q. Clearly, $\beta^0 \in \mathbf{J}(K^Q)$ and β^0 is noncomparable with $\alpha^x \vee \beta^z = \varphi(a^0 \vee b^y)$. Then $\varphi^{-1}(\beta^0) = b^v$ for some $v \in Q$ satisfying $0 < v < y$. In addition, $a^v = a^0 \wedge b^v$ and $\beta^x = \alpha^x \wedge \beta^0$. Let $c \in L$ such that $a \geqq c \succ 0$ and let $\varphi(c^0) = \gamma^w$. Then $\gamma \leqq \alpha$ and $x \leqq w$. Since $v > 0$, $c^0 \nleqq a^v$. It now follows that

$$[0, c]^Q \cong [0^0, c^0] \cong [0^0, \gamma^w] \cong [0, \gamma]^{[w,1]}.$$

In the light of Corollary 3.2 this implies that

$$\mathbf{J}([0, c]) \times Q^d \cong \mathbf{J}([0, \gamma]) \times (w, 1]^d.$$

As $c \succ 0$, *sinister* is isomorphic to Q^d which, by hypothesis, is direct product nondecomposable. Hence, either $\mathbf{J}([0, \gamma])$ or $[w, 1]^d$ is isomorphic to Q^d. Since $0 < x \leqq w$ it follows that $\mathbf{J}([0, \gamma]) \cong Q^d$ and $w = 1$.

Finally, we consider the elements

$$a^v \wedge c^0 = c^v > c^y = a^y \wedge c^0,$$

which are join irreducible in L^Q.

On the other hand, we then have that

$$\varphi(c^v) = \beta^x \wedge \gamma^1 = (\beta \wedge \gamma)^1 = \beta^t \wedge \gamma^1 = \varphi(c^y),$$

which is impossible.

Proof of Theorem 4.1. We proceed by induction on $|Q|$. If Q is direct product decomposable, say $Q = Q_1Q_2$ then

$$(L^{Q_1})^{Q_2} \cong L^{Q_1Q_2} \cong K^{Q_1Q_2} \cong (K^{Q_1})^{Q_2},$$

whence, by the induction hypothesis, $L^{Q_1} = K^{Q_2}$ and, in turn, $L \cong K$. Therefore, we may assume that Q is direct product nondecomposable.

We need only show that there is a weak embedding of $\mathbf{P}(\boldsymbol{\Delta}(L^Q))$ into $\mathbf{P}(\boldsymbol{\Delta}(K^Q))$. Indeed, we define a map ψ of $\mathbf{P}(\boldsymbol{\Delta}(L^Q))$ to $\mathbf{P}(\boldsymbol{\Delta}(K^Q))$ by

$$\psi(a^0) = \beta^0$$

for $a \in \mathbf{J}(L)$, where $\beta \in \mathbf{J}(K)$ is the element guaranteed by Lemma 4.2, and

$$\psi({}^1a) = {}^1\beta$$

for $a \in \mathbf{M}(L)$, where $\beta \in \mathbf{M}(K)$ is the element guaranteed by the dual of Lemma 4.2.

We claim that ψ is order-preserving.

Let $a_1 < a_2$ in $\mathbf{J}(L)$. Then there exist $b_1, b_2 \in \mathbf{J}(L)$, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbf{J}(K)$ and sequences $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m$ of nonzero elements of Q such that $\varphi(a_1{}^0) = \alpha_1{}^{x_1}$, $\varphi(a_2{}^0) = \alpha_2{}^{y_1}$, $\varphi(b_1{}^0) = \alpha_1{}^0$, $\varphi(b_2{}^0) = \alpha_2{}^0$, $\varphi(b_1{}^{x_1}) = \alpha_1{}^{x_2}$, $\varphi(b_1{}^{x_2}) = \alpha_1{}^{x_3}, \ldots, \varphi(b_1{}^{x_{n-1}}) = \alpha_1{}^{x_n}$, $\varphi(b_2{}^{y_1}) = \alpha_2{}^{y_2}$, $\varphi(b_2{}^{y_2}) = \alpha_2{}^{y_3}, \ldots, \varphi(b_2{}^{y_{m-1}}) = \alpha_2{}^{y_m}$, and $\varphi(b_1{}^{x_n}) = \beta_1{}^0$, $\varphi(b_2{}^{y_m}) = \beta_2{}^0$. Now, $a_1 < a_2$ implies that $a_1{}^0 < a_2{}^0$. If $n > 0$ and $m > 0$, then $\alpha_1{}^{x_1} < a_2{}^{y_1}$, whence, $\alpha_1 \leqq \alpha_2$ and $y_1 \leqq x_1$ so that $b_1 < b_2$. In turn, this implies that $b_1{}^{x_1} < b_2{}^{y_1}$ and $\alpha_1{}^{x_2} < \alpha_2{}^{y_2}$ and $y_2 \leqq x_2$. Let $n \geqq m \geqq 0$. If also $m > 0$ then $b_1{}^{x_m} < b_2{}^{y_m}$. Now, $\alpha_1{}^{x_{m+1}} < \alpha_2{}^0$ or $\alpha_1{}^0 < \alpha_2{}^0$ and, in any case, $\alpha_1{}^0 < \alpha_2{}^0$. Let $m > n \geqq 0$. If also $n > 0$, then $b_1{}^{x_n} < b_2{}^{y_n}$. Then $\beta_1{}^0 < \beta_2{}^{y_{n+1}}$, which is impossible. Let $a_1 \in \mathbf{M}(L)$, $a_2 \in \mathbf{J}(L)$ and $a_1 < a_2$. Let $b_1 \in \mathbf{M}(L)$, α_1, $\beta_1 \in \mathbf{M}(K)$, and $x_1, x_2, \ldots, x_n$ be nonunit elements of Q such that $\varphi({}^1a_1) = {}^{x_1}\alpha_1$, $\varphi({}^1b_1) = {}^1\alpha_1$, $\varphi({}^{x_1}b_1) = {}^{x_2}\alpha_1$, $\varphi({}^{x_2}b_1) = {}^{x_3}\alpha_1, \ldots, \varphi({}^{x_{n-1}}b_1) = {}^{x_n}\alpha_1$, and $\varphi({}^{x_n}b_1) = {}^1\beta_1$, and again let $b_2 \in \mathbf{J}(L)$, $\alpha_2, \beta_2 \in \mathbf{J}(K)$, and $y_1, y_2, \ldots, y_m$ be nonzero elements of Q such that $\varphi(a_2{}^0) = \alpha_2{}^{y_1}$, $\varphi(b_2{}^0) = \alpha_2{}^0$, $\varphi(b_2{}^{y_1}) = \alpha_2{}^{y_2}$, $\varphi(b_2{}^{y_2}) = \alpha_2{}^{y_3}, \ldots, \varphi(b_2{}^{y_{m-1}}) = \alpha_2{}^{y_m}$, and $\varphi(b_2{}^{y_m}) = \beta_2{}^0$. Observe that $n = 0$ if and only if $m = 0$. Let $n > 0$ and $m > 0$, then ${}^{x_1}\alpha_1 < \alpha_2{}^{y_1}$ so that $\alpha_1 = 0$, $\alpha_2 = 1$, and $y_1 \leqq x_1$. Then $b_1 = 0$, $b_2 = 1$, and $Q = [0, x_1] \cup [y_1, 1]$, so that ${}^{x_1}b_1 < b_2{}^{y_1}$ and ${}^{x_2}\alpha_1 < \alpha_2{}^{y_2}$. If $n < m$ then ${}^{x_n}b_1 < b_2{}^{y_n}$ and ${}^1\beta_1 < \alpha_2{}^{y_{n+1}}$ whence $\beta_1 = 0$; if $n > m$ then ${}^{x_m}b_1 < b_2{}^{y_m}$ and ${}^{x_{m+1}}\alpha_1 < \beta_2{}^0$ whence $\beta_2 = 1$; if $n = m$ then ${}^{x_n}b_1 < b_2{}^{y_n}$ and ${}^1\beta_1 < \beta_2{}^0$.

The remaining cases are similar—straightforward yet tedious.

That ψ^{-1} is order-preserving follows by symmetry. This completes the proof of the theorem.

We now give a second application of the logarithmic property of exponents. This result was obtained jointly with R. McKenzie.

THEOREM 4.3. *Let P be a finite partially ordered set that is not unordered and let Q and R be finite partially ordered sets. If $P^Q \cong P^R$, then $Q \cong R$.*

Proof. For the purposes of this proof we define the *depth* $\delta(x)$ of an element x of a partially ordered set X to be the maximum of the lengths of all chains in X whose least element is x. Note that an element x is of maximum depth in X if and only if $\delta(x) = l(X)$. Also, we let $[x)$ denote the set $\{y | y \geqq x\}$.

Let $Q = Q_1 + Q_2 + \ldots + Q_n$ and $R = R_1 + R_2 + \ldots + R_m$ be non-decomposable sum representations of Q and R.

Let f be an element of maximum depth in P^Q. Since $P^Q \cong P^{Q_1}P^{Q_2} \ldots P^{Q_n}$ we may identify f with the n-tuple $(f_1, f_2, \ldots, f_n)$ where f_i is the restriction of f to Q_i $(i = 1, 2, \ldots, n)$. Then f_i is an element of maximum depth in P^{Q_i}; by

Corollary 2.2, $\delta(f_i) = l(P)\cdot|Q_i|$. It follows that $f_i(Q_i)$ contains only elements of maximum depth in P. Since a set of elements of maximum depth must be unordered and $f_i(Q_i)$ is connected, $f_i(Q_i)$ must be a singleton. Let $f_i(Q_i) = \{a_i\}$ for $a_i \in P$ satisfying $\delta(a_i) = l(P)$ $(i = 1, 2, \ldots, n)$. Clearly $[f_i) \cong [a_i)^{Q_i}$. Observe that $[f) \cong \prod_{i=1}^n [f_i)$; whence, applying Theorem 3.1, we have

$$\begin{aligned}\mathbf{V}([f)) &\cong \mathbf{V}\left(\prod_{i=1}^{n} [f_i)\right)\\ &\cong \sum_{i=1}^{n} \mathbf{V}([f_i))\\ &\cong \sum_{i=1}^{n} \mathbf{V}([a_i)) \times Q_i^{\,d}.\end{aligned}$$

(A few remarks concerning this calculation are in order: the hypothesis that P is not unordered ensures that $\delta(a_i) > 0$; whence, $V([a_i)) \neq \emptyset$ $(i = 1, 2, \ldots, n)$; also $\mathbf{V}(\prod_{i=1}^n [f_i)) \cong \sum_{i=1}^n \mathbf{V}([f_i))$ is a consequence of the fact that an element $(x_1, x_2, \ldots, x_n)$ of a direct product of finite partially ordered sets $X_1, X_2, \ldots, X_n$, each with a least element O_{X_i}, has a unique lower cover if and only if there exists i_0 such that $x_i = O_{X_i}$ for all $i \neq i_0$ and $x_{i_0} \in \mathbf{V}(X_{i_0})$.)

Let φ be an isomorphism of P^Q onto P^R and let f be as above. Then $\varphi(f) = g = (g_1, g_2, \ldots, g_m)$ is a function of maximum depth in P^R. Moreover, g_j must be a constant function on R_j; $g_j(R_j) = \{b_j\}$, say, where $\delta(b_j) = l(P)$ $(j = 1, 2, \ldots, m)$. Again

$$\mathbf{V}([g)) \cong \sum_{j=1}^{m} \mathbf{V}([b_j)) \times R_j^{\,d}.$$

Since $\varphi(f) = g$, $[f) \cong [g)$; hence,

$$(1)\quad \sum_{i=1}^{n} \mathbf{V}([a_i)) \times Q_i^{\,d} \cong \mathbf{V}([f)) \cong \mathbf{V}([g)) \cong \sum_{j=1}^{m} \mathbf{V}([b_j)) \times R_j^{\,d}.$$

We claim that Q and R have the same number of connected components. Each n-tuple $(a_1, a_2, \ldots, a_n)$ of elements of maximum depth in P induces a function of maximum depth in P^Q: namely, the map f of Q to P defined by $f(y) = a_i$ for all $y \in Q_i$ $(i = 1, 2, \ldots, n)$. Of course, the same observations hold for functions of maximum depth in P^R. If D denotes the set of all elements of maximum depth in P, then the number of elements of maximum depth in P^Q (and P^R) is $|D|^n = |D|^m$. If $|D| > 1$ then $n = m$; that is, Q and R have the same number of connected components. If $|D| = 1$ then $Q \cong R$ follows from (1).

Summing (1) over all $\boldsymbol{a} = (a_1, a_2, \ldots, a_n) \in D^n$ and, using the distributivity of product over disjoint sum, we obtain

$$(2)\quad \sum_{i=1}^{n} \left(\sum_{\boldsymbol{a}\in D^n} \mathbf{V}([a_i))\right) \times Q_i^{\,d} \cong \sum_{j=1}^{n} \left(\sum_{\boldsymbol{b}\in D^n} \mathbf{V}([b_j))\right) \times R_j^{\,d}.$$

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(Here we identify $f \in P^Q$ of maximum depth with the ordered n-tuple $\boldsymbol{a} \in D^n$ of its images and we take $\varphi(\boldsymbol{a}) = \boldsymbol{b}$.) From this it follows that

$$(3) \quad \left(|D|^{n-1} \cdot \sum_{a \in D} \mathbf{V}([a))\right) \times Q^d \cong \left(|D|^{n-1} \cdot \sum_{b \in D} \mathbf{V}([b))\right) \times R^d.$$

Applying the cancellation law for products to (3) we conclude that $Q \cong R$.

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