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ISOMETRIES OF NONCOMPACT LIPSCHITZ SPACES

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ABSTRACT. We show that under reasonable restrictions on the metric spaces X and Y, every surjective isometric isomorphism between Lip(X) and Lip(Y) arises in a simple manner from an isometry between X and Y. Our result differs from several previous results along these lines in that we do not require X and Y to be compact.

A map $f: X \longrightarrow Y$ between metric spaces is called *Lipschitz* if its *Lipschitz number*

$$L(f) = \sup_{\substack{p,q \in X \\ p \neq q}} \frac{\rho^Y\big(f(p), f(q)\big)}{\rho^X(p,q)}$$

is finite. For any metric space X the *Lipschitz space* Lip(X) is defined to be the set of all bounded scalar-valued Lipschitz functions on X, with norm

$$\|f\|_L = \max\big(\|f\|_\infty, L(f)\big).$$

We allow either real or complex scalars. It is standard that Lip(X) is a Banach space.

Let $\mathbf{F}$ be the scalar field, $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$, and let $\mathbf{U} \subset \mathbf{F}$ be the set of elements of modulus 1. If $g: Y \longrightarrow X$ is a surjective isometry and $\alpha \in \mathbf{U}$, the map $f \longmapsto \alpha f \circ g$ is an isometric isomorphism from Lip(X) onto Lip(Y); a good deal of attention has been focused on finding conditions under which every isometric isomorphism from Lip(X) onto Lip(Y) is of this form.

This is certainly not true in general. For instance, it is easy to see ([V], [W]) that if X is any metric space and Y is the completion of the metric space whose underlying set is X and whose metric is $\min\big(2, \rho(p,q)\big)$, then Lip($X$) and Lip($Y$) are naturally isometrically isomorphic. If X is not complete or has diameter > 2, this isometric isomorphism cannot arise from an isometry from Y onto X because there are no such isometries.

The preceding shows that it is worthwhile to restrict attention to the class $\mathcal{M}^2$ of complete metric spaces of diameter ≤ 2. However, even if X and Y belong to $\mathcal{M}^2$ there are counterexamples. For instance, let $X = Y$ be a metric space consisting of two elements p, q such that $\rho(p,q) = 1$. Then Lip(X) is (isometrically isomorphic to) $\mathbf{F}^2$ with the norm

$$\|(a,b)\|_L = \max(|a|, |b|, |a-b|)$$

and the map taking (a,b) to $(a, a-b)$ is an isometric isomorphism of Lip(X) onto itself which does not arise from composition with an isometry of X.

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The papers [Je], [JP], [R], and [V] all deal with the classification of isometric isomorphisms. Vasavada's result in [V] implies all the others (except that of [JP], which is false); it states that if $X, Y \in \mathcal{M}^2$ are compact and β-connected for some $\beta < 1$, then every isometric isomorphism from Lip(X) onto Lip(Y) arises in the desired manner from an isometry from Y onto X. Here "β-connected" means that the space cannot be decomposed into two disjoint sets whose distance is $\geq \beta$. (This is not exactly the stated result, but is trivially equivalent to it.)

(The argument given in [JP] fails on p. 200, where it is falsely claimed that a certain condition distinguishes "good" extreme points of the dual unit ball of Lip(X) from "bad" ones. The argument does hold under the assumption that X and Y have diameters < 1, but in this case the result follows from Vasavada's.)

We find that we can weaken Vasavada's hypothesis to require only that $X, Y \in \mathcal{M}^2$ be 1-connected. The passage from β to 1 is perhaps a minor improvement, but the removal of the compactness assumption seems more significant. All published research known to this author which deals with the classification of isometric isomorphisms, depends heavily on the assumption that the underlying metric spaces are compact. Dispensing with this assumption requires a new technique, which we develop in Section 1.

(One should also mention the paper of Mayer-Wolf [MW], which classifies the isometric isomorphisms of the so-called "Lip$^\alpha$" spaces for $0 < \alpha < 1$. Mayer-Wolf also assumes compactness and this assumption can be removed by a technique similar to that given here. We shall give more details on this in a separate publication.)

1. **Normality of dual extreme points.** The following construction is one of the basic tools in the study of Lipschitz spaces; it derives from the seminal paper of de Leeuw [dL]. For any metric space X let $\hat{X} = X^2 - \{(p,p) : p \in X\}$ and let W be the topological space which is the disjoint union of X and $\hat{X}$. Then we have an isometry Φ from Lip(X) into $C_b(W)$ (= the bounded continuous scalar-valued functions on W) defined by $\Phi f(p) = f(p)$ for $p \in X$ and

$$\Phi f(p,q) = \frac{f(p) - f(q)}{\rho(p,q)}$$

for $(p,q) \in \hat{X}$.

The embedding Φ is useful because it allows us to classify the extreme points of the dual unit ball $\mathcal{B}\big(\mathrm{Lip}(X)^*\big)$. (Note: for any Banach space E we write $\mathcal{B}(E)$ for its closed unit ball.) Namely, by a standard extension theorem (*e.g.* see [C], Proposition V.7.9), every extreme point of $\mathcal{B}\big(\mathrm{Lip}(X)^*\big)$ extends to an extreme point of $\mathcal{B}(C_b(W)^*)$. Now $C_b(W) \cong C(\beta W)$, where βW is the Stone-Čech compactification of W, and the dual of the latter can be identified with $M(\beta W)$, the space of finite Borel measures on βW. The extreme points of the unit ball of $M(\beta W)$ are precisely the measures $\alpha\mu_\theta$ where $\alpha \in \mathbf{U}$ and μ_θ is the point mass at $\theta \in \beta W$. Thus, for every extreme point x of $\mathcal{B}\big(\mathrm{Lip}(X)^*\big)$ we can find $\alpha \in \mathbf{U}$ and $\theta \in \beta W$ such that $x = \Phi^*(\alpha\mu_\theta)$.

For $\theta \in W$ it is easy to describe the action of the linear functional $\Phi^*(\mu_\theta)$. If $\theta = p \in X$, then $\Phi^*(\mu_\theta) = \chi_p$, the "evaluation at p" functional defined by $\chi_p(f) = f(p)$; if $\theta = (p,q) \in \hat{X}$ then $\Phi^*(\mu_\theta) = (\chi_p - \chi_q)/\rho(p,q)$.

The key point of the following theorem is that if $x = \Phi^*(\alpha\mu_\theta)$ for some $\alpha \in \mathbf{U}$ and $\theta \in \beta W$, then one can tell whether $\theta \in W$ by examining the action of x on the order structure of Lip(X). The relevant concept is the following. Let (f_λ) be a net of real-valued functions in Lip(X); then we write $f_\lambda \searrow 0$ if (f_λ) is decreasing (*i.e.* $\lambda \leq \kappa$ implies $f_\lambda \geq f_\kappa$) and (f_λ) converges pointwise to 0. Equivalently, $f_\lambda \searrow 0$ if (f_λ) is decreasing and $\bigwedge f_\lambda = 0$. (As we noted in [W], every bounded set of real-valued functions in Lip(X) has a meet and a join, which satisfy $\|\bigwedge f_\lambda\|_L, \|\bigvee f_\lambda\|_L \leq \sup\|f_\lambda\|_L$.) We say that $x \in \mathrm{Lip}(X)^*$ is *normal* if $x(f_\lambda) \longrightarrow 0$ whenever (f_λ) is a bounded net of real-valued functions such that $f_\lambda \searrow 0$. By the second definition of $f_\lambda \searrow 0$, it follows that normality of x can be defined purely in terms of the order structure of Lip(X).

THEOREM A. *Let X be a complete metric space with finite diameter and let x be an extreme point of $\mathcal{B}(\mathrm{Lip}(X)^*)$. Then the following are equivalent:*

a) x is in the linear span of the evaluation functionals χ_p ($p \in X$);

b) x is normal;

c) $x = \alpha\chi_p$ for some $\alpha \in \mathbf{U}$ and $p \in X$ or $x = \alpha(\chi_p - \chi_q)/\rho(p,q)$ for some $\alpha \in \mathbf{U}$ and $(p,q) \in \hat{X}$.

PROOF. a) $\Rightarrow$ b). Trivial.

b) $\Rightarrow$ c). Find $\alpha \in \mathbf{U}$ and $\theta \in \beta W$ such that $x = \Phi^*(\alpha\mu_\theta)$. We are going to prove the contrapositive and therefore assume that $\theta \notin W$. We will show that $\Phi^*\mu_\theta$ is not normal, which will imply that x is also not normal.

For the first two cases below, suppose $\theta \in \beta\hat{X} - \hat{X}$ and find a net of elements $(p_\lambda, q_\lambda) \in \hat{X}$ such that $(p_\lambda, q_\lambda) \longrightarrow \theta$. By taking subnets we may suppose that $p_\lambda \longrightarrow \theta_1$ and $q_\lambda \longrightarrow \theta_2$ for some $\theta_1, \theta_2 \in \beta X$. Also, since x is not zero, there exists $g \in \mathrm{Lip}(X)$ such that $\Phi^*\mu_\theta(g) = \widetilde{\Phi g}(\theta) \neq 0$, where $\widetilde{\Phi g}$ is the continuous extension of Φg to βW. Writing g as a linear combination of positive Lipschitz functions, this shows that $\widetilde{\Phi f}(\theta) \neq 0$ for some positive $f \in \mathrm{Lip}(X)$. Dividing by a positive scalar, we may assume that $\|f\|_L = 1$.

CASE 1. Suppose $\theta \in \beta\hat{X} - \hat{X}$ and $\tilde{f}(\theta_1) = \tilde{f}(\theta_2)$ where $\tilde{f}$ is the continuous extension of f to βX. Let k be this common value and define a sequence of functions

$$f_n = [(f - k + k/n) \vee 0] \wedge 2k/n.$$

Clearly the sequence (f_n) is bounded in Lipschitz norm and $f_n \searrow 0$. However, for all $n \in \mathbf{N}$

$$\widetilde{\Phi f_n}(\theta) = \lim_\lambda \Phi f_n(p_\lambda, q_\lambda) = \lim_\lambda \Phi f(p_\lambda, q_\lambda) = \widetilde{\Phi f}(\theta),$$

since for sufficiently large λ we have $|f(p_\lambda) - k|, |f(q_\lambda) - k| < k/n$ hence $\Phi f_n(p_\lambda, q_\lambda) = \Phi f(p_\lambda, q_\lambda)$. Thus, since $\widetilde{\Phi f}(\theta) \neq 0$, $\Phi^*\mu_\theta(f_n)$ does not converge to zero, hence $\Phi^*\mu_\theta$ is not normal, which is what we wanted to show.

CASE 2. Suppose $\theta \in \beta\hat{X} - \hat{X}$ and $\tilde{f}(\theta_1) \neq \tilde{f}(\theta_2)$. This implies $\theta_1 \neq \theta_2$, and since $\theta \notin \hat{X}$ it follows that θ_1 and θ_2 cannot both be in X. Without loss of generality suppose that $\theta_1 \notin X$. Then p_λ does not cluster at any point of X, hence (since X is complete) it has

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no Cauchy subnet. Taking a universal subnet, this implies that there exists $\epsilon' > 0$ such that for every $p \in X$, the ϵ'-ball about p is eventually disjoint from the subnet. Thus, taking subnets, we may assume that for every $p \in X$ we eventually have $\rho(p_\lambda, p) \geq \epsilon'$.

Let $\epsilon = \min\big(\epsilon', |\tilde{f}(\theta_1) - \tilde{f}(\theta_2)|/2\big)$. Then the net f_κ defined by

$$f_\kappa(p) = \bigvee_{\lambda \geq \kappa} \max\big(0, \epsilon - \rho(p, p_\lambda)\big)$$

is bounded in Lipschitz norm and decreasing pointwise to zero. Also, since $\|f\|_L = 1$ and $f(p_\lambda) \longrightarrow \tilde{f}(\theta_1)$ and $f(q_\lambda) \longrightarrow \tilde{f}(\theta_2)$, it follows that eventually $\rho(p_\lambda, q_\kappa) \geq \epsilon$, *i.e.* this holds for all $\lambda, \kappa \geq$ some λ_0. Thus for each $\kappa \geq \lambda_0$ we have $\lim_\lambda f_\kappa(p_\lambda) = \epsilon$ and $\lim_\lambda f_\kappa(q_\lambda) = 0$, hence

$$\begin{aligned}\widetilde{\Phi f_\kappa}(\theta) &= \lim_\lambda \Phi f_\kappa(p_\lambda, q_\lambda) \\ &= \lim_\lambda \big(f_\kappa(p_\lambda) - f_\kappa(q_\lambda)\big) / \rho(p_\lambda, q_\lambda) \\ &= \epsilon / \tilde{\rho}(\theta),\end{aligned}$$

where $\tilde{\rho}$ is the continuous extension of the distance function ρ to $\beta\hat{X}$. We conclude that $\Phi^*\mu_\theta$ is not normal since $\Phi^*\mu_\theta(f_\kappa) = \widetilde{\Phi f_\kappa}(\theta)$ evidently does not converge to 0.

CASE 3. Finally, suppose $\theta \in \beta X - X$. Then we can find a net $(p_\lambda) \subset X$ which converges to θ; as in Case 2 we may assume that for every $p \in X$ we eventually have $\rho(p_\lambda, p) \geq \epsilon$, for some $\epsilon > 0$.

Then the net f_κ defined by

$$f_\kappa(p) = \bigvee_{\lambda \geq \kappa} \max\big(0, \epsilon - \rho(p, p_\lambda)\big)$$

is bounded in Lipschitz norm and decreasing pointwise to zero. But

$$\tilde{f}_\kappa(\theta) = \lim_\lambda f_\kappa(p_\lambda) = \epsilon,$$

so once again $\Phi^*\mu_\theta$ is not normal.

c) $\Rightarrow$ a). Vacuous. ∎

According to ([Jo], Corollary 4.2), the closed span of the evaluation functionals χ_p in $\mathrm{Lip}(X)^*$ is a predual of Lip(X), *i.e.* the dual of this space can be identified with Lip(X). It is easy to see that every element of this space is a normal linear functional on Lip(X) and it is natural to ask whether this property characterizes the space. That is, if $x \in \mathrm{Lip}(X)^*$ is normal does it follow that x is in the closed span of the evaluation functionals? We do not know the answer to this question but conjecture it to be no. (It is fairly easy to see that the answer is yes if x is assumed to be decomposable into positive functionals, but not every x is so decomposable.)

We also wish to include the following two facts for reference. The first is trivial and appeared in [V]; the second is well-known in the compact case, and the non-compact proof is not much different, but we give it just to be safe.

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PROPOSITION B. *Let X be a metric space of diameter ≤ 2. Then for any $p, q \in X$, $\rho(p,q) = \|\chi_p - \chi_q\|$ (taking the norm in* $\mathrm{Lip}(X)^*$*).*

PROPOSITION C. *Let X be a metric space of diameter ≤ 2 and let $p \in X$. Then χ_p is an extreme point of $\mathcal{B}\big(\mathrm{Lip}(X)^*\big)$.*

PROOF. Define the function $f \in \mathrm{Lip}(X)$ by $f(q) = 1 - \rho(p,q)/2$. Then $|\Phi f| \leq 1/2$ on $\hat{X}$, and for any $\epsilon > 0$ we have $|\Phi f(q)| \leq 1 - \epsilon/2$ for all $q \in X$ outside the ϵ-ball about p.

Suppose $\chi_p = tx_1 + (1-t)x_2$ for some $x_1, x_2 \in \mathcal{B}\big(\mathrm{Lip}(X)^*\big)$ and $t \in (0,1)$. We can find measures $\mu_1, \mu_2 \in \mathcal{B}\big(M(\beta W)\big)$ such that $x_1 = \Phi^*\mu_1$ and $x_2 = \Phi^*\mu_2$. Now since $\Phi f(p) = 1$ we have

$$1 = t\int(\Phi f)\,d\mu_1 + (1-t)\int(\Phi f)\,d\mu_2.$$

Since $\|\Phi f\|_\infty = 1$ and $\|\mu_1\|, \|\mu_2\| \leq 1$ we must have

$$\int(\Phi f)\,d\mu_1 = \int(\Phi f)\,d\mu_2 = 1.$$

But $|\Phi f(\theta)| < 1$ for all $\theta \in \beta W$ except p, so μ_1 and μ_2 must be supported on this point. It follows that $\mu_1 = \mu_2$ is the point mass at p, hence $x_1 = x_2 = \chi_p$. So χ_p is an extreme point. ∎

The basic technique of the preceding proof comes from [dL].

2. **Isometries of 1-connected spaces.** Recall that $\mathcal{M}^2$ is the class of all complete metric spaces with diameter ≤ 2. The goal of this section is to prove that if $X, Y \in \mathcal{M}^2$ are 1-connected then every isometric isomorphism from $\mathrm{Lip}(X)$ onto $\mathrm{Lip}(Y)$ arises in a simple way from an isometry of Y onto X. The proof proceeds through a series of lemmas; the general idea is that the adjoint of the given isometric isomorphism preserves a lot of the structure of the dual space.

In Lemmas 1–6 let $X, Y \in \mathcal{M}^2$ be 1-connected metric spaces, let $T\colon \mathrm{Lip}(X) \longrightarrow \mathrm{Lip}(Y)$ be a surjective isometric isomorphism, and let $T^*\colon \mathrm{Lip}(Y)^* \longrightarrow \mathrm{Lip}(X)^*$ be the adjoint map (also a surjective isometric isomorphism, of course).

LEMMA 1. *$|T(1)(p)| = 1$ for all $p \in Y$, where 1 denotes the constant function on X.*

PROOF. The function $\Phi(1)$ takes only the values 0 and 1 on W, so

$$\begin{aligned}\big\{x(1) : x \text{ is an extreme point of } \mathcal{B}\big(\mathrm{Lip}(X)^*\big)\big\} &\subset \{\alpha(\widetilde{\Phi 1})(\theta) : \alpha \in \mathbf{U}, \theta \in \beta W\}\\ &\subset \mathbf{U} \cup \{0\}.\end{aligned}$$

Therefore, letting $f = T(1)$, we also have that $\big\{x(f) : x$ is an extreme point of $\mathcal{B}\big(\mathrm{Lip}(Y)^*\big)\big\} \subset \mathbf{U} \cup \{0\}$, since this set is evidently preserved by surjective isometric isomorphisms. By Proposition C we get $f(p) = \chi_p(f) \in \mathbf{U} \cup \{0\}$ for all $p \in Y$. However, since $\|f\|_L = \|1\|_L = 1$ hence $L(f) \leq 1$, the sets $f^{-1}(0)$ and $f^{-1}(\mathbf{U})$ contradict 1-connectedness of Y unless one of them is empty. Clearly we cannot have $f(p) = 0$ for all $p \in Y$, hence $f^{-1}(0) = \emptyset$ and so $|f(p)| = 1$ for all $p \in Y$. ∎

We call an extreme point x of $\mathcal{B}\big(\mathrm{Lip}(X)^*\big)$ *simple* if $x = \alpha\chi_\theta = \Phi^*(\alpha\mu_\theta)$ for some $\alpha \in \mathbf{U}$ and $\theta \in \beta X$. (The evaluation functional χ_θ is, of course, defined by $\chi_\theta(f) = \tilde{f}(\theta)$.) Clearly, every simple extreme point x satisfies $x(1) \in \mathbf{U}$. Conversely, a non-simple extreme point x must equal $\Phi^*(\alpha\mu_\theta)$ for some $\theta \in \beta\hat{X}$, and as $\Phi(1)$ is 0 on $\hat{X}$ this implies $x(1) = 0$. Thus, if x is simple then $x(1) \in \mathbf{U}$, and otherwise $x(1) = 0$.

LEMMA 2. *T^* carries simple extreme points of $\mathcal{B}\big(\mathrm{Lip}(Y)^*\big)$ to simple extreme points of $\mathcal{B}\big(\mathrm{Lip}(X)^*\big)$.*

PROOF. As in Lemma 1 let 1 denote the constant function on X and let $f = T(1)$. If the extreme point x of $\mathcal{B}\big(\mathrm{Lip}(Y)^*\big)$ is simple, say $x = \Phi^*(\alpha\mu_\theta)$ for some $\theta \in \beta Y$, then $x(f) = \alpha\tilde{f}(\theta) \in \mathbf{U}$ by Lemma 1. Thus $(T^*x)(1) = x(f) \neq 0$ which implies that T^*x is simple. ∎

LEMMA 3. *Let $\alpha \in \mathbf{U}$, $\alpha \neq 1$, and $t \in [0, 1)$. Then $t < \big|\alpha(1-t) - 1\big|$.*

PROOF. The lemma is trivial in the real case. In the complex case we have $|\alpha(1-t) - 1| = |\alpha^{-1} - (1-t)|$, and as α ranges over the unit circle the complex number $\alpha^{-1} - (1-t)$ ranges over the unit circle shifted to the left by $1 - t$. Excepting the point corresponding to $\alpha = 1$, the latter is strictly outside the disk about the origin of radius t. ∎

LEMMA 4. *Let $\alpha \in \mathbf{U}$, $\alpha \neq 1$, let $p, q \in X$, and let $\theta, \phi \in \beta X$. Then (taking all norms in $\mathrm{Lip}(X)^*$)*

a) $\rho(p,q) = \|\chi_p - \chi_q\| < 1$ *implies* $\|\chi_p - \chi_q\| < \|\alpha\chi_p - \chi_q\|$;
b) $\|\chi_\theta - \chi_\phi\| < 1$ *implies* $\|\chi_\theta - \chi_\phi\| \leq \|\alpha\chi_\theta - \chi_\phi\|$; *and*
c) $\|\chi_\theta - \chi_\phi\| \geq 1$ *implies* $\|\alpha\chi_\theta - \chi_\phi\| \geq 1$.

PROOF. a) We noted in Proposition B that $\rho(p,q) = \|\chi_p - \chi_q\|$. Suppose $\rho(p,q) < 1$. Then the function $f(r) = 1 - \rho(r,q)$ is in $\mathcal{B}\big(\mathrm{Lip}(X)\big)$ and so

$$\|\alpha\chi_p - \chi_q\| \geq |(\alpha\chi_p - \chi_q)(f)| = \Big|\alpha\big(1 - \rho(p,q)\big) - 1\Big| > \rho(p,q),$$

by Lemma 3.

b) Suppose $\|\chi_\theta - \chi_\phi\| < 1$. Then for any $\epsilon > 0$ we can choose $g \in \mathcal{B}\big(\mathrm{Lip}(X)\big)$ such that

$$\|\chi_\theta - \chi_\phi\| \leq |(\chi_\theta - \chi_\phi)(g)| + \epsilon = |\tilde{g}(\theta) - \tilde{g}(\phi)| + \epsilon.$$

Define $f(p) = 1 - |g(p) - \tilde{g}(\phi)|$. Then $f \in \mathcal{B}\big(\mathrm{Lip}(X)\big)$ and since

$$|\tilde{g}(\theta) - \tilde{g}(\phi)| = |(\chi_\theta - \chi_\phi)(g)| \leq \|\chi_\theta - \chi_\phi\| < 1,$$

Lemma 3 then shows that

$$\begin{aligned}\|\alpha\chi_\theta - \chi_\phi\| &\geq |(\alpha\chi_\theta - \chi_\phi)(f)| = \Big|\alpha\big(1 - |\tilde{g}(\theta) - \tilde{g}(\phi)|\big) - 1\Big| \\ &> |\tilde{g}(\theta) - \tilde{g}(\phi)| \geq \|\chi_\theta - \chi_\phi\| - \epsilon.\end{aligned}$$

Taking $\epsilon \longrightarrow 0$ yields the desired.

c) If $\|\chi_\theta - \chi_\phi\| \geq 1$ then for any $\epsilon > 0$ we can choose $g \in \mathcal{B}\big(\mathrm{Lip}(X)\big)$ so that

$$1 - \epsilon < |(\chi_\theta - \chi_\phi)(g)| = |\tilde{g}(\theta) - \tilde{g}(\phi)| < 1.$$

Then reasoning just as in case b) we get

$$\|\alpha\chi_\theta - \chi_\phi\| \geq |\tilde{g}(\theta) - \tilde{g}(\phi)| > 1 - \epsilon,$$

which is enough. ∎

Now let $x = \alpha\chi_\theta$ and $y = \beta\chi_\phi$ ($\alpha, \beta \in \mathbf{U}$ and $\theta, \phi \in \beta X$) be simple extreme points of $\mathcal{B}\big(\mathrm{Lip}(X)^*\big)$; we say they are *aligned* if $\alpha = \beta$, *i.e.* if $x(1) = y(1)$. The point of Lemma 4 is that if x and y are close enough then one can tell whether they are aligned by looking at the norms of linear combinations of x and y. This idea is used in the proof of the next lemma.

LEMMA 5. *Let $p, q \in Y$, $p \neq q$, $\rho(p,q) < 1$. Then $T^*\chi_p$ and $T^*\chi_q$ are aligned.*

PROOF. Let $x = T^*\chi_p$ and $y = T^*\chi_q$. Since χ_p and χ_q are simple extreme points of $\mathcal{B}\big(\mathrm{Lip}(Y)^*\big)$, Lemma 2 shows that x and y are simple extreme points of $\mathcal{B}\big(\mathrm{Lip}(X)^*\big)$. Thus let $x = \alpha\chi_\theta$ and $y = \beta\chi_\phi$ for $\alpha, \beta \in \mathbf{U}$ and $\theta, \phi \in \beta X$.

By Lemma 4 c), $\|x - y\| = \|\chi_p - \chi_q\| = \rho(p,q) < 1$ implies that $\|\chi_\theta - \chi_\phi\| < 1$. If $\alpha \neq \beta$ then by Lemma 4 a),

$$\begin{aligned}\|(\alpha/\beta)\chi_\theta - \chi_\phi\| = \|x - y\| = \|\chi_p - \chi_q\| < \|(\beta/\alpha)\chi_p - \chi_q\| = \|(\beta/\alpha)x - y\| \\ = \|\chi_\theta - \chi_\phi\|,\end{aligned}$$

which together with $\|\chi_\theta - \chi_\phi\| < 1$ contradicts Lemma 4 b). So $\alpha = \beta$ as desired. ∎

LEMMA 6. *$T(1) = \alpha$ is a constant function and $\alpha^{-1}T$ is an isometric isomorphism of* $\mathrm{Lip}(X)$ *onto* $\mathrm{Lip}(Y)$ *which is also an order-isomorphism. Its adjoint $\alpha^{-1}T^*$ takes every evaluation functional χ_q ($q \in Y$) to an evaluation functional χ_p ($p \in X$).*

PROOF. For any $p, q \in Y$, $\rho(p,q) < 1$, we have

$$T(1)(p) = (T^*\chi_p)(1) = (T^*\chi_q)(1) = T(1)(q)$$

since $T^*\chi_p$ and $T^*\chi_q$ are aligned by Lemma 5. Since Y is 1-connected, this shows that $T(1)$ is a constant function; say $T(1) = \alpha$. Then $\alpha \in \mathbf{U}$ by Lemma 1.

Since $|\alpha| = 1$, $\alpha^{-1}T$ is clearly an isometric isomorphism of $\mathrm{Lip}(X)$ onto $\mathrm{Lip}(Y)$. To see that it preserves order, suppose $f \in \mathrm{Lip}(X)$, $f \geq 0$. Then for every $p \in Y$, letting $T^*\chi_p = \alpha\chi_\theta$ by Lemma 2 (the coefficient is α since $\alpha = T(1)(p) = (T^*\chi_p)(1)$) we get

$$(\alpha^{-1}Tf)(p) = (\alpha^{-1}T^*\chi_p)(f) = \tilde{f}(\theta) \geq 0,$$

so $\alpha^{-1}Tf \geq 0$ also. To see that the inverse map αT^{-1} preserves order, simply interchange X and Y and apply the same argument. Thus $\alpha^{-1}T$ is an order-isomorphism.

Now to show $\alpha^{-1}T^*\chi_p = \chi_\theta$ satisfies $\theta = q \in X$ it suffices by Theorem A to show that χ_θ is normal. But χ_p is normal, so since $\alpha^{-1}T$ is an order isomorphism so is χ_θ. ∎

If one assumes that X is compact then $X = \beta X$ and so the last part of Lemma 6 is trivial. This is the crucial step where the noncompact case requires extra work.

We can now prove our main result.

THEOREM D. *Let* $X, Y \in \mathcal{M}^2$ *be* 1*-connected and let* $T\colon \mathrm{Lip}(X) \longrightarrow \mathrm{Lip}(Y)$ *be a surjective isometric isomorphism. Then for some* $\alpha \in \mathbf{U}$ *and some isometry* g *of* Y *onto* X*, we have* $Tf = \alpha f \circ g$ *for all* $f \in \mathrm{Lip}(X)$.

PROOF. The scalar α is defined as in Lemma 6, and $g\colon Y \longrightarrow X$ is defined by $\alpha^{-1} T^* \chi_q = \chi_{g(q)}$. This is an isometry since $\alpha^{-1} T^*$ is an isometry by Lemma 6 and since X and Y can be isometrically identified with the evaluation functionals by Proposition B. The desired formula holds since

$$(Tf)(q) = \chi_q(Tf) = (T^* \chi_q)(f) = \alpha \chi_{g(q)}(f) = \alpha f\big(g(q)\big)$$

for all $f \in \mathrm{Lip}(X)$ and $q \in Y$. Finally, g is onto since otherwise there would exist $p \in X$ such that $\epsilon = \rho\big(p, g(Y)\big) > 0$, and by the above formula for T we would have $T(f) = 0$ for the function $f(r) = \max\big(0, \epsilon - \rho(p, r)\big)$, contradicting the fact that T is an isometry. ∎

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