PRIME IDEALS IN VECTOR LATTICES

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1. Introduction. Projectors, spectral functions, carriers, and collections of these objects are some of the tools which have been used to study vector lattices. One of our objectives in this paper is to show that these various approaches are not essentially different. We do this by proving that each of the above-mentioned objects can be identified with a collection of prime ideals.

A linear subspace I of a vector lattice is called an *ideal* if $|x| \leqslant |y|$ and $y \in I$ imply that $x \in I$. A proper ideal I in a vector lattice is called *prime* if $x \wedge y \in I$ implies that $x \in I$ or $y \in I$. The theorem established in § 5 shows that an archimedean vector lattice E may be represented as a vector lattice of extended functions on a large variety of topological spaces. The spaces in question are subspaces of the space of all prime ideals in E, equipped with the hull-kernel topology.

The study of vector lattices via prime ideals is not new. To cite just one example, Yosida **(13)**, using such ideals, proved that every archimedean vector lattice is isomorphic to a vector lattice of extended functions on some Hausdorff space. As will be shown, the space which he used is locally compact.

In **(11)**, Nakano proved that every σ-complete vector lattice E is isomorphic to a vector lattice of extended functions on some totally disconnected Hausdorff space X. The space X is obtained by providing the collection of all maximal dual ideals in the distributive lattice of projectors on E with the dual hull-kernel topology. Yosida commented on the difference between the topological space which he used and the one used by Nakano. In § 6, we show that the representations of Yosida and Nakano can each be obtained by a suitable specialization of the space of prime ideals in Theorem 5.1. Our Theorem 6.7 provides a generalization of Nakano's representation to arbitrary (not necessarily σ-complete) archimedean vector lattices.

Using the concept of spectral function, Amemiya **(1)** developed a spectral theory for vector lattices, generalizing Nakano's theory for the σ-complete and complete cases **(10)**. In § 4, we indicate how Amemiya's spectral theory can be obtained by ideal-theoretic methods. We do this by showing that the set of all spectral functions defined on a vector lattice E is essentially the same as the set of all prime ideals in E. Section 2 is devoted to a brief sketch of part of Amemiya's spectral theory, while some of the properties of prime ideals which we will use are recorded in § 3.

The lattice $\mathfrak{C}$ of carriers of a vector lattice E, an object of recent origin **(7)**,

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has been useful in the study of lattice-ordered algebraic systems (for example, **4**; **5**; **7**; **12**). In § 7, we show that $\mathfrak{C}$ is isomorphic to a basis for the open sets in $\mathfrak{M}$, the set of all minimal prime ideals in E equipped with the hull-kernel topology. Our proof of this result depends upon a characterization of minimal prime ideals obtained in § 6. We make another application of this characteristic property of minimal prime ideals to reverify a theorem of Lorenzen (**8**).

The setting for our results is a vector lattice, but the reader will observe that most of the development, with obvious modifications, is valid for any commutative lattice-ordered group.

2. Spectral theory. Throughout this paper, E will denote a vector lattice, and E_+ its cone of positive elements. We use the terminology of (**2**), except that x^- denotes the positive element $(-x)^+$. This section is devoted to a brief sketch, without proofs, of part of the development in (**1**).

Denote the points of the three element chain **3** by the symbols $-\infty$, $\mathbf{0}$, $+\infty$, and let the order be given by $-\infty < \mathbf{0} < +\infty$. A commutative addition, with $\mathbf{0}$ acting as an additive identity, is defined in **3** such that $(+\infty) + (+\infty) = +\infty$, and $(-\infty) + (-\infty) = -\infty$, except that $(+\infty) + (-\infty)$ has no meaning. The elements of **3** are multiplied by real numbers subject to the following rules:

$\alpha\mathbf{0} = \mathbf{0}$ for each real number α, and $0(\pm\infty) = \mathbf{0}$;
$\alpha(+\infty) = +\infty$, and $\alpha(-\infty) = -\infty$ for each positive real number α;
$\alpha(+\infty) = -\infty$, and $\alpha(-\infty) = +\infty$ for each negative real number α.

A function f with domain a vector lattice E and range contained in **3** is called a *spectral function* if it satisfies the following conditions:

(1) $f(x) \neq \mathbf{0}$ for at least one x in E;
(2) $f(\alpha x) = \alpha f(x)$ for each x in E, and each real number α;
(3) $f(x \vee y) = \max\{f(x), f(y)\}$ for each pair x, y in E.

Let $\mathfrak{F}$ denote the collection of all spectral functions defined on E. Partially order $\mathfrak{F}$ by writing $f \leqslant g$ in case $f(x) \leqslant g(x)$ for all x in E_+. For each x in E, put $\mathfrak{F}_x = \{f \in \mathfrak{F} : f(x) \neq \mathbf{0}\}$. For any pair x, y in E, and any non-zero scalar α, we have $\mathfrak{F}_{\alpha x} = \mathfrak{F}_x = \mathfrak{F}_{|x|}$; $\mathfrak{F}_{|x| \wedge |y|} = \mathfrak{F}_x \cap \mathfrak{F}_y$; and $\mathfrak{F}_{|x| \vee |y|} = \mathfrak{F}_x \cup \mathfrak{F}_y$. Thus the collection of sets $\{\mathfrak{F}_x : x \in E_+\}$ is a basis for the open sets for a topology on $\mathfrak{F}$, and the latter set, equipped with this topology, and the given partial order, is called the *spectral space* of E.

If a is a non-zero element of E, and if $f \in \mathfrak{F}_a$, then for each x in E, the *relative spectrum* of x with respect to a at f, denoted by $(x/a, f)$, is defined as the infimum of all real numbers α such that $f(\alpha a - x) = f(a)$, where the infimum of the empty set is understood to be $+\infty$.

A vector lattice is called *archimedean* if $nb \leqslant a$ for all non-negative integers n implies that $b \leqslant 0$. The following theorem summarizes some of the properties of the relative spectrum.

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THEOREM 2.1. *Let a be a non-zero element in E, let $f \in \mathfrak{F}_a$, and let x and y be arbitrary elements of E. Then*

(a) $$\left(\frac{\alpha x + \beta y}{a}, f\right) = \alpha(x/a, f) + \beta(y/a, f)$$

for all real numbers α and β, provided the right-hand side makes sense;

(b) *if $f(a) = +\infty$, then*

$$\left(\frac{x \vee y}{a}, f\right) = \max\{(x/a, f), (y/a, f)\}$$

and

$$\left(\frac{x \wedge y}{a}, f\right) = \min\{(x/a, f), (y/a, f)\};$$

(c) *for each fixed x, $(x/a, .)$ is a continuous function on $\mathfrak{F}_a$, where the latter set is given the relative topology;*

(d) *if E is archimedean, then for each x in E, $(x/a, .)$ is real valued on an open, everywhere dense subset of $\mathfrak{F}_a$;*

(e) *if E is archimedean, and if both $\mathfrak{F}_x$ and $\mathfrak{F}_y$ are contained in $\mathfrak{F}_a$, then $x \leqslant y$ if, and only if, $(x/a, f) \leqslant (y/a, f)$ for all f in $\mathfrak{F}_a$.*

A spectral function f is said to be *maximal* if $f \leqslant g$, where $g \in \mathfrak{F}$, implies that $f = g$; it is called *minimal* if $g \leqslant f$ implies that $g = f$. In general, there are no minimal spectral functions on E. The set of all maximal spectral functions is called the *proper space* of E. A proof of the following result will be given in § 6.

THEOREM 2.2. *The proper space of any vector lattice is a Hausdorff space in its relative topology, has a basis consisting of open and closed sets, and is dense in the spectral space.*

3. Prime ideals. A *homomorphism* of the vector lattice E is a linear transformation f of E into a vector lattice F such that $f(x^+) = (f(x))^+$ for all x in E; a one-one homomorphism is called an *isomorphism*. If I is an ideal in E, then let $I(x)$ denote the element of the quotient vector space E/I which contains the element x. The vector space E/I is partially ordered by agreeing that a coset $I(x)$ is positive if there is an element y in E_+ such that $x - y \in I$. With this ordering, E/I is a vector lattice, and the canonical mapping $x \to I(x)$ of E onto E/I is a homomorphism.

THEOREM 3.1. *The following statements are equivalent for an ideal P in the vector lattice E.*

(a) *P is a prime ideal.*

(b) *If $x \wedge y = 0$, then either $x \in P$ or $y \in P$.*

(c) *The quotient vector lattice E/P is linearly ordered.*

(d) *If $P \supseteq I \cap J$, where I and J are ideals in E, then either $P \supseteq I$ or $P \supseteq J$.*

Proof. That (a) implies (b) is obvious.

(b) *implies* (c). For every x in E, we have $x^+ \wedge x^- = 0$, and so by (b), either $x^+ \in P$ or $x^- \in P$. Thus, either $P(x^+) = (P(x))^+ = 0$, or $P(x^-) = (P(x))^- = 0$, that is, either $P(x) \leqslant 0$ or $P(x) \geqslant 0$.

(c) *implies* (d). Suppose $P \supseteq I \cap J$, where I and J are ideals in E. Then, in E/P, we have $P(I) \cap P(J) = \{0\}$. But, $P(I)$ and $P(J)$ are ideals in E/P, and the ideals in a linearly ordered vector lattice form a chain, so this means that either $P(I) = \{0\}$ or $P(J) = \{0\}$. Thus, either $P \supseteq I$ or $P \supseteq J$.

(d) *implies* (a). If, for an element w in E, we let (w) denote the ideal generated by w, then this set consists of all elements v such that $|v| \leqslant n|w|$ for some positive integer n. Now let x and y be elements of E_+ such that $x \wedge y \in P$. It is easy to see that $(x \wedge y) = (x) \cap (y)$. Thus, $(x) \cap (y) \subseteq P$ implies that $(x) \subseteq P$ or $(y) \subseteq P$, that is, $x \in P$ or $y \in P$. If x and y are any elements of E such that $x \wedge y \in P$, then $(x - z) \wedge (y - z) = 0$, where $z = x \wedge y$. Since $x - z$ and $y - z$ are elements of E_+, the above case implies that $x \in P$ or $y \in P$. This completes the proof.

Now let $\mathfrak{P}$ denote any collection of prime ideals in E. For any subset $\mathfrak{S}$ of $\mathfrak{P}$, the *kernel* of $\mathfrak{S}$, denoted by $k(\mathfrak{S})$, is defined to be the set of elements in E that are common to all of the ideals in $\mathfrak{S}$. For any subset A of E, the *hull* of A, denoted by $h(A)$, is the set of all P in $\mathfrak{P}$ such that $P \supseteq A$. If $\mathfrak{S}$ is a subset of $\mathfrak{P}$, then put $\mathfrak{S}^- = h(k(\mathfrak{S}))$. In virtue of the equivalence of statements (a) and (d) of the above theorem, the correspondence $\mathfrak{S} \to \mathfrak{S}^-$ is a closure operator on $\mathfrak{P}$ which makes the latter set into a topological space. The topology so defined on $\mathfrak{P}$ is called the *hull-kernel topology*. If x is an element of E, then let $\mathfrak{P}_x = \{P \in \mathfrak{P} : x \notin P\}$. It is easy to see that the collection $\{\mathfrak{P}_x : x \in E\}$ is a basis for the open sets for the hull-kernel topology on $\mathfrak{P}$.

4. Spectral functions and prime ideals. Let P be a prime ideal in E, and x an element of E. If $|x| \notin P$, then since $|x| = x^+ + x^-$, either $x^+ \notin P$ or $x^- \notin P$. But $x^+ \wedge x^- = 0$, and so either $x^+ \in P$ or $x^- \in P$. Hence, the three cases $|x| \in P$; $x^+ \notin P$; $x^- \notin P$ are mutually exclusive and exhaustive. It follows that the function $\bar{P}$, where

$$\bar{P}(x) = \begin{cases} \mathbf{0} & \text{if } |x| \in P \\ +\infty & \text{if } x^+ \notin P \\ -\infty & \text{if } x^- \notin P \end{cases}$$

is well defined.

We show that $\bar{P}$ is a spectral function on E. Condition (1) in the definition of spectral function follows since P is proper. We have $|\alpha x| = |\alpha||x|$ for all real numbers α, $(\alpha x)^+ = \alpha x^+$ for $\alpha \geqslant 0$, and $(\alpha x)^+ = -\alpha x^-$ for $\alpha < 0$. It follows that $\bar{P}$ satisfies condition (2). To verify condition (3), we first show that $\bar{P}(x) \leqslant \bar{P}(y)$ whenever $x \leqslant y$. Suppose that $\bar{P}(y) = \mathbf{0}$. Since $0 \leqslant x^+ \leqslant y^+ \in P$, we have $x^+ \in P$, or $\bar{P}(x) \leqslant \mathbf{0}$. Suppose that $\bar{P}(y) = -\infty$; then since $0 \leqslant y^- \leqslant x^-$, and $y^- \notin P$, we must have $x^- \notin P$, that is, $\bar{P}(x) = -\infty$.

This shows that $\bar{P}$ is order preserving, and so $\bar{P}(x \vee y) \geqslant \max\{\bar{P}(x), \bar{P}(y)\}$. If $\bar{P}(x)$ and $\bar{P}(y)$ are both $< +\infty$, then $(x \vee y)^+ = x^+ \vee y^+ \in P$, or $\bar{P}(x \vee y) < +\infty$; if $\bar{P}(x) = \bar{P}(y) = -\infty$, then $(x \vee y)^- = x^- \wedge y^- \notin P$, or $\bar{P}(x \vee y) = -\infty$. Condition (3) now follows.

If P and Q are distinct prime ideals, then there is a positive element x such that $x \in P$ but $x \notin Q$, that is, $\bar{P}(\mathrm{x}) = \mathbf{0}$ and $\bar{Q}(x) = +\infty$. Thus, the mapping $P \to \bar{P}$ is one-to-one.

To show that the mapping $P \to \bar{P}$ is onto, let f be a spectral function on E, and put $P = \{x \in E : f(x) = \mathbf{0}\}$. It is shown in **(1)** that a spectral function satisfies the equality $f(x + y) = f(x) + f(y)$, provided the right-hand side makes sense. From this, and the defining properties of spectral function, it is easy to see that P is a prime ideal. To complete the proof, we show that $\bar{P} = f$. If $|x| \in P$, then also $x \in P$, and so $f(x) = \bar{P}(x) = \mathbf{0}$. If $x^+ \notin P$, then $f(x^+) = +\infty$; since $f(x^+) = \max\{f(x), \mathbf{0}\}$, we have $f(x) = \bar{P}(x) = +\infty$. The argument for the remaining case is similar.

Let $\mathfrak{P}$ denote the set of all prime ideals in E, equipped with the hull-kernel topology. If P and Q are elements of $\mathfrak{P}$, then it is obvious that $Q \subseteq P$ if and only if $\bar{P} \leqslant \bar{Q}$. Moreover, since the set $\mathfrak{F}_a$ is in one-to-one correspondence with the set $\mathfrak{P}_a$, the mapping $P \to \bar{P}$ of $\mathfrak{P}$ onto $\mathfrak{F}$ is clearly a homeomorphism. We summarize the preceding statements in

THEOREM 4.1. *The set of all prime ideals in the vector lattice E, equipped with the hull-kernel topology, and partially ordered by set inclusion, is homeomorphic and order anti-isomorphic with the spectral space of E.*

5. Vector lattices of extended functions. Let P be a prime ideal in E, and let a be an element of E_+ which is not in P. For each element x in E, put $\bar{x}(P) = \inf\{\alpha : P(x) < \alpha P(a)\}$. Since $\bar{P}(y) = +\infty$ if and only if $y^+ \notin P$, it follows that $\bar{x}(P)$ is the relative spectrum of x with respect to a at the spectral function $\bar{P}$. It is easily verified that $\bar{x}(P)$ is also given by the infimum of all real numbers α such that $P(x) \leqslant \alpha P(a)$, that is, $\bar{x}(P) = \inf\{\alpha : (\alpha a - x)^- \in P\}$.

For x in E, let $x\perp$ denote the set of elements y such that $|x| \wedge |y| = 0$. If S is any subset of E, then let $S\perp = \bigcap \{s\perp : s \in S\}$. A subset A of E_+ is called *orthogonal* if $a \wedge b = 0$ whenever $a, b \in A$, and $a \neq b$. By Zorn's lemma, E contains a maximal orthogonal set.

By an *extended (real-valued) function* on the topological space X, we mean a continuous mapping of X into the two-point compactification of the real line R which is real valued on an everywhere dense subset of X. Let $D(X)$ denote the set of all extended functions on X. If $f, g \in D(X)$ and $\alpha \in R$, then the functions αf, $f \vee g$, and $f \wedge g$, which are defined pointwise, are in $D(X)$. Let $R(f)$ denote the set of points at which f is real valued. If there is a function h in $D(X)$ such that $h(x) = f(x) + g(x)$ for each x in $R(f) \cap R(g)$, then h is called the *sum* of f and g. Since $R(f) \cap R(g)$ is dense in X, the sum is uniquely defined, provided it exists. A subset S of $D(X)$ which is closed under these four operations is called a *vector lattice of extended functions* on X.

Although a proof of the following result could be based on Theorem 2.1, we shall use the techniques in **(13)**.

THEOREM 5.1. *Let $\mathfrak{P}$ be a collection of prime ideals in the archimedean vector lattice E such that $\bigcap \{P: P \in \mathfrak{P}\} = \{0\}$, let A be a maximal orthogonal subset of E, and let the set $\mathfrak{X} = \bigcup \{\mathfrak{P}_a: a \in A\}$ be equipped with the hull-kernel topology. Then E is isomorphic to a vector lattice of extended functions on the topological space $\mathfrak{X}$.*

Proof. First observe that the sets $\mathfrak{P}_a$ are pairwise disjoint. This follows since A is an orthogonal set. If $P_0 \in \mathfrak{X}$, then let a be the unique element of A such that $P_0 \in \mathfrak{P}_a$. For x in E_+, put $\bar{x}(P_0) = \inf\{\alpha: P_0(x) \leqslant \alpha P_0(a)\}$. If x is an arbitrary element of E, put $\bar{x}(P_0) = \overline{x^+}(P_0) - \overline{x^-}(P_0)$. Since $x^+ \wedge x^- = 0$, $\bar{x}(P_0)$ is well defined. For the remainder of this proof, we assume that x is an element of E_+. We show that $\bar{x}$ is continuous at each point of $\mathfrak{X}$. Suppose first that $\bar{x}(P_0) = +\infty$. If α is any real number, then $(\alpha a - x)^- \notin P_0$. Put $c = a \wedge (\alpha a - x)^-$; then $c \notin P_0$, that is, $P_0 \in \mathfrak{P}_c$ and $\mathfrak{P}_c \subseteq \mathfrak{P}_a$. If $P \in \mathfrak{P}_c$, then $(\alpha a - x)^- \notin P$, and so $\bar{x}(P) \geqslant \alpha$. Now let $\bar{x}(P_0) = \alpha_0$ be finite. If $\epsilon > 0$, then $((\alpha_0 + \epsilon)a - x)^+ \notin P_0$, and $((\alpha_0 - \epsilon)a - x)^- \notin P_0$. Put $c = a \wedge ((\alpha_0 + \epsilon)a - x)^+ \wedge ((\alpha_0 - \epsilon)a - x)^-$; then $P_0 \in \mathfrak{P}_c \subseteq \mathfrak{P}_a$. If $P \in \mathfrak{P}_c$, then $((\alpha_0 + \epsilon)a - x)^+ \notin P$, and $((\alpha_0 - \epsilon)a - x)^- \notin P$, that is, $\bar{x}(P) \leqslant \alpha_0 + \epsilon$, and $\bar{x}(P) \geqslant \alpha_0 - \epsilon$.

We use the following lemmas to show that $\bar{x}$ is an extended function on $\mathfrak{X}$.

LEMMA 5.2. *If $\mathfrak{P}$ is a set of prime ideals in E such that $\bigcap \{P: P \in \mathfrak{P}\} = \{0\}$, then $b\perp = k(\mathfrak{P}_b)$ for any element b of E.*

Proof. If $|y| \wedge |b| = 0$, and if $P \in \mathfrak{P}_b$, then $y \in P$ since P is a prime ideal: thus, $y \in k(\mathfrak{P}_b)$, that is, $b\perp \subseteq k(\mathfrak{P}_b)$. Conversely, let $y \in k(\mathfrak{P}_b)$. If $P \in \mathfrak{P}_b$, then $|y| \wedge |b| \in P$; if $P \notin \mathfrak{P}_b$, then also $|y| \wedge |b| \in P$. Thus, $|y| \wedge |b| \in P$ for all $P \in \mathfrak{P}$, and so $|y| \wedge |b| = 0$.

A proof of the next lemma can be found in **(3)**.

LEMMA 5.3. *If $b \in E$, and if $\{y_\lambda: \lambda \in \Lambda\}$ is a collection of elements in $b\perp$ such that $y = \sup\{y_\lambda: \lambda \in \Lambda\}$ exists, then $y \in b\perp$.*

Now let $\mathfrak{X}_b$ be a basic open set in $\mathfrak{X}$. We may assume that both x and b are non-zero elements of E_+. We have $\mathfrak{X}_b = \mathfrak{P}_b \cap \mathfrak{X} = \bigcup \{\mathfrak{P}_{a\wedge b}: a \in A\}$. Suppose that $\bar{x}(P) = +\infty$ for all P in $\mathfrak{X}_b$. Then for each a in A, and each P in $\mathfrak{P}_{a\wedge b}$, we have $(na - x)^+ \in P$, that is, $(a - (x/n))^+ \in P$, for $n = 1, 2, \ldots$. By Lemma 5.2, $(a - (x/n))^+ \in (a \wedge b)\perp$ for all positive integers n. Now $a \geqslant (a - (x/n))^+$ for all n; if $y \geqslant (a - (x/n))^+$, then $a - y \leqslant x/n$ for all n, and hence $a \leqslant y$ since E is archimedean. Thus, $a = \sup\{(a - (x/n))^+: n = 1, 2, \ldots,\}$, and so by Lemma 5.3, $a \in (a \wedge b)\perp$, that is, $b \in A\perp$. Since

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A is a maximal orthogonal set, this implies that $b = 0$, a contradiction. Hence, $R(\bar{x})$ is a dense subset of $\mathfrak{X}$.

We have shown that the set $\bar{E} = \{\bar{x}: x \in E\}$ is a subset of $D(\mathfrak{X})$. It is easy to see that $\overline{x + y} = \bar{x} + \bar{y}$, $\overline{\alpha x} = \alpha\bar{x}$, and $\overline{x \vee y} = \bar{x} \vee \bar{y}$ for all x, y in E, and all real numbers α. Hence, $\bar{E}$ is a vector lattice of extended functions on $\mathfrak{X}$, and the mapping $x \to \bar{x}$ is a homomorphism. To show that this homomorphism is an isomorphism, let $\bar{x} = 0$. Then for each $a \in A$, and each $P \in \mathfrak{B}_a$, we have $((a/n) - x)^- \in P$ for all positive integers n. By Lemma 5.2, $((a/n) - x)^- \in a\perp$ for all n. Since E is archimedean, $x = \sup\{((a/n)-x)^-: n = 1, 2, \ldots,\}$, and so Lemma 5.3 implies that $x \in a\perp$. It follows that $x \in A\perp$, and hence $x = 0$. This completes the proof of the theorem.

As a parenthetical remark, we note that the above result leads to a representation theorem for Φ-algebras. A Φ-algebra $\mathscr{A}$ is an archimedean lattice-ordered algebra containing an identity element 1 which is a weak order unit. Let $\mathscr{M}$ be the set of all maximal $\mathfrak{l}$-ideals in $\mathscr{A}$. It is known that the zero of $\mathscr{A}$ is the only element that is common to all of the $\mathfrak{l}$-ideals in $\mathscr{M}$, and that each element of $\mathscr{M}$ is a prime (vector lattice) ideal **(6)**. Let $\mathfrak{B} = \mathscr{M}$, and $A = \{1\}$ in the above theorem; then $\mathfrak{X} = \mathscr{M}$. It is easy to see that $\overline{xy} = \bar{x}\,\bar{y}$ for all x and y in $\mathscr{A}$. Thus, every Φ-algebra $\mathscr{A}$ is isomorphic (as a lattice-ordered algebra) t o an algebra of extended functions on $\mathscr{M}$. This is a result of Henriksen and Johnson **(6)**.

6. Minimal prime ideals. A non-empty proper subset K of E_+ is called a *positive ideal* if it is closed under addition and multiplication by non-negative scalars, and if $x \in K$ whenever $0 \leqslant x \leqslant y$ with $y \in K$. The mapping defined by $I \to I \cap E_+$, where I is an ideal in E, is a one-to-one correspondence between the set of all ideals in E, and the set of all positive ideals. With respect to this mapping, the inverse image of the positive ideal K is the ideal $K - K$, that is, the set of all elements in E which can be expressed as the difference of two elements in K. A positive ideal K is said to be *prime* if the ideal $K - K$ is prime.

Let I be an ideal in E. A prime ideal P such that $I \subseteq P$ is called a *minimal prime ideal belonging to the ideal* I if there is no prime ideal containing I and properly contained in P. A minimal prime positive ideal belonging to a given positive ideal is defined analogously. It is easy to see that P is minimal prime ideal belonging to the ideal I if and only if $P \cap E_+$ is a minimal prime positive ideal belonging to the positive ideal $I \cap E_+$. Therefore, when characterizing the minimal prime ideals belonging to a given ideal, it is sufficient to consider only positive ideals. This characterization (Theorem 6.5) will be obtained via a series of lemmas. Part of the development in this section was suggested by the theory of minimal prime ideals in commutative rings **(9)**.

A non-empty subset S of E_+ is called a *lower sublattice* of the vector lattice E provided that $x \wedge y \in S$ whenever both x and y are in S. Note that the positive ideal K is prime if and only if K', the complement of K in E_+, is a lower sublattice. The following result is an easy consequence of Zorn's lemma.

LEMMA 6.1. *Let K be a positive ideal in the vector lattice E, and let S be a lower sublattice which does not meet K. Then S is contained in a lower sublattice which is maximal with respect to the property of not meeting K.*

LEMMA 6.2. *Let S be a lower sublattice in E, and let K be a positive ideal which does not meet S. Then K is contained in a positive ideal M which is maximal with respect to the property of not meeting S; moreover, M is prime.*

Proof. The existence of such an M is a consequence of Zorn's lemma. It is clear that a positive ideal L is prime if, and only if, $x \wedge y = 0$ implies that $x \in L$ or $y \in L$. We use this characterization of prime positive ideals to show that M is prime. Let $a \wedge b = 0$, but suppose that $a \notin M$ and $b \notin M$. The positive ideal generated by M and a is denoted by (M, a); it consists of all $x \in E_+$ such that $x \leqslant na + v$ for some positive integer n and some element v in M. Because of the maximal property of M, there exist elements $x \in (M, a) \cap S$ and $y \in (M, b) \cap S$. Thus, $0 \leqslant x \leqslant ma + v$, and $0 \leqslant y \leqslant nb + w$ for positive integers m, n; and v, w in M. We may choose $m = n$ and $v = w$, and therefore, $x \wedge y - v = (x - v) \wedge (y - v) \leqslant m(a \wedge b) = 0$. It follows that $x \wedge y \in S \cap M$, a contradiction, and thus M is prime.

In what follows, if B is a subset of E_+, then B' will denote its complement in E_+.

LEMMA 6.3. *A subset J of positive elements of the vector lattice E is a minimal prime positive ideal belonging to the positive ideal K if and only if J' is a lower sublattice which is maximal with respect to the property of not meeting K.*

Proof. Let J be a subset of E_+ such that J' is a lower sublattice which is maximal with respect to the property of not meeting K. By Lemma 6.2, there is a prime positive ideal M such that $K \subseteq M \subseteq J$. Since M' is a lower sublattice which doesn't meet K, the maximal property of J' insures that $J = M$. This shows that J is a minimal prime positive ideal belonging to K.

Conversely, let J be a minimal prime positive ideal belonging to K. Then J' is a lower sublattice which does not meet K. By Lemma 6.1, J' is contained in a lower sublattice S which is maximal with respect to the property of not meeting K. By the above case, S' is a minimal prime positive ideal belonging to K. By the minimal property of J, we conclude that $S = J'$. This completes the proof.

LEMMA 6.4. *A prime positive ideal J is a minimal prime positive ideal belonging to the positive ideal K if and only if whenever $x \in J$ then there exists $y \in J'$ such that $x \wedge y \in K$.*

Proof. To prove sufficiency, let $K \subseteq N \subset J$, where N is prime, and choose $x \in J$ such that $x \notin N$. By hypothesis, there exists an element $y \in J'$ such that

$x \wedge y \in K$. Hence, $x \wedge y \in N$, but $x \notin N$ and $y \notin N$. Since N is prime, this is impossible.

Conversely, let J be a minimal prime positive ideal belonging to K. By Lemma 6.3, J' is a lower sublattice which is maximal with respect to the property of not meeting K. Let x be an element of J, and put $S = J' \cup \{a \wedge x : a \in J'\}$. Since $a \wedge x \in J$ for all $a \in J'$, the set S is a lower sublattice which properly includes J', and so there is an element y in J' such that $x \wedge y \in K$. This completes the proof.

As an immediate consequence of the above result and the remarks at the beginning of this section, we obtain

THEOREM 6.5. *A prime ideal P in the vector lattice E is a minimal prime ideal belonging to the ideal I if and only if whenever $x \in P$, then there is an element $y \notin P$ such that $|x| \wedge |y| \in I$.*

By a *minimal prime ideal*, we mean a minimal prime ideal belonging to the ideal $\{0\}$. Let $\mathfrak{M}$ denote the collection of all minimal prime ideals in E. In virtue of the order anti-isomorphism between spectral functions and prime ideals obtained in § 4, $\mathfrak{M}$ is in one-to-one correspondence with the proper space of E. We now use Theorem 6.5 to give an independent proof of Theorem 2.2. We shall make use of the following result.

LEMMA 6.6. *Every ideal is the intersection of all minimal prime ideals belonging to it.*

Proof. It is sufficient to consider only positive ideals. Let K be a positive ideal, and let a be an element of E_+ such that $a \notin K$. Then the single element set $\{a\}$ is a lower sublattice which does not meet K. In virtue of Lemma 6.1, there is a lower sublattice S, containing a, and which is maximal with respect to the property of not meeting K. By Lemma 6.3, the set S' is a minimal prime positive ideal belonging to K.

It is an immediate consequence of this lemma that $k(\mathfrak{M}) = \{0\}$, that is, $\mathfrak{M}$ is dense in $\mathfrak{P}$, the set of all prime ideals in E. This proves one part of Theorem 2.2. To show that $\mathfrak{M}$ is a Hausdorff space, first recall that the collection $\{\mathfrak{M}_a : a \in E\}$, where $\mathfrak{M}_a = \{P \in \mathfrak{M} : a \notin P\}$, is a basis for the open sets when $\mathfrak{M}$ is equipped with the hull-kernel topology. Now let P_1 and P_2 be distinct elements of $\mathfrak{M}$, and choose $x \in P_1 \cap E_+$ such that $x \notin P_2$; and $y \in P_2 \cap E_+$ such that $y \notin P_1$. By Theorem 6.5, there is a positive element $s \notin P_1$ such that $s \wedge x = 0$. Put $t = s \wedge y$; then $t \notin P_1$ and $x \notin P_2$, that is, $P_1 \in \mathfrak{M}_t$ and $P_2 \in \mathfrak{M}_x$. Moreover, $\mathfrak{M}_t \cap \mathfrak{M}_x = \mathfrak{M}_{s \wedge y \wedge x} = \mathfrak{M}_0 = \phi$, and hence $\mathfrak{M}$ is a Hausdorff space. Finally, let $P \notin \mathfrak{M}_a$, where $a \in E_+$. Choose a positive element $b \notin P$ such that $a \wedge b = 0$. Then $\mathfrak{M}_b \subseteq \mathfrak{M}_a'$, the complement of $\mathfrak{M}_a$ in $\mathfrak{M}$, and so $\mathfrak{M}_a$ is both open and closed.

If we let $\mathfrak{B} = \mathfrak{M}$ in Theorem 5.1, then it follows from the preceding discussion that the space $\mathfrak{X} = \bigcup \{\mathfrak{V}_a : a \in A\}$ is a Hausdorff space which has a basis consisting of open-closed sets. Thus, we have

THEOREM 6.7. *Every archimedean vector lattice is isomorphic to a vector lattice of extended functions on a Hausdorff space having a basis consisting of sets which are both open and closed.*

If f and g are linear operators acting on the vector lattice E, then we write $f \leqslant g$ in case $f(x) \leqslant g(x)$ for all x in E_+. A *projection* on E is an idempotent linear operator f such that $0 \leqslant f \leqslant 1$, where 0 is the null operator, and 1 is the identity operator. A projection f is called a *projector* if there is an element a in E such that $a\perp$ coincides with the nullspace of f; in this case, we say that a *belongs* to the projector f.

Now let E be a σ-complete vector lattice. It is known **(10)** that each element in E belongs to a projector, and that the set $\mathfrak{C}$ of all projectors, partially ordered as above, is a conditionally σ-complete, relatively complemented, distributive lattice. It follows from work of Amemiya **(1)** that the collection of all maximal dual ideals in $\mathfrak{C}$, equipped with the dual hull-kernel topology, is homeomorphic with $\mathfrak{M}$, and that the collection of sets $\{\mathfrak{M}_a : a \in E\}$, partially ordered by set inclusion, is isomorphic with $\mathfrak{C}$. Consequently, in the case of a σ-complete vector lattice, Theorem 6.7 reduces to a representation theorem of Nakano **(11)**.

For each element z in the vector lattice E, let $\mathfrak{X}^z$ be the set of all ideals which are maximal with respect to the property of not containing z. By Lemma 6.2, such ideals exist and, moreover, each is prime, that is, $\mathfrak{X}^z \subseteq \mathfrak{P}_z$. It again follows from work of Amemiya that E is archimedean if and only if $\mathfrak{X}^z$ is dense in $\mathfrak{P}_z$ for each z in E. Now let E be archimedean, and let A be a maximal orthogonal subset of E. Put $\mathfrak{V} = \bigcup \{\mathfrak{X}^a : a \in A\}$. It is easy to see that $\mathfrak{V}$ is dense in $\mathfrak{P}$. With the notation as in Theorem 5.1, $\mathfrak{V}_a = \mathfrak{X}^a$, and so $\mathfrak{X} = \mathfrak{V}$. The space $\mathfrak{X}$ is the one which Yosida **(13)** used for his representation theorem.

THEOREM 6.8 (Yosida). *Every archimedean vector lattice is isomorphic to a vector lattice of extended functions on a locally compact Hausdorff space.*

Proof. Yosida proved that the space $\mathfrak{X}$ as defined above is a Hausdorff space. We show that it is locally compact by proving that each set $\mathfrak{X}^a$ is compact.

We first prove that each set $\mathfrak{P}_a$ is compact. Consider a collection of relatively closed subsets of $\mathfrak{P}_a$ whose intersection is empty. We may suppose that they have the form $h(b) \cap \mathfrak{P}_a$, where b ranges over some subset B of E_+, and where $h(b) = \{P \in \mathfrak{P} : b \in P\}$. Thus, $h(B) \cap \mathfrak{P}_a = \phi$, or $h(B) \subseteq h(a)$. By Lemma 6.6, each ideal is the intersection of all prime ideals containing it. It follows that $a \in (B)$, where the latter set denotes the ideal generated by B. Hence, there exist positive integers $n_1, \ldots, n_k$ and elements $b_1, \ldots, b_k$ in B

such that $a \leqslant n_1 b_1 + \ldots + n_k b_k$. Thus, $\mathfrak{P}_a \subseteq \bigcup_1^k \mathfrak{P}_{b_i}$, or $\bigcap_1^k h(b_i) \cap \mathfrak{P}_a = \phi$. This proves that $\mathfrak{P}_a$ is compact.

To show that $\mathfrak{X}^a$ is compact, let $\mathfrak{X}^a \subseteq \bigcup \{\mathfrak{P}_b : b \in B\}$, where B is some subset of E_+. If $P \in \mathfrak{P}_a$, then there is an element Q in $\mathfrak{X}^a$ such that $P \subseteq Q$. Moreover, there is an element b in B such that $Q \in \mathfrak{P}_b$. Hence, the collection $\{\mathfrak{P}_b : b \in B\}$ is an open covering of the compact set $\mathfrak{P}_a$, and so there exist elements $b_1, \ldots, b_k$ in B such that $\mathfrak{P}_a \subseteq \bigcup_1^k \mathfrak{P}_{b_i}$. But $\mathfrak{X}^a \subseteq \mathfrak{P}_a$, and so the former set is compact.

7. The carrier space. The elements a and b of E_+ are called *equivalent* if $a\perp = b\perp$. It is clear that this is an equivalence relation on E_+; let a^* denote the equivalence class containing a, and let $\mathfrak{C}$ denote the collection of all equivalence classes a^* as a ranges over E_+. The set $\mathfrak{C}$, when partially ordered by writing $a^* \leqslant b^*$ in case $b\perp \subseteq a\perp$, is called the *carrier space* of E, and its elements are called *carriers*. (We are using the terminology in **(4)**; Jaffard **(7)** calls the elements of $\mathfrak{C}$ *filets*.) The following result is proved in **(7)**.

THEOREM 7.1. *The carrier space of the vector lattice E is a distributive, disjunctive lattice.*

We now proceed to give another characterization of $\mathfrak{C}$, together with a proof of the above theorem.

THEOREM 7.2. *The collection of sets $\{\mathfrak{M}_a : a \in E_+\}$, when partially ordered by set inclusion, is a distributive, disjunctive lattice which is isomorphic to the carrier space of E.*

Proof. Let $\mathfrak{X}$ denote the collection of sets $\mathfrak{M}_a$ as a ranges over E_+. Since $\mathfrak{M}_{a\wedge b} = \mathfrak{M}_a \cap \mathfrak{M}_b$, $\mathfrak{M}_{a\vee b} = \mathfrak{M}_a \cup \mathfrak{M}_b$, and $\mathfrak{M}_0 = \phi$, the collection $\mathfrak{X}$, when partially ordered by set inclusion, is a distributive lattice with a smallest element. We now show that $\mathfrak{X}$ is disjunctive. For this, let $\mathfrak{M}_a \not\subseteq \mathfrak{M}_b$, and choose $P \in \mathfrak{M}_a$ such that $P \notin \mathfrak{M}_b$. Since $b \in P$, by Theorem 6.5, there is a positive element $c \notin P$ for which $b \wedge c = 0$. Put $d = a \wedge c$; then $d \notin P$. Thus, we have $\mathfrak{M}_b \cap \mathfrak{M}_d = \phi$, and $\phi \neq \mathfrak{M}_d \subseteq \mathfrak{M}_a$, that is, $\mathfrak{X}$ is disjunctive.

For a^* in $\mathfrak{C}$, put $\beta(a^*) = \mathfrak{M}_a$. Let $a^* \leqslant b^*$, that is, $b\perp \subseteq a\perp$. Since $\bigcap \{P : P \in \mathfrak{M}\} = \{0\}$, Lemma 5.2 insures that $x\perp = k(\mathfrak{M}_x)$ for each x, and so $k(\mathfrak{M}_b) \subseteq k(\mathfrak{M}_a)$. Hence, $h(k(\mathfrak{M}_a)) \subseteq h(k(\mathfrak{M}_b))$, and since $\mathfrak{M}_x$ is closed in $\mathfrak{M}$, we have $\mathfrak{M}_a \subseteq \mathfrak{M}_b$. This shows that the mapping β is both well defined and order-preserving. Now if $\mathfrak{M}_a \subseteq \mathfrak{M}_b$, then $b\perp = k(\mathfrak{M}_b) \subseteq k(\mathfrak{M}_a) = a\perp$, and so $a^* \leqslant b^*$. Hence, $a^* \leqslant b^*$ if, and only if, $\beta(a^*) \leqslant \beta(b^*)$. This completes the proof.

Let θ be an isomorphism of the vector lattice E onto a subdirect sum F of the linearly ordered vector lattices $\{F_\alpha : \alpha \in \mathfrak{A}\}$. For $y = \{\ldots, y_\alpha, \ldots\}$ in F, let $pr_\alpha(y) = y_\alpha \in F_\alpha$, and for x in E, put $\sigma(x) = \{\alpha \in \mathfrak{A} : pr_\alpha(\theta x) \neq 0\}$. The isomorphism θ is called *completely regular* if for each $\alpha \in \mathfrak{A}$, and each $x \in E_+$

such that $\alpha \notin \sigma(x)$, there exists an element y in E_+ such that $\alpha \in \sigma(y)$, and $\sigma(x) \cap \sigma(y) = \phi$.

Ribenboim **(12)** makes use of the set $\mathfrak{D}$ of all maximal dual ideals of the distributive lattice $\mathfrak{C}$ to give a new proof of the following result of Lorenzen **(8)**. It is not hard to show that $\mathfrak{D}$ is in one-to-one correspondence with $\mathfrak{M}$. (In fact, if $\mathfrak{D}$ is equipped with the dual hull-kernel topology, then this correspondence is a homeomorphism.)

THEOREM 7.3. *Every vector lattice admits a completely regular isomorphism onto a subdirect sum of linearly ordered vector lattices.*

Proof. Let the elements of $\mathfrak{M}$ be indexed by the set $\mathfrak{A}$, and for $P_\alpha \in \mathfrak{M}$, put $F_\alpha = E/P_\alpha$. Since $\bigcap \{P_\alpha : \alpha \in \mathfrak{A}\} = \{0\}$, the mapping $x \rightarrow \{P_\alpha(x) : \alpha \in \mathfrak{A}\}$ is an isomorphism of E onto a subdirect sum of the linearly ordered vector lattices $\{F_\alpha : \alpha \in \mathfrak{A}\}$. To see that this isomorphism is completely regular, note that $\sigma(x) = \mathfrak{M}_x$ for each x. Theorem 6.5 completes the proof.

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