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APPROXIMATION ON BOUNDARY SETS

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ABSTRACT. Let U be a bounded open subset of the complex plane. By a well known result of A. M. Davie, $C(bU)$ is the uniformly-closed linear span of $A(U)$ and the powers $(z-z_i)^{-n}$, $n=1, 2, 3, \ldots$ with z_i a point in each component of U. We show that if $A(U)$ is a Dirichlet algebra and bU is of infinite length, then one power of $(z-z_i)$ is superfluous.

Let U be a bounded open subset of the complex plane $\mathbb{C}$; let $A(U)$ be the algebra of all continuous functions on $\bar{U}$ which are analytic on U. Let U_1, $U_2, \ldots$ be the components of U, and choose a point $z_j \in U_j$ for each $j=1$, $2, \ldots$. In [2] Davie proved that $C(bU)$ is the uniformly-closed linear span of $A(U)$ and the powers $(z-z_j)^{-n}$, $n=1, 2, \ldots$, where bU is the topological boundary of U.

If bU_i has finite one-dimensional Hausdorff measure, then it is not hard to construct a (complex Borel) measure on bU_i which annihilates $A(U)$, $(z-z_j)^{-n}$ and $(z-z_i)^{-(n+1)}$, $j\neq i$, $n=1, 2, \ldots$ while $\int(z-z_i)^{-1}\,d\mu\neq 0$. Hence the power $(z-z_i)^{-1}$ is not in the closed span of $A(U)$ and other powers. The same is true for every power $(z-z_i)^{-n}$.

In 1957, Werner [9] observed that for every Jordan curve Γ of infinite length, at least one power of z is superfluous in spanning $C(\Gamma)$. An extension of this result has been given by Korevaar and Pfluger [6]. Recently Pietz [8] has some more general results for the algebras $R(K)$. In his proof, however, he made the assumption that the closure of each component of $\mathring{K}$, the interior of K, has connected complement.

In this note, we show that if $A(U)$ is Dirichlet and bU_i has infinite one-dimensional Hausdorff measure, then at least one power of $(z-z_i)^{-n}$ is superfluous in spanning $C(bU)$.

THEOREM 1. *Let $A(U)$ be a Dirichlet algebra. Assume there is a measure μ on bU such that μ annihilates $A(U)$, $(z-z_j)^{-n}$ and $(z-z_i)^{-(n+1)}$, $j\neq i$, $n=1, 2, \ldots$ while $\int(z-z_i)^{-1}\,d\mu=1$. Then bU_i has finite one-dimensional Hausdorff measure.*

Proof. The proof is along the line of [8].

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Theorem 3.1 of [5] implies that each U_j is simply connected. A well-known decomposition theorem for Dirichlet algebras (Theorem 1.1 of [5]) gives $\mu = \sum \mu_j$ where μ_j is supported on U_j, $\mu_j \perp A(U_j)$ and μ_j is absolutely continuous with respect to harmonic measure for z_j on bU_j for each j. We may therefore restrict our attention to the pair (μ_i, U_i), which we relabel (μ, U), and assume $z_i = 0$.

Let ϕ be the Riemann map of $\Delta = \{|z| < 1\}$ onto U. The map ϕ has a measurable one-to-one extension ϕ^* to a subset of $b\Delta$ of full measure, i.e., U is nicely connected (see, e.g., [1]). Write ρ for the harmonic measure for 0 on $b\Delta$, and λ the same for 0 on bU. Since $A(U)$ is Dirichlet, there exists $h \in H_0^1(\rho) = \{h \in L^1(\rho)\colon \int z^k h\, d\rho = 0 \text{ for all } k \geq 0\}$ such that $\int f\, d\mu = \int (f \circ \phi^*) h\, d\rho$ for all Borel function f on bU, by a theorem of Davie ([1], p. 352). Let $w \in H^1$ so that $h\, d\rho = w\, dz$. Then for any k, $0 < r < 1$,

$$\int_{|z|=r} \phi^k(z) w(z)\, dz = \int_{|z|=r} \phi^{*k}(z) w(z)\, dz = \delta_{-1,k}$$

But

$$\frac{1}{2\pi i} \int_{|z|=r} \phi^k(z) \phi'(z)\, dz = \delta_{-1,k}.$$

Hence $(w(z) - \phi'(z)/2\pi i)\, dz$ annihilates all integral powers of ϕ, which are uniformly dense on $|z| = r$ by a theorem of Walsh, so that $w(z) = \phi'(z)/2\pi i$. Therefore $\phi' \in H^1$. Theorem 3.11 of [3] shows that ϕ has an absolutely continuous extension to $|z| = 1$ which implies bU has finite one-dimensional Hausdorff measure.

Let Ψ be a homeomorphism of $\mathbb{C}$ to $\mathbb{C}$. Let $V = \Psi(U)$ and $V_j = \Psi(U_j)$ for $j = 1, 2, \ldots$. We say Ψ is singular on bU if Ψ carries a set of full harmonic measure on bU_j to a set of zero harmonic measure on bV_j for each j. In [7], O'Farrell proved that if a compact K has connected complement and Ψ is a homeomorphism singular on bK then $C(bK)$ is the uniformly-closed linear span of z^n and Ψ^n, $n = 0, 1, 2, \ldots$. By imitating his technique, we can show the following.

THEOREM 2. *Let $A(U)$ be a Dirichlet algebra. Let Ψ be a homeomorphism singular on bU such that $A(V)$ is also Dirichlet. Then $C(bU)$ is the uniformly-closed linear span of $A(U)$ and $A(V) \circ \Psi$.*

Proof. Let μ be a measure on bU which annihilates $A(U)$ and $A(V) \circ \Psi$. Then the measure $\Psi_{\#}\mu$ on bV defined by $\int f\, d\Psi_{\#}\mu = \int f \circ \Psi\, d\mu$ annihilates $A(V)$ and $A(U) \circ \Psi^{-1}$. Again the decomposition theorem for Dirichlet algebras gives $\mu = \sum \mu_j$ where μ_j is supported on $\bar{U}_j$, $\mu_j \perp A(U_j)$ and $\Psi_{\#}\mu_j \perp A(V_j)$. Also μ_j and $\Psi_{\#}\mu_j$ are absolutely continuous with respect to the harmonic measures for U_j and V_j, respectively. Hence $\Psi_{\#}\mu_j = 0$, which in turn implies $\mu_j = 0$ for all j and so $\mu = 0$.

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Note that there are Dirichlet algebras $A(U)$ such that $A(V)$ is not Dirichlet for a homeomorphism Ψ, e.g., the "string of beads" ([4], p. 145) sets. It would be interesting to know whether Theorem 2 is still true if we replace the singular homeomorphism by an orientation-reversing homeomorphism. This would be a generalized Walsh–Lebesgue Theorem.

These same methods in this note can be used to obtain similar results for a hypo-Dirichlet algebra.

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