Can J. Math., Vol. XXXVIII, No. 5, 1986, pp. 1228-1238

THE IDEAL GENERATION CONJECTURE FOR 28 POINTS IN $\mathbf{P}^3$

LESLIE G. ROBERTS

1. Introduction. The ideal generation conjecture has recently been proved for general points in $\mathbf{P}_k^3$ (k a field) [**1**], [**6**]. The proof in [**1**] is by induction. One of the starting points for the induction (called A(5) in [**1**]) is proved in [**6**]. The theoretical proof of A(5) in [**6**] seems to be very difficult, apparently even more difficult than the induction. Because of this, and also because [**6**] is not publically available, I feel it is worth knowing that A(5) can be proved numerically with modest readily available computing facilities. In this note I discuss the computation involved, and give a few explicit examples. In the course of working out these examples I found 26 points in $\mathbf{P}_k^3$ that satisfy the ideal generation conjecture, but which cannot be extended to 27 or 28 points satisfying the ideal generation conjecture. This phenomenon can be interpreted combinatorially, leading to an infinite number of similar examples. These examples are perhaps surprising because n points in generic position in $\mathbf{P}^n$ can always be extended to $s + 1$ points in generic position (k infinite).

First let me review some definitions. Suppose we have a set X of s points in $\mathbf{P}^n$, with homogeneous co-ordinate ring

$$A = \bigoplus_{i \geqq 0} A_i.$$

The Hilbert function of X is

$$a_i = \dim_k A_i \quad (i \geqq 0).$$

Let $I = \bigoplus_{i \geqq 0} I_i$ be the homogeneous ideal of X, with d the smallest integer such that $I_d \neq 0$. That is, $A = R/I$, $R = k[x_0, \ldots, x_n]$. The set X is in generic position (more properly generic s-position) [**3**] if

$$a_i = \min\left(s, \binom{n+i}{n}\right).$$

($\binom{n+i}{n}$ is the number of monomials of degree i in $n + 1$ variables, so this means that a_i is "as large as possible".) The set X satisfies the ideal generation conjecture if X is in generic position and if the map

$$f: I_d^{n+1} \rightarrow I_{d+1}$$

Received July 4, 1985.

https://doi.org/10.4153/CJM-1986-062-x Published online by Cambridge University Press

given by

$$f(c_0, \ldots, c_n) = x_0c_0 + x_1c_1 + \ldots + x_nc_n$$

is of maximal rank (i.e., injective or surjective). That is, I_d generates "as much as possible" of I_{d+1}. To prove that the ideal generation conjecture holds "generically" for s points in $\mathbf{P}^n$ it suffices to find one set of points where it holds.

A(5) is stated in [1] in the following manner: there exist subsets (X, Y), $X \subset Y \subset \mathbf{P}^3$, X containing 28 points, and Y containing 29, such that

$$h^0(\mathbf{P}^3, \Omega(5) \otimes \mathscr{I}_X) = 0 \quad \text{and}$$
$$h^0(\mathbf{P}^3, \Omega(5) \otimes \mathscr{I}_Y) = 0.$$

($\mathscr{I}_X$ or $\mathscr{I}_Y$ is the sheaf of ideals of X, respectively Y, in $\mathbf{P}^3$, and Ω is the sheaf of differentials $\Omega_{\mathbf{P}^3}$). There is an exact sequence ([5], p. 176)

$$0 \to \Omega(5) \to 4O_{\mathbf{P}^3}(4) \xrightarrow{g} O_{\mathbf{P}^3}(5) \to 0$$

which yields an exact sequence

$$0 \to \Gamma(\mathbf{P}^3, \Omega(5) \otimes \mathscr{I}_X) \to I_4^4 \xrightarrow{f} I_5.$$

For 28 points in generic position in $\mathbf{P}^3$,

$$\dim_k I_4 = 7 \quad \text{and} \quad \dim_k I_5 = 28.$$

Thus (as noted in [1]) the ideal generation conjecture (i.e., f is one-to-one) holds for 28 points in generic position in $\mathbf{P}^3$ if and only if

$$\dim_k \Gamma(\mathbf{P}^3, \Omega(5) \otimes \mathscr{I}_X) = h^0(\mathbf{P}^3, \Omega(5) \otimes \mathscr{I}_X) = 0.$$

If $X \subset Y$ then $\mathscr{I}_Y \subset \mathscr{I}_X$, so we have an inclusion

$$\Gamma(\mathbf{P}^3, \Omega(5) \otimes \mathscr{I}_Y) \to \Gamma(\mathbf{P}^3, \Omega(5) \otimes \mathscr{I}_X).$$

Thus if

$$h^0(\mathbf{P}^3, \Omega(5) \otimes \mathscr{I}_X) = 0,$$

then also

$$h^0(\mathbf{P}^3, \Omega(5) \otimes \mathscr{I}_Y) = 0.$$

Thus proving A(5) is equivalent to finding 28 points in $\mathbf{P}^3$ satisfying the ideal generation conjecture.

2. The brute force method. In the rest of the paper k will either be the rationals $\mathbf{Q}$, or $\mathbf{F}_p$ ($= \mathbf{Z}/p\mathbf{Z}$, $\mathbf{Z}$ the integers, p a prime). (One might prefer to work theoretically over an algebraically closed field. However a_i and $\dim_k I_i$ are preserved under field extensions, so an example over k yields an example over any larger field.) 28 points X in $\mathbf{P}_k^3$ can be represented

https://doi.org/10.4153/CJM-1986-062-x Published online by Cambridge University Press

by a 4×28 matrix M whose i^{th} column is the homogeneous co-ordinates of the i^{th} point. (If $k = \mathbf{Q}$ we can clear denominators and do all calculations over $\mathbf{Z}$.) Let $M(j)$ be the matrix whose rows are all monomials of degree j in the rows of M. That is, the columns of $M(j)$ are the homogeneous co-ordinates of the images of the points of X under the j-uple embedding ([**5**], Exercise 7.1(a), p. 54). To prove that X is in generic position it suffices to show that rank $M(1) = 4$, rank $M(2) = 10$, rank $M(3) = 20$, and rank $M(4) = 28$. The matrices $M(i)$ $(1 \leqq i \leqq 4)$ are respectively of sizes 4×28, 10×28, 20×28, and 35×28. I_4 is the null space of $M(4)^t$ (t = transpose). Let N be a 7×35 matrix whose rows are a basis for I_4. From N we can construct the matrix T for f, which will be 56×28 (taking as basis for I_4 the rows of N, and regarding I_5 as a subspace of R_5, which is of dimension 56. The basis for R_5 could be the monomials of degree 5 in 4 variables, ordered lexicographically.) It then suffices to prove that T is of rank 28.

I used APL on an IBM PC to carry out the above calculations, (although practically any language should work). Ranks of matrices of the sizes above can be found easily by row reduction to upper triangular form, and a system of linear equations can be solved by futher row reduction. It took me less than 2.5 minutes to verify the ideal generation conjecture for a set of 28 "randomly chosen" points in $\mathbf{P}^3$, over $\mathbf{F}_{11}$. A more powerful computer would of course be faster, but already on the small computer the computation time is negligible compared with the time to write the programs. I do all calculations over $\mathbf{Q}$ by working over $\mathbf{Z}$. Elements of $\mathbf{Z}$ can be represented exactly in APL up to about 2^{56}. A "randomly chosen" set of 28 points in $\mathbf{P}^3_{\mathbf{Q}}$ quickly leads to integers larger than this when the above calculations are attempted. One can get around this problem by reducing mod p, by reducing modulo a non-zero divisor, or by choosing the points carefully, as I discuss in the next three sections. Some implementations of some computer languages do exact calculation over $\mathbf{Z}$, but the need for such is not crucial here.

3. Reduction mod p. Let M be an $r \times s$ matrix with integer coefficients. Let $M(p)$ be the matrix over $\mathbf{F}_p$ obtained from M by taking the images in $\mathbf{F}_p$ of the coefficients. The following are readily established.

LEMMA 3.1. Rank $M(p) \leqq$ Rank M, *with equality at all but a finite number of* p. (Rank M *is computed over* $\mathbf{Q}$ *and* rank $M(p)$ *over* $\mathbf{F}_p$.)

LEMMA 3.2. *M can be reduced to upper triangular form $\bar{M}$ by invertible row operations over $\mathbf{Z}$. If M is square then* Rank $M(p)$ = Rank M *for all primes except those that divide a diagonal entry of $\bar{M}$.*

By "upper triangular form" I mean that the first non-zero entry of row i $(2 \leqq i \leqq r)$ lies to the right of the first non-zero entry in the preceding row. An "invertible row operation" over $\mathbf{Z}$ means one of the following (a) multiplication of a row by -1 (b) interchanging two rows (c) adding a multiple of one row to another. Performing a sequence of invertible row operations is equivalent to multiplying on the left by an element of $GL(r, \mathbf{Z})$. The number of non-zero rows of $\bar{M}$ is the rank of M.

Definition 3.3. Let V be an r dimensional subspace of $\mathbf{Q}^s$ ($r \leqq s$), and W an r dimensional subspace of $\mathbf{F}_p^s$. Then we say that W is obtained from V *by reduction* mod p if there is an $r \times s$ integer matrix M such that the rows of M are a basis for V, and the rows of $M(p)$ are a basis of W. ($\mathbf{Q}^s$ and $\mathbf{F}_p^s$ are being thought of as s-tuples, with the "standard basis", rather than as abstract s-dimensional vector spaces.)

LEMMA 3.4. *Let M be an $s \times r$ matrix with integer coefficients. Suppose that* rank M = rank $M(p)$. *Then the null space of $M(p)$ can be obtained from the null space of M by reduction* mod p.

Proof. Let d = rank M. First interchange columns so that the first d colums of $M(p)$ are linearly independent. Then if we row reduce M as in Lemma 3.2, $\bar{M}$ has d non-zero rows, and the $d \times d$ matrix at the upper left corner of $\bar{M}$ has none of its diagonal entries divisible by p. The null spaces of $\bar{M}$ and $\bar{M}(p)$ can now be found by corresponding computations over $\mathbf{Q}$ and $\mathbf{F}_p$ respectively. Write the resulting basis for the null space of $\bar{M}$ as the rows of a $d \times r$ matrix, clear denominators (which can be done by multiplying by an integer prime to p), and undo the original column interchange. The resulting matrix expresses the null space of $M(p)$ as being obtained from the null space of M by reduction mod p.

THEOREM 3.5. *Let M be an $r \times s$ matrix with integer coefficients* ($s \geqq r + 1$). *The following are equivalent*:

(a) *The ideal generation conjecture holds for the s points in $\mathbf{P}^r_{\mathbf{Q}}$ given by the columns of M.*

(b) *The ideal generation conjecture holds for the s points in $\mathbf{P}^r_{\mathbf{F}_p}$ given by the columns of $M(p)$, for some p.*

(c) *The ideal generation conjecture holds for the s points in $\mathbf{P}^r_{\mathbf{F}_p}$ given by the columns of $M(p)$, for all but a finite number of p.*

Proof. (c) $\Rightarrow$ (b) is obvious. The generic position part of the equivalences follows from Lemma 3.1. To finish the proof for (a) $\Rightarrow$ (c) one need only observe that the procedure described in Section 2 can be carried out over $\mathbf{Q}$ by inverting only a finite number of elements. That (b) $\Rightarrow$ (a) follows from Lemma 3.4 and Lemma 3.1. For let N and T be as in Section 2, over $\mathbf{Q}$, and N_p, T_p be over $\mathbf{F}_p$. Lemma 3.4 says that we can choose N and N_p such that $N(p) = N_p$. Then $T(p) = T_p$. By (b), T_p is of maximal rank. Hence so is T by Lemma 1, so (a) holds.

As indicated in Section 2, the computation time for verifying the ideal generation conjecture over $\mathbf{F}_p$ is not large, and Theorem 3.5 then yields an example over $\mathbf{Q}$. However, if one can do the complete calculation over $\mathbf{Z}$, then one can see which are the "bad" primes in 3.5(c).

4. Reduction by a non-zero divisor. As in Section 1 let $A = R/I$ be the homogeneous co-ordinate ring of s points in generic position in $\mathbf{P}^3_k$ $R = k[x_0, x_1, x_2, x_3]$. Assume that A contains a non-zero divisor of degree 1, which by change of variable we can assume is x_0. Let

$$S = k[x_1, x_2, x_3] \quad \text{and} \quad J = (I, x_0)/x_0.$$

Then as indicated in [**2**, Section 1], we have

LEMMA 4.1. *For A, I, S, J as above, and d the smallest integer such that $I_d \neq 0$, we have*

(a) $J \cong I/x_0 I$

(b) $I_d \cong J_d$

(c) $f_1, \dots, f_r$ *generate I if and only if* $\bar{f}_1, \dots, \bar{f}_r$ *generate J* (*where the* $f_i \in R$ *are homogeneous, and* $\bar{f}_i$ *is the image of* f_i *in* S).

(d) $f: I_d^{n+1} \rightarrow I_{d+1}$ *is of maximal rank if and only if the induced map*

$$\bar{f}: J_d^n \rightarrow J_{d+1}$$

is of maximal rank. (*Explicitly*

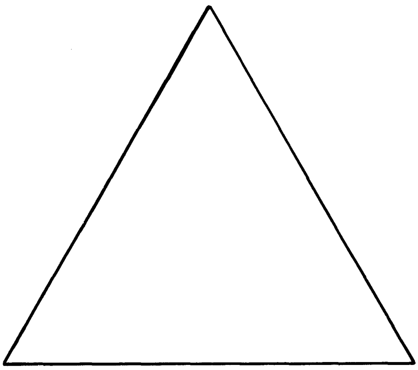
$$f(c_0, c_1, \dots, c_n) = x_0 c_0 + x_1 c_1 + \dots + x_n c_n \quad \textit{and}$$

$$\bar{f}(d_1, \dots, d_n) = x_1 d_1 + \dots + x_n d_n).$$

Using Lemma 4.1(d) we can reduce the sizes of some of the matrices that we have to work with. For example, let $s = 28$, and refer to the notation of the "brute force" methods of Section 2. Then we still have to calculate N as before, before factoring out by x_0. But then we can drop the first 20 columns of N (corresponding to monomials which in the lexicographical ordering contain x_0). This leaves a 7×15 matrix $\widetilde{N}$, and the matrix $\widetilde{T}$ for $\bar{f}$ will then be 21×21 (in this case $J_5 = S_5$ is of dimension 21). Thus for the final verification of the ideal generation conjecture we need only show that a 21×21 matrix has non-zero determinant. One can expect the largest integer occurring in the row reduction of $\widetilde{T}$ to be smaller than the largest integer occurring in the row reduction of T, thus improving the chances of completing an exact calculation over $\mathbf{Z}$. In one typical example the size of the largest integer was reduced by a factor of 100 by reduction mod x_0.

5. Special sets of points. In [**2**] a method ("lifting of monomial ideals" based on [**4**]) is given for producing explicit sets of points satisfying the ideal generation conjecture. The method also gives explicit generators for the ideal of the points. Unfortunately, this method does not work for 27 or 28 points in $\mathbf{P}^3$, but it does work for 26 points. The idea explored in this section is to start with 26 points as in [**2**], and then make a "random" choice of two more points. In this way 26 of the points are chosen with "small" co-ordinates, and the risk of integers becoming too large in the row reductions is reduced.

First let me review the construction. The monomials of degree 4 in S are represented as the upward pointing triangles, and the monomials of degree 5 are represented as the vertices in the following graph:



For example, the top triangle is x_1^4, the top vertex is x_1^5, triangle A is $x_1^2x_2x_3$, and the top, left, and right vertices of A are $x_1^3x_2x_3$, $x_1^2x_2^2x_3$, and $x_1^2x_2x_3^2$. We define J to be the ideal in S generated by elements in S_4 corresponding to the dotted triangles, that is by

$$x_1^4,\ x_1^3x_2,\ x_1^2x_2^2,\ x_2^4,\ x_2^3x_3,\ x_2^2x_3^2,\ x_3^4,\ x_3^3x_1,\ x_3^2x_1^2.$$

Because every vertex is the corner of one of these triangles, J_i is all of S_i for $i \geqq 5$. The monomials that are non-zero in S/J are all those of degree $\leqq 3$ (20 in all) and the 6 undotted triangles

$$x_1^3x_3,\ x_1^2x_2x_3,\ x_1x_2^3,\ x_1x_2^2x_3,\ x_1x_2x_3^2,\ x_2x_3^3.$$

Now choose 3 distinct non-zero elements of k (say 1, 2, 3). (To work over the prime field we thus need characteristic $\geqq 5$). To the monomials $x_1^ax_2^bx_3^c$ in S/J we let correspond the point $(1, a, b, c) \in \mathbf{P}_k^3$. To the 9 generators of J we let correspond respectively

$$f_1 = x_1\,(x_1 - x_0)(x_1 - 2x_0)(x_1 - 3x_0),$$

$$f_2 = x_1(x_1 - x_0)(x_1 - 2x_0)x_2,$$

$$f_3 = x_1(x_1 - x_0)x_2(x_2 - x_0),$$

$$f_4 = x_2(x_2 - x_0)(x_2 - 2x_0)(x_2 - 3x_0),$$

$$f_5 = x_2(x_2 - x_0)(x_2 - 2x_0)x_3, f_6 = x_2(x_2 - x_0)x_3(x_3 - x_0),$$

$$f_7 = x_3(x_3 - x_0)(x_3 - 2x_0)(x_3 - 3x_0),$$
$$f_8 = x_3(x_3 - x_0)(x_3 - 2x_0)x_1,$$
$$f_9 = x_3(x_3 - x_0)x_1(x_1 - x_0).$$

Let I be the ideal in R generated by f_i $(1 \leqq i \leqq 9)$. Then Theorem 2.2 (and Definition 1.7) of **[2]** say that I is the ideal in R of the 26 points obtained from the monomials in S/J. Explicitly these 26 points are the columns of the matrix

$$M = \left(\begin{array}{cccccccccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 2 & 1 & 1 & 0 & 0 & 0 & 3 & 2 & 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 2 & 1 & 0 & 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 2 & 0 & 0 & 1 & 0 & 1 & 2 \end{array}\right.$$

$$\left.\begin{array}{cccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 3 & 2 & 1 & 1 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 & 3 & 2 & 1 & 1 \\ 0 & 1 & 2 & 3 & 1 & 1 & 0 & 1 & 2 & 3 \end{array}\right)$$

Now pick two more points. Let H be the ideal of all 28 points. Then H_4 is a subspace of I_4, and a basis for H_4 can be obtain by solving a system of 2 equations in 9 unknowns. If $\dim H_4 = 7$ then our 28 points are in generic position. The rest of the ideal generation conjecture can be verified for all 28 points, by constructing $\widetilde{T}$ as in Section 4, and determining its rank. Recall that $\widetilde{T}$ is 21×21. Thus the amount of computation required to verify the ideal generation conjecture for 28 points has been reduced to solving one system of 2 equations in 9 unknowns, and verifying that one 21×21 matrix has rank 21; much less than required in Section 2.

Example 5.1. Let $P_{27} = (1, 6, -4, 5)$ and $P_{28} = (1, 7, 5, -3)$. Then the 28 points are in generic position for all p except 2, 3, 5 and satisfy the ideal generation conjecture for all p except 2, 3, 5, 673, and 2113.

Example 5.2. Let $P_{27} = (1, 2, 3, 4)$ and $P_{28} = (1, 4, 1, 3)$. Then the 28 points are in generic position for all p except $p = 2$ and $p = 3$. They satisfy the ideal generation conjecture for all p except 2 and 3. (I was surprised to find an example with no large primes.)

Example 5.3. Let $P_{27} = (2, -3, 1, -1)$ and $P_{28} = (1, -1, -5, 3)$. Then the 28 points are in generic position for all p except $p = 2$ and $p = 3$. They satisfy the ideal generation conjecture for all p except 2, 3, 5, and 135141739.

Note that $\mathbf{P}^3_k$ ($k = \mathbf{F}_p$) contains $(p^4 - 1)/(p - 1)$ k-rational points. If $p = 2$ then $\mathbf{P}^3_k$ contains only 15 points, so there is no hope of satisfying the ideal generation conjecture with k-rational points. If $p = 3$, then $\mathbf{P}^3_k$ contains 40 k-rational points, so one might expect to satisfy the ideal generation conjecture with k-rational points. However, there are only 27 points in which x_0 is a non-zero divisor, and also $\mathbf{F}_p$ does not contain 3 distinct non-zero elements, so the methods of Sections 4-5 cannot be used. I have tried to find 28 k-rational points in $\mathbf{F}_3$ satisfying the ideal generation conjecture by the "brute force" method of Section 2, and have not succeeded. Enough examples were tried to suggest there might not be 28 such points, but I have not been able to prove this. There appear not to be even 27 points satisfying the ideal generation conjecture, but 26 points satisfying the ideal generation are readily found. (Ballico in [**1**] has observed that $Y \subset X$, $\# X = i$, $\# Y = j$, $20 \leqq j < i \leqq 28$, X, Y sets of points in generic position $\mathbf{P}^3_k$, and X satisfies the ideal generation conjecture, then so does Y.)

6. Inappropriate choices of points. Suppose in the method of Section 5 that we choose $J \subset S$ to be the ideal generated by the monomials of degree 4 corresponding to the dotted triangles

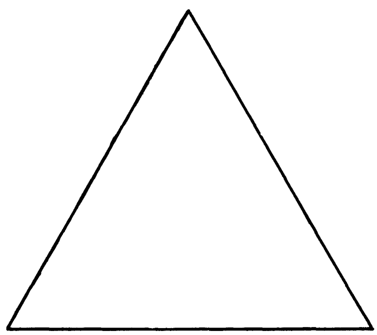


Figure 6.1

That is,

$$J = (x_1^4,\ x_1^2x_2^2,\ x_1^2x_2x_3,\ x_1^2x_3^2,\ x_2^4,\ x_2^3x_3,\ x_2^2x_3^2,\ x_2x_3^3,\ x_3^4).$$

The undotted triangles correspond to

$$x_1^3x_2,\ x_1^3x_3,\ x_1x_2^3,\ x_1x_2^2x_3,\ x_1x_2x_3^2,\ x_1x_3^3.$$

The monomials in S/J yield 26 points satisfying the ideal generation conjecture, namely 20 points with homogeneous co-ordinates the first 20 columns of M (of Section 5), and corresponding to the above 6 monomials, points with co-ordinates the columns of

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 1 & 1 & 1 & 1 \\ 1 & 0 & 3 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 & 2 & 3 \end{pmatrix}$$

I claim that these 26 points cannot be extended to 27 or 28 points satisfying the ideal generation conjecture. Note from Figure 6.1 that 9 of the monomials in S_5 are multiples of a generator of J only by x_1 (the top vertices of the dotted triangles). If we let these 9 monomials come first then the matrix $(\widetilde{T})^t$ (for any choice of two additional points) looks like

$$\begin{pmatrix} A & B \\ 0 & C \\ 0 & D \end{pmatrix}$$

where the columns correspond to the 21 monomials of S_5. The rows of (A, B) correspond to multiples of the rows of $\widetilde{N}$ by x_1, $(0, C)$ corresponds to multiples of the rows of $\widetilde{N}$ by x_2, and $(0, D)$ corresponds to multiples by x_3. Since A is 7×9 $\widetilde{T}$ cannot be of rank 21, so our 26 points cannot be extended to 28, satisfying the ideal generation conjecture. If we were trying to extend to 27 points A would be 8×9, $\widetilde{T}$ would be 24×21, and again $\widetilde{T}$ could not be of maximal rank, so our 26 points cannot be extended to 27 satisfying the ideal generation conjecture.

As usual let $S = k[x_1, x_2, x_3]$ and $S_d =$ degree d part of S. The above example can then be generalized in the following manner:

THEOREM 6.2. *Suppose we have m elements X of S_d which cover S_{d+1} (that is, every element of S_{d+1} is a multiple by x_i $(1 \leqq i \leqq 3)$ of some element of X). Suppose $l \leqq m$ of the vertices of S_{d+1} are multiples of an element of X only by x_1 (i.e., the "free tops" in the graph). Suppose there exists an integer $h \geqq 1$ such that*

(a) $m - h < l$

(b) $3(m - h) \geqq \binom{d+3}{2} = \#S_{d+1}$.

Then there exist $N = \binom{d+3}{3} - m$ points in generic position in $\mathbf{P}^3$ satisfying the ideal generation conjecture, which cannot be extended to $N + h$ points satisfying the ideal generation conjecture.

Proof. J generated by m elements of S_d corresponds to

$$(\# R_d) - m = \binom{d+3}{3} - m$$

points in $\mathbf{P}^3$, under the construction of Section 5, so yielding the expression for N. As in the discussion of 28 points, the matrix $(\widetilde{T})^t$ for $N + h$ points looks like

$$\begin{pmatrix} A & B \\ 0 & C \\ 0 & D \end{pmatrix}$$

where A is $(m - h) \times l$, C and D each have $m - h$ rows, and B, C, D have $\binom{d+3}{2} - l$ colums ($\# S_{d+1} = \binom{d+3}{2}$). Condition (b) ensures that the matrix has more rows than columns (so to be of maximal rank the columns must be linearly independent). Condition (a) says that A has more columns than rows. Thus $(\widetilde{T})^t$ is not of maximal rank, and Theorem 6.2 follows.

Example 6.3. Examples satisfying 6.2 can be obtained in the following manner.

(a) If d is even choose as generators of J alternate rows of triangles in the graph of S_d (i.e.,

$$J = (x_1^a x_2^b x_3^c),$$

with $a + b + c = d$, a even). Then

$$m = l = \left(\frac{d+2}{2}\right)^2,$$

and 6.2 is satisfied with $h = 1$, for $d \geqq 4$ ($d = 4$ being the example earlier in this section). One must check that

$$3\left(\left(\frac{d+2}{2}\right)^2 - 1\right) \geqq \binom{d+3}{2}$$

which simplifies to $d(d + 2) \geqq 12$.

(b) If d is odd choose as generators of J alternate rows of triangles, starting at the bottom, together with the top vertex (that is,

$$J = (x_1^d, x_1^a x_2^b x_3^c),$$

with $a + b + c = d$, a even). Then

$$m = (d^2 + 4d + 7)/4 \quad \text{and} \quad l = m - 2.$$

In order to satisfy 6.2(a), h must be at least 3. Finally 6.2 is satisfied for $h = 3$, so long as $d \geqq 5$. The inequality 6.2(b) is

$$3\left(\frac{d^2 + 4d + 7}{4} - 3\right) \geqq \binom{d + 3}{2}$$

which simplifies to $d(d + 2) \geqq 27$. Note that this fails for $d = 3$.

(c) More specifically there exist 26 points satisfying the ideal generation conjecture which cannot be extended to 27 or 28 (also satisfying the ideal generation conjecture), 43 points which cannot be extended to 46, 68 points that cannot be extended to 69, 70, 71, or 72 and 99 points that cannot be extended to 102, 103, 104, 105. The number of permissible h goes to ∞ as $d \to \infty$.

(d) Note that the N points in Theorem 6.2 cannot be extended to $N + h$ points satisfying the ideal generation conjecture, even if we are allowed to choose the new points over a ground field extension.

References

1. E. Ballico, *Generators for the homogeneous ideal of s general points in* $\mathbf{P}^3$, (preprint).
2. A. V. Geramita, D. Gregory and L. Roberts, *Monomial ideals and points in projective space*, Journal of Pure and Applied Algebra *40* (1986), 33-62.
3. A. V. Geramita and F. Orecchia, *On the Cohen-Macaulay type of s lines in* A^{n+1}, J. Alg. *70* (1981), 116-140.
4. R. Hartshorne, *Connectedness of the Hilbert scheme*, Publ. Math. Inst. des Hautes Etudes Sci. *29* (1966), 261-304.
5. ——— *Algebraic geometry* (Springer-Verlag, 1977).
6. A. Hirschowitz, letter to R. Hartshorne, (1983).

Queen's University,
Kingston, Ontario