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ON NORMED LIE ALGEBRAS WITH SUFFICIENTLY MANY SUBALGEBRAS OF CODIMENSION 1

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0. Introduction

Let H be a finite or infinite dimensional Lie algebra. Barnes [**2**] and Towers [**5**] considered the case when H is a finite-dimensional Lie algebra over an arbitrary field, and all maximal subalgebras of H have codimension 1. Barnes, using the cohomology theory of Lie algebras, investigated solvable algebras, and Towers extended Barnes's results to include all Lie algebras. In [**4**] complex finite-dimensional Lie algebras were considered for the case when all the maximal subalgebras of H are not necessarily of codimension 1 but when

$$\bigcap_{M\in S(H)} M=\{0\} \tag{1}$$

where $S(H)$ is the set of all Lie subalgebras in H of codimension 1. Amayo [**1**] investigated the finite-dimensional Lie algebras with core-free subalgebras of codimension 1 and also obtained some interesting results about the structure of infinite dimensional Lie algebras with subalgebras of codimension 1.

By $\mathfrak{X}$ we shall denote the class of complex finite or infinite dimensional normed Lie algebras for which (1) holds. In Section 2 the results of Amayo will be applied in order to prove that for every complex normed Lie algebra H and for every subalgebra $M\in S(H)$ the largest Lie ideal $I(M)$ of H contained in M has codimension less or equal to 3. Using this result for the case when $H\in\mathfrak{X}$ we shall show that, if $S_k(H)=\{M\in S(H)\colon \operatorname{codim} I(M)=k\}$, for $k=1,2,3$, then $L(H)=\bigcap_{M\in S_1(H)\cup S_2(H)}I(M)$ is a semi-simple ideal in H and $R(H)=\bigcap_{M\in S_3(H)}I(M)$ is the radical of H. We shall also prove that $H_{(2)}\subseteq L(H)$, so that $R(H)_{(2)}=0$. If H is finite-dimensional, then it was proved in [**4**] that $H=L(H)\dot{+}R(H)$ and that $L(H)=L_1\dot{+}\cdots\dot{+}L_n$, where all L_i are Lie ideals in H and isomorphic to $sl(2,\mathbb{C})$. If H is infinite dimensional, then this does not necessarily hold any longer. We shall consider an example of a normed Lie algebra H from $\mathfrak{X}$ such that $R(H)=0$ but $L(H)\neq H$. We shall also show that the property of belonging to $\mathfrak{X}$ is inherited by all closed subalgebras of H and by all quotient algebras $H/\mathscr{T}_S$ where S is any subset of $S(H)$ and where $\mathscr{T}_S=\bigcap_{M\in S}I(M)$. Finally, we shall consider the set T of all ideals $I(M)$ such that $\operatorname{codim} I(M)=3$ and shall introduce a Jacobson's topology on T.

In Section 3 the structure of solvable algebras from $\mathfrak{X}$ is investigated. For every $R\in\mathfrak{X}$ we consider a special set Σ of functionals on R from $R^0_{(1)}$ ($R^0_{(1)}$ is the polar of $R_{(1)}$) and the corresponding set of ideals $R^g_{(1)}=\{r'\in R:[r,r']=g(r)r'$ for every $r\in R\}$ in $R_{(1)}(g\in\Sigma)$. If R is a finite-dimensional solvable Lie algebra from $\mathfrak{X}$, then it was shown in [4] that

(T_1) the nil-radical N of R is commutative and a commutative subalgebra Γ of R exists such that $R=\Gamma\dot{+}N$,

(T_2) $N=Z\dot{+}R_{(1)}$, where Z is the centre of R, and $R_{(1)}=\sum_{i=1}^{n}\dot{+}R^{g_i}_{(1)}$, where $g_i\in\Sigma$.

For the case when R is infinite dimensional but Σ is a finite set, we shall prove in Theorem 3.6 that (T_1) and (T_2) hold. (This is the main result of the section). If Σ is not finite, then the structure of R is more complicated. In particular, (T_1) and (T_2) may no longer hold. To illustrate this we shall consider a solvable algebra R such that $N=R_{(1)}$, that $\dim(R/R_{(1)})=2$ and therefore $\dim(R^0_{(1)})=2$, but Σ is infinite. We shall show that in this case (T_1) and (T_2) do not hold and that there is not even a commutative algebra Γ such that $\Gamma\cap N=0$ and such that linear combinations of elements from Γ and N are dense in R. We shall also prove that $R^g_{(1)}=0$ in this example for all $g\in\Sigma$.

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1. Preliminaries and notation

Let m and n be arbitrary integers. Then $\binom{m}{n}$ is the usual binomial coefficient with the understanding that $\binom{m}{n}=0$ if $m<0$ or $n<0$ or $m<n$. But we take $\binom{m}{o}=1$ if $m\geqq -1$. As in [1] we define, for arbitrary integers,

$$\lambda_{ij}=\binom{i+j}{i+1}-\binom{i+j}{j+1}. \tag{2}$$

If a linear space B is the direct sum of its subspaces B_i, for $i=1,\dots,n$ we shall denote it by

$$B=B_1\dot{+}\cdots\dot{+}B_n.$$

Let H be a complex Lie algebra of finite or infinite dimension and let there exist a subalgebra M of codimension 1. By $I(M)$ we denote the largest Lie ideal of H contained in M. Then $I(M)$ contains any Lie ideal of H contained in M. Now put $I_0=M$ and let h_- be an element in H which does not belong to M. For every $i\geqq 0$ let us define by induction

$$I_{i+1}=\{h\in H:[h,h_-]\in I_i\}. \tag{3}$$

If $h\in H$ by $\{h\}$ we shall denote the one-dimensional subspace generated by h.

Theorem 1.1. [1] (Amayo) *If M is a Lie subalgebra in H of codimension 1 then three possibilities exist:*

(1) $I(M)=M$;

(2) $\dim(H/I(M))=2$, *$H/I(M)$ is solvable but not commutative and there exist elements h_- and h_0 in H such that*

$$H=\{h_-\}\dot{+}M,\ M=\{h_0\}\dot{+}I(M)$$

and

$$[h_-,h_0]\equiv h_- \bmod I(M); \tag{4}$$

(3) (i) *all I_i are Lie ideals of M, $I_{i+1}\subseteq I_i$ and*

$$I(M)=\bigcap_{i=0}^{\infty} I_i,$$

(ii) *there exist elements h_i, possibly zero, such that*

$$I_i=\{h_i\}\dot{+}I_{i+1} \text{ and } [h_i,h_j]\equiv\lambda_{ij}h_{i+j} \bmod I_{i+j+1},$$

(iii) *if $I_i=I_{i+1}$ for some i, then $I_j=I_{j+1}$ for all $j\geqq 2$, $I(M)=I_2$ and there exist elements h_-, h_0 and h_+ in H such that*

$$H=\{h_-\}\dot{+}M,\ M=\{h_0\}\dot{+}\{h_+\}\dot{+}I(M) \tag{5}$$

and

$$[h_-,h_0]\equiv 2h_- \bmod I(M),\ [h_0,h_+]\equiv 2h_+ \bmod I(M),$$

$$[h_+,h_-]\equiv h_0 \bmod I(M). \tag{6}$$

Now let a complex Lie algebra H be a Banach space. We shall call H a normed Lie algebra if a constant C exists such that

$$\|[h_1,h_2]\|\leqq C\|h_1\|\|h_2\| \tag{7}$$

for every $h_1,h_2\in H$. We say that a closed subalgebra M of a normed Lie algebra H has codimension 1 if there exists $h\in H$ such that $h\notin M$ and that $H=M\dot{+}\{h\}$. By $S(H)$ we denote the set of all closed Lie subalgebras of codimension 1 in H. We shall often make use of the following property of Lie algebras from $\mathfrak{X}$ which follows easily from (1): for every $h\in H$ there exists $M\in S(H)$ such that $H=M\dot{+}\{h\}$.

By $H^2=[H,H]$ we shall denote the closed Lie subalgebra of H spanned by all Lie products of pairs of elements of H. H^k, for $k>1$, is the closed Lie subalgebra which is defined inductively by the rule $H^k=[H^{k-1},H]$. H is said to be nilpotent if $H^k=0$ for some k. The closed subalgebras $H_{(k)}$ are also defined by the inductive rule that $H_{(1)}=H^2$ and $H_{(k+1)}=[H_{(k)},H_{(k)}]$ for $k\geqq 1$. H is called solvable if $H_{(k)}=0$ for some k. A solvable (nilpotent) ideal $R(N)$ is called the radical (nil-radical) of H if it contains every solvable (nilpotent) ideal of H. If $R=0$, then H is called semisimple.

For every linear subspace G in H let $\bar{G}$ be its closure. Using (7) one can easily prove that $[G_1, G_2] = [\bar{G}_1, \bar{G}_2]$ for all subspaces G_1 and G_2 in H and that, if G is a subalgebra of H, then $\bar{G}$ is also a subalgebra of H. Therefore, if G is a solvable (nilpotent) subalgebra of H, then $\bar{G}$ is also a solvable (nilpotent) subalgebra of H. Thus R and N are closed ideals of H.

The simple Lie algebra of complex matrices $\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)$ is denoted by $sl(2, \mathbb{C})$. Set $h = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, $h_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $h_+ = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Then

$$[h_0, h_+] = 2h_+, \ [h_0, h_-] = -2h_-, \ [h_+, h_-] = h_0. \tag{8}$$

2. The structure of normed Lie algebras from $\mathfrak{X}$

Theorem 2.1. *Let H be a complex normed Lie algebra and let there exist a closed Lie subalgebra M of codimension* 1. *Then*

(i) *$I(M)$ is closed,*

(ii) *if* $\dim(H/I(M)) > 2$, *then* $\dim(H/I(M)) = 3$, *$H/I(M)$ is isomorphic to* $sl(2, \mathbb{C})$ *and there exist elements h_-, h_0 and h_+ in H such that formulae* (5) *and* (6) *hold.*

Proof. Since $I(M)$ is the largest Lie ideal of H contained in M, the proof of (i) follows from the fact that, if G is a Lie ideal of H, then $\bar{G}$ is also a Lie ideal of H.

If H is finite-dimensional, then the proof of (ii) follows immediately from Theorem 1.1 (3) (iii). Now let H be infinite dimensional. First we shall show by induction that all I_i are closed. $I_0 = M$ is closed. Suppose that I_k is closed. Let elements $h^{(p)}$ belong to I_{k+1} and converge to h. By (7),

$$\|[h, h_{-1}] - [h^{(p)}, h_{-1}]\| = \|[h - h^{(p)}, h_{-1}]\| \leqq C\|h - h^{(p)}\| \|h_{-1}\| \to 0.$$

Hence the elements $[h^{(p)}, h_{-1}]$ converge to $[h, h_{-1}]$. By (3), $[h^{(p)}, h_{-1}]$ belong to I_k and, since I_k is closed, we have that $[h, h_{-1}]$ belongs to I_k. Therefore $h \in I_{k+1}$ and I_{k+1} is closed.

It follows from Theorem 1.1 (3) (ii) and from (2), that for $j \geqq 0$

$$[h_0, h_j] = jh_j + a_{j+1}$$

where $a_{j+1} \in I_{j+1}$. Since, by Theorem 1.1 (3) (i), all I_i are ideals of M and since $h_0 \in M$, we have that $[h_0, h] \in I_{j+1}$ for every $h \in I_{j+1}$ and that

$$[h_0, h_j + h] = [h_0, h_j] + [h_0, h] = jh_j + h'$$

where $h' = a_{j+1} + [h_0, h]$ belongs to I_{j+1}. By (7),

$$\|jh_j + h'\| = \|[h_0, h_j + h]\| \leqq C\|h_0\| \|h_j + h\|.$$

Dividing by j we get that for every $h \in I_{j+1}$ the element $f(h) = h'/j$ in I_{j+1} exists such that

$$\|h_j + f(h)\| \leqq C_1 \|h_j + h\| \tag{9}$$

where $C_1 = C\|h_0\|/j$. Let us choose an element $h^{(1)}$ in I_{j+1} and let us put by induction $h^{(k)} = f(h^{(k-1)})$. Then, by (9),

$$\|h_j + h^{(k)}\| = \|h_j + f(h^{(k-1)})\| \leqq C_1 \|h_j + h^{(k-1)}\|$$

$$= C_1 \|h_j + f(h^{(k-2)})\| \leqq C_1^2 \|h_j + h^{(k-2)}\| \leqq \cdots \leqq C_1^{k-1} \|h_j + h^{(1)}\|.$$

If j is large enough so that $C_1 < 1$, then we obtain that $\|h_j + h^{(k)}\| \to 0$. Since all $h^{(k)}$ belong to I_{j+1} and since I_{j+1} is closed, we have that $h_j \in I_{j+1}$. Hence, by Theorem 1.1 (3) (ii), $I_j = \{h_j\} + I_{j+1} = I_{j+1}$ and the proof of the theorem follows from Theorem 1.1 (3) (iii) and from formula (8).

For $k = 1, 2$ and 3 we put

$$S_k(H) = \{M \in S(H): \operatorname{codim} I(M) = k\}.$$

Then $S(H) = \bigcup_{k=1}^{3} S_k(H)$. Now put

$$L(H) = \bigcap_{M \in S_1(H) \cup S_2(H)} I(M) \quad \text{and} \quad R(H) = \bigcap_{M \in S_3(H)} I(M).$$

If $S_1(H) \cup S_2(H) = \varnothing$, then put $L(H) = H$. If $S_3(H) = \varnothing$, then put $R(H) = H$. If $H \in \mathfrak{X}$, then it follows from (1) that

$$L(H) \cap R(H) = \bigcap_{M \in S(H)} I(M) \subseteq \bigcap_{M \in S(H)} M = 0.$$

Since all $I(M)$ are closed, $L(H)$ and $R(H)$ are also closed.

Lemma 2.2. *If $H \in \mathfrak{X}$, then $H_{(2)} \subseteq L(H)$.*

Proof. Let $M \in S_1(H)$. Then $I(M) = M$ and an element h_- exists such that $H = \{h_-\} \dotplus M$. Therefore $H_{(1)} = [H, H] \subseteq I(M)$. Since M is an arbitrary subalgebra in $S_1(H)$, we obtain that

$$H_{(2)} \subseteq H_{(1)} \subseteq \bigcap_{M \in S_1(H)} I(M).$$

Now let $M \in S_2(H)$. Then, by Theorem 1.1 (2), the elements h_- and h_0 exist such that

$$H = \{h_-\} \dotplus M, \; M = \{h_0\} \dotplus I(M)$$

and

$$[h_-, h_0] \equiv h_- \bmod I(M).$$

Therefore $H_{(1)} \subseteq \{h_-\} \dot{+} I(M)$ and $H_{(2)} = [H_{(1)}, H_{(1)}] \subseteq I(M)$. Since M is an arbitrary subalgebra in $S_2(H)$, we obtain that

$$H_{(2)} \subseteq \bigcap_{M \in S_2(H)} I(M).$$

Thus $H_{(2)} \subseteq \bigcap_{M \in S_1(H) \cup S_2(H)} I(M) = L(H)$ which completes the proof.

For every closed subalgebra G of H set

$$S_G(H) = \{M \in S(H): G \nsubseteq M\} \text{ and } S^G(H) = \{M \in S(H): G \subseteq M\}.$$

Then $S(H) = S_G(H) \cup S^G(H)$. Now let S be a subset in $S(H)$. Set

$$\mathscr{T}_S = \bigcap_{M \in S} I(M).$$

Then $\mathscr{T}_S$ is a closed ideal of H.

Theorem 2.3. *Let $H \in \mathfrak{X}$.*

(i) *If G is a closed subalgebra of H, then $G \in \mathfrak{X}$.*

(ii) *For every subset S in $S(H)$ the quotient algebra $H/\mathscr{T}_S$ belongs to $\mathfrak{X}$.*

Proof. Let G be a closed subalgebra of H. For every $M \in S_G(H)$ set $M_G = G \cap M$. Then all M_G are closed subalgebras in G of codimension 1 and

$$\bigcap_{M' \in S(G)} M' \subseteq \bigcap_{M_G \in S(G)} M_G = G \cap \left(\bigcap_{M \in S_G(H)} M \right) = \bigcap_{M \in S(H)} M = 0.$$

Therefore $G \in \mathfrak{X}$ and (i) is proved.

Now let $S \subseteq S(H)$ and let f be the homomorphism of H onto $H/\mathscr{T}_S$. If $M \in S^{\mathscr{T}_S}(H)$, then $f(M)$ is a closed subalgebra of codimension 1 in $H/\mathscr{T}_S$. Therefore in order to prove (ii) it is sufficient to prove that

(a) if $h \notin \mathscr{T}_S$, then there exists $M \in S^{\mathscr{T}_S}(H)$ such that $h \notin M$.

Since $h \notin \mathscr{T}_S$, there exists $M_0 \in S$ such that $h \notin I(M_0)$.

If $M_0 \in S_1(H)$, then $I(M_0) = M_0$ and therefore $h \notin M_0$. Since $M_0 \in S^{\mathscr{T}_S}(H)$, we obtain that (a) holds for h.

If $M_0 \in S_2(H)$, then there exist elements h_- and h_0 such that (4) holds. Therefore $h = ah_- + bh_0 + i$ where a and b are complex and where $i \in I(M_0)$. If $a \neq 0$, then $h \notin M_0$ and (a) holds for h. Let $a = 0$, that is, $h = bh_0 + i$ and $b \neq 0$. Then $h \in M_0$. Set $x = h_- + h_0$ and $M = I(M_0) \dot{+} \{x\}$. We have that M is a Lie subalgebra of H and that $h \notin M$. Since $I(M_0)$ is closed and since $\dim(H/I(M_0)) = 2$, we obtain easily that M is closed and that codim $M = 1$. Since $\mathscr{T}_S \subseteq I(M_0) \subset M$, (a) holds for h.

Finally, let $M_0 \in S_3(H)$. Then there exist elements h_-, h_0 and h_+ such that (5) and (6) hold. Therefore $h = ah_- + bh_0 + ch_+ + i$ where a, b and c are complex and where $i \in I(M_0)$. If $a \neq 0$, then $h \notin M_0$ and (a) holds for h. Let $a = 0$, that is, $h = bh_0 + ch_+ + i$ and $|b| + |c| \neq 0$. Then $h \in M_0$. If $c \neq 0$, then set $M = I(M_0) \dotplus \{h_-\} \dotplus \{h_0\}$. Since $I(M_0)$ is closed and since $\dim(H/I(M_0)) = 3$, we obtain easily that M is closed. It follows from (5) and (6) that M is a Lie subalgebra of H, that $h \notin M$ and that $\operatorname{codim} M = 1$. Since $\mathscr{T}_S \subseteq I(M_0) \subset M$, (a) holds for h. Now let $c = 0$, so that $h = bh_0 + i$ and $b \neq 0$. Set

$$x = h_- + h_0, \; y = h_- + 2h_0 - 4h_+ \text{ and } M = I(M_0) \dotplus \{x\} \dotplus \{y\}.$$

It follows from (6) that $[x, y] = 2y \bmod I(M_0)$. Therefore M is a closed Lie subalgebra of codimension 1 in H and h does not belong to M. Thus (a) holds for h and the proof of (ii) is complete.

From Lemma 2.2 and from Theorem 2.3 we obtain immediately the following corollary.

Corollary 2.4. *If $H \in \mathfrak{X}$ and if $H' = H/L(H)$, then H' is solvable, $H'_{(2)} = 0$ and $L(H)$ and H' belong to $\mathfrak{X}$.*

Lemma 2.5. *Let $M \in S_3(H)$.*

(i) *The elements h_-, h_0 and h_+ in formulae (5) and (6) can be chosen from $L(H)$.*

(ii) *Subalgebra $M_L = M \cap L(H)$ has codimension 1 in $L(H)$, $I(M_L) = L(H) \cap I(M)$ and $M_L \in S_3(L(H))$.*

Proof. Let $M \in S_3(H)$. By Theorem 1.1 (3) (iii), the elements h'_-, h'_0 and h'_+ exist in H such that $H = \{h'_-\} \dotplus M$, $M = \{h'_0\} \dotplus \{h'_+\} \dotplus I(M)$ and elements g_-, g_0 and g_+ exist in $I(M)$ such that

$$[h'_-, h'_0] = 2h'_- + g_-, \; [h'_0, h'_+] = 2h'_+ + g_+, \; [h'_+, h'_-] = h'_0 + g_0. \tag{10}$$

Put $h_0 = [[h'_0, h'_+], \; [h'_-, h'_0]]/4$, $h_- = [[h'_-, h'_0], \; [h'_+, h'_-]]/4$ and $h_+ = [[h'_+, h'_-], [h'_0, h'_+]]/4$. By Lemma 2.2, h_0, h_- and h_+ belong to $L(H)$. It follows from (10) that

$$h_0 = [2h'_+ + g_+, 2h'_- + g_-]/4 \equiv [h'_+, h'_-] \bmod I(M) \equiv h'_0 \bmod I(M).$$

In the same way we can show that

$$h_- \equiv h'_- \bmod I(M)$$

and that

$$h_+ \equiv h'_+ \bmod I(M).$$

from which the rest of the proof of (i) follows immediately.

Let $M_L = M \cap L(H)$. Since $h_- \in L(H)$ and $h_- \notin M$ and since $\operatorname{codim} M = 1$, M_L is a closed subalgebra of codimension 1 in $L(H)$. Let $I = L(H) \cap I(M)$. Then I is a closed ideal of $L(H)$ contained in M_L and, by (i) and by Theorem 1.1 (3) (iii),

$$L(H) = \{h_-\} \dotplus M_L$$

and

$$M_L = \{h_0\} \dotplus \{h_+\} + I.$$

Since h_-, h_0 and h_+ belong to $L(H)$, all identities in (6) hold *modulo I*. Therefore the quotient algebra $L(H)/I$ is isomorphic to $sl(2, \mathbb{C})$. From this it follows easily that I is the maximal ideal of $L(H)$ contained in M_L, that is, $I = I(M_L)$. Thus $M_L \in S_3(L(H))$ and the proof is complete.

Theorem 2.6. (i) *$R(H)$ is the radical of H and* $R(H)_{(2)} = 0$. (ii) *$L(H)$ is semisimple.*

Proof. It follows from Lemma 2.2 that $R(H)_{(2)} \subseteq L(H)$. Since $R(H)$ is a Lie ideal of H, we have that $R(H)_{(2)} \subseteq R(H)$. But $L(H) \cap R(H) = 0$. Hence $R(H)_{(2)} = 0$. Therefore $R(H)$ is a solvable ideal in H.

Now suppose that R is another closed solvable ideal in H. Let $M \in S_3(H)$ and let f be the homomorphism of H onto $H/I(M)$. Then $f(R)$ is a solvable Lie ideal in $H/I(M)$. But, by Theorem 2.1 $H/I(M)$ is isomorphic to $sl(2, \mathbb{C})$ which is simple, Therefore every solvable ideal in $H/I(M)$ is trivial. Hence $R \subseteq I(M)$. Since M is an arbitrary subalgebra in $S_3(H)$, we obtain that

$$R \subseteq \bigcap_{M \in S_3(H)} I(M) = R(H)$$

and (i) is proved.

It follows from the definition of the radical and from Lemma 2.5 (ii) that

$$R(L(H)) = \bigcap_{M' \in S_3(L(H))} M' \subseteq \bigcap_{M \in S_3(H)} M_L = L(H) \cap \left(\bigcap_{M \in S_3(H)} M \right) = L(H) \cap R(H) = 0.$$

Therefore $L(H)$ is semisimple.

Remark. If H is finite-dimensional, then it was proved in [4] that $H = L(H) \dotplus R(H)$ and that $L(H) = L_1 \dotplus \cdots \dotplus L_k$ where all L_i are Lie ideals of H and isomorphic to $sl(2, \mathbb{C})$.

Now we shall consider an example of a normed infinite dimensional Lie algebra H from $\mathfrak{X}$ such that $R(H) = 0$ but $L(H) \neq H$.

Example 1. Let $H = \{A = \{A_n\}_{n=1}^{\infty}$: (i) $A_n \in sl(2, \mathbb{C})$, (ii) there exists a matrix $A_0 = \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}$ such that $\lim A_n = A_0\}$.

Set $\|A\|=\sup_n\|A_n\|$ and set $[A,B]=\{[A_n,B_n]\}_{n=1}^{\infty}$. Then H is a normed Lie algebra.

It is well-known that $sl(2,\mathbb{C})\in\mathfrak{X}$ and that $S(sl(2,\mathbb{C}))=S_3(sl(2,\mathbb{C}))$. For every subalgebra $\mathcal{M}$ of codimension 1 in $sl(2,\mathbb{C})$ and for every positive integer k let

$$M_k=\{A=\{A_n\}_{n=1}^{\infty}\in H: A_k\in\mathcal{M}\}.$$

Then M_k are subalgebras of codimension 1 in H and

$$\bigcap_{M\in S(H)} M\subseteq \bigcap_{\substack{k=1\\ \mathcal{M}\in S(sl(2,\mathbb{C}))}}^{\infty} M_k=0.$$

Thus $H\in\mathfrak{X}$. Let $I_k=\{A\in H:A_k=0\}$. Then I_k are ideals of H and $I(M_k)=I_k$ for every subalgebra $\mathcal{M}\in S(sl(2,\mathbb{C}))$. Every ideal I_k has codimension 3 in H and H/I_k is isomorphic to $sl(2,\mathbb{C})$. Therefore all M_k belong to $S_3(H)$. Then we have that

$$R(H)=\bigcap_{M\in S_3(H)} I(M)\subseteq\bigcap_{k=1}^{\infty} I_k=0.$$

Now let $I_\infty=\{A\in H:\lim A_n=0\}$. By G we shall denote the two-dimensional solvable Lie algebra of all complex matrices $\begin{pmatrix} a & b\\ 0 & -a\end{pmatrix}$. Then $G\in\mathfrak{X}$ and $S(G)=S_1(G)\bigcup S_2(G)$. For every $\mathcal{M}\in S(G)$ let $M_\infty=\{A\in H:\lim A_n\in\mathcal{M}\}$. Then M_∞ are subalgebras of codimension 1 in H and $I_\infty\subset M_\infty$. If $I(M_\infty)$ is the corresponding maximal ideal of H in M_∞ then $I_\infty\subseteq I(M_\infty)$. Since $\operatorname{codim} I_\infty=2$ we have that $\operatorname{codim} I(M_\infty)\leqq 2$. Hence $M_\infty\in S_1(H)\cup S_2(H)$. Therefore

$$L(H)=\bigcap_{M\in S_1(H)\cup S_2(H)} I(M)\subseteq\bigcap_{\mathcal{M}\subset G} I(M_\infty)=I_\infty.$$

In fact one can easily prove that $L(H)=I_\infty$. Thus $L(H)\neq H$, $R(H)=0$ and $H/L(H)$ is isomorphic to G. It can also be proved easily that $H_{(2)}=L(H)$ and that $L(H)_{(1)}=L(H)$.

Remark. If H is a finite-dimensional semisimple Lie algebra, then $H_{(1)}=H$. In the example above H is infinite dimensional and, although it is semisimple, we have that $H_{(1)}\neq H$. But we also have that $L(H)_{(1)}=L(H)$. The question arises as to whether $L(H)_{(1)}=L(H)$ for every $H\in\mathfrak{X}$.

By T we shall denote the set of all ideals $I(M)$ such that $\operatorname{codim} I(M)=3$. Let τ be any subset in T. Put $I(\tau)=\bigcap_{I(M)\in\tau} I(M)$ and put $\bar{\tau}=\{I(M)\in T: I(\tau)\subseteq I(M)\}$. Now suppose that $\mathcal{T}_1$ and $\mathcal{T}_2$ are Lie ideals such that $\mathcal{T}_1\cap\mathcal{T}_2\subseteq I(M)$ where $I(M)\in T$. Let f be the homomorphism of H onto $H/I(M)$. Then $f(\mathcal{T}_1)$ and $f(\mathcal{T}_2)$ are Lie ideals in $H/I(M)$. But since $H/I(M)$ is simple, we have that $f(\mathcal{T}_1)$ and $f(\mathcal{T}_2)$ are either trivial ideals or coincide with $H/I(M)$. Taking into account that $[h_1,h_2]\in\mathcal{T}_1\cap\mathcal{T}_2\subseteq I(M)$ for every $h_1\in\mathcal{T}_1$ and for every $h_2\in\mathcal{T}_2$ we obtain that $[f(h_1), f(h_2)]=f([h_1,h_2])=0$. Since $H/I(M)$ is not commutative we obtain that at least one of these ideals is trivial. Thus if $\mathcal{T}_1$ and $\mathcal{T}_2$ are ideals such that $\mathcal{T}_1\cap\mathcal{T}_2\subseteq I(M)\in T$ then either $\mathcal{T}_1\subseteq I(M)$ or $\mathcal{T}_2\subseteq I(M)$.

Using this argument and repeating the proof of Lemma 3.1.1 [3] we can easily prove the following lemma:

Lemma 2.7.

(i) $\bar{\varnothing}=\varnothing$ *and* $\tau \subseteq \bar{\tau}$ *for every* $\tau \subset T$

(ii) $\bar{\bar{\tau}}=\bar{\tau}$ *for every* $\tau \subset T$ *and* $\overline{\tau_1 \cup \tau_2}=\bar{\tau}_1 \cup \bar{\tau}_2$ *if* $\tau_1, \tau_2 \subset T$.

From Lemma 2.7. it follows that there exists a unique topology (Jacobson's topology) on T such that for every $\tau \subset T$ the set $\bar{\tau}$ is its closure in this topology. Since every $I(M)$ in T is maximal we have that it is closed.

3. The structure of normed solvable algebras from $\mathfrak{X}$

We shall start the section with a well-known lemma.

Lemma 3.1. *Let N be a normed nilpotent algebra from $\mathfrak{X}$. Then N is commutative.*

Let R be a normed solvable algebra from $\mathfrak{X}$ and let N be its nil-radical. It follows from Theorem 2.6 that $R_{(2)}=0$. Hence $R_{(1)}$ is a commutative ideal of R. Therefore $R_{(1)} \subseteq N$. By Theorem 2.3 (i), N belongs to $\mathfrak{X}$ and hence, by Lemma 3.1, N is commutative.

By R^* we shall denote the dual space of R which consists of all bounded functionals on R. For every $r \in R$ we denote by A_r the operator on R^* which is defined by the formula

$$(A_r f)(r_1)=f([r,r_1]). \tag{11}$$

Then A_r is a linear operator and it is bounded since

$$\|A_r f\|=\sup_{\|r_1\|=1}|(A_r f)(r_1)| \leqq \|f\| \sup_{\|r_1\|=1}\|[r,r_1]\| \leqq C\|f\|\,\|r\|.$$

By $R^0_{(1)}$ we denote the polar of $R_{(1)}$ which consists of all functionals f in R^* such that $f|_{R_{(1)}}=0$.

Lemma 3.2.

(i) $A_r f=0$ *for every* $r \in R$ *if and only if* $f \in R^0_{(1)}$.

(ii) *If* $A_r f=g(r)f$ *for every* $r \in R$, *where* g *is a functional on* R, *then* $f \in R^0_{(1)}$ *and* $g(r) \equiv 0$.

(iii) *Every operator* A_r *is continuous in* $\sigma(R^*, R)$*-topology*

Proof. If $f \in R^0_{(1)}$, then, by (11), $(A_r f)(r_1)=0$ for all r, $r_1 \in R$. Hence $A_r f=0$. If, on the other hand, we have that $A_r f=0$ for all $r \in R$, then, by (11), $f([r,r_1])=0$ for every $r_1 \in R$. Hence $R_{(1)} \subseteq \operatorname{Ker} f$ and therefore $f \in R^0_{(1)}$. Thus (i) is proved.

Now let $A_r f=g(r)f$ for every $r \in R$ and let $r_1 \in \operatorname{Ker} f$. Then

$$f([r,r_1]=(A_r f)(r_1)=g(r)f(r_1)=0.$$

Hence $[r, r_1] \in \operatorname{Ker} f$. Therefore $\operatorname{Ker} f$ is an ideal in R. Let $r_0 \in R$ be such that $R = \{r_0\} \dot{+} \operatorname{Ker} f$ and that $f(r_0)=1$. Then for every $r \in R$ there exists a complex t such that $r = tr_0 + r_1$ where $r_1 \in \operatorname{Ker} f$. Then

$$(A_r f)(r_0) = g(r) f(r_0) = f([r, r_0]) = f([r_1, r_0]) = 0,$$

since $[r_1, r] \in \operatorname{Ker} f$. Hence $g(r)=0$ for all $r \in R$. Therefore it follows from (i) that $f \in R_{(1)}$ and (ii) is proved.

Let $r \in R$ and let (f_α) be a directed set of elements in R^* converging to 0 in $\sigma(R^*, R)$-topology. For every finite set $(r_i)_{i=1}^n$ put $r_i' = [r, r_i]$. Let $\varepsilon > 0$ and let us choose α_0 such that $|f_\alpha(r_i')| < \varepsilon$ for all i and for $\alpha > \alpha_0$. Then

$$|A_r f_\alpha(r_i)| = |f_\alpha([r, r_i])| < \varepsilon$$

and $(A_r f_\alpha)$ converges to 0 in $\sigma(R^*, R)$-topology. Hence (iii) is proved.

Lemma 3.3. *Let $r_- \in R_{(1)}$ and let $M \in S(R)$ be a subalgebra such that $r_- \notin M$. Then $M \in S_2(R)$ and there exist $r_0 \in M$ and functionals $g \in R_{(1)}^0$ and $f \notin R_{(1)}^0$ such that $r_0 \notin R_{(1)}$, $g(r_0) \neq 0$, $f(r_-) \neq 0$, $[r_0, r_-] \equiv r_- \bmod I(M)$ and for every $r \in R$*

$$A_r f = g(r) f - f(r) g.$$

Proof. Since $R \in \mathfrak{X}$, there exists a subalgebra $M \in S(R)$ such that $r_- \notin M$. If M is an ideal in R, then $R_{(1)} \subseteq M$ which contradicts the fact that r_- does not belong to M. Hence $M \notin S_1(R)$. Since R is solvable, we have that $S_3(R) = \varnothing$. Hence $M \in S_2(H)$. By Theorem 1.1 (2), an element $r_0 \in M$ exists such that $M = \{r_0\} \dot{+} I(M)$ and that $[r_0, r_-] \equiv r_- \bmod I(M)$. Therefore $R_{(1)} \subseteq \{r_-\} \dot{+} I(M)$. Hence r_0 does not belong to $R_{(1)}$.

Since M is closed and $\operatorname{codim}(M) = 1$, there exists a functional f such that $\operatorname{Ker} f = M$. Hence $f(r_-) \neq 0$ and therefore $f \notin R_{(1)}^0$. The subspace $\{r_-\} \dot{+} I(M)$ is closed and has codimension 1 in R. Therefore a functional f_1 exists such that $\operatorname{Ker} f_1 = \{r_-\} \dot{+} I(M)$. Hence $f_1(r_0) \neq 0$. Since $R_{(1)} \subseteq \{r_-\} \dot{+} I(M)$, we have that $f_1 \in R_{(1)}^0$.

Let $I^0(M) = \{f \in R^* : f|_{I(M)} = 0\}$ be the polar of $I(M)$ in R^*. Since $\operatorname{codim} I(M) = 2$, we have that $\dim I^0(M) = 2$. The functionals f and f_1 belong to $I^0(M)$ and, since $f(r_-) \neq 0$ and $f_1(r_-) = 0$, they are linearly independent. Hence f and f_1 form a basis in $I^0(M)$. Since $I(M)$ is an ideal in R, it follows from (11) that $I^0(M)$ is invariant under all operators $A_r, r \in R$. Hence

$$A_r f = g(r) f + h(r) f_1$$

where g and h are linear bounded functionals on R. Then, since $f(r_0)=0$ and since $f_1(r_-)=0$, we have that

$$(A_r f)(r_0) = f([r, r_0]) = g(r) f(r_0) + h(r) f_1(r_0) = h(r) f_1(r_0), \tag{12}$$

$$(A_r f)(r_-) = f([r, r_-]) = g(r) f(r_-) + h(r) f_1(r_-) = g(r) f(r_-). \tag{13}$$

If $r \in I(M)$, then $[r, r_0] \in I(M)$ and $f([r, r_0]) = 0$. Since $f_1(r_0) \neq 0$, we get from (12) that $I(M) \subseteq \operatorname{Ker} h$. If $r = r_0$, then, by (12), $h(r_0) = 0$. Hence $M = \{r_0\} \dot{+} I(M) = \operatorname{Ker} f \subseteq \operatorname{Ker} h$. Therefore $h = af$ where a is a complex number. If $r = r_-$ then $[r_-, r_0] \equiv -r_- \bmod I(M)$ and, by (12),

$$-f(r_-) = af(r_-)f_1(r_0).$$

Hence $a = -1/f_1(r_0)$.

Now if $r \in I(M)$, then $[r, r_-] \in I(M)$ and $f([r, r_-]) = 0$. Since $f(r_-) \neq 0$, we get from (13) that $I(M) \subseteq \operatorname{Ker} g$. If $r = r_-$, then, by (13), $g(r_-) = 0$. Hence $\{r_-\} \dot{+} I(M) = \operatorname{Ker} f_1 \subseteq \operatorname{Ker} g$. Therefore $g = bf_1 \in R^0_{(1)}$ where b is a complex number. If $r = r_0$, then $[r_0, r_-] = r_- \bmod I(M)$ and, by (13),

$$f(r_-) = bf_1(r_0)f(r_-).$$

Hence $b = 1/f_1(r_0) = -a$. Therefore

$$h(r)f_1 = af(r)f_1 = -bf(r)f_1 = -f(r)g.$$

Hence $A_r f = g(r)f - f(r)g$ which concludes the proof of the lemma.

Let $g \in R^0_{(1)}$. By T_g we shall denote the set of all functionals f such that for every $r \in R$

$$A_r f = g(r)f - f(r)g. \tag{14}$$

Then $\lambda g \in T_g$, where λ is complex, since, by Lemma 3.2,

$$A_r g = 0 = g(r)\lambda g - \lambda g(r)g.$$

For some $g \in R^0_{(1)}$, $T_g = \{g\}$ where $\{g\}$ is one-dimensional subspace generated by g.

Let

$$T = \bigcup_{g \in R^0_{(1)}} T_g$$

and let

$$\Sigma = \{g \in R^0_{(1)} : T_g \neq \{g\}\}.$$

We shall denote by $[T]$ the linear span of T closed in the norm topology and by $[T]_\sigma$ the linear span of T closed in $\sigma(R^*, R)$-topology.

Lemma 3.4.

(i) *T_g is a $\sigma(R^*, R)$-closed linear subspace in R^*, and $T_g \cap R^0_{(1)} = \{g\}$.*

(ii) *$T_g \cap T_{\lambda g} = \{g\}$, and $T_{g_1} \cap T_{g_2} = 0$ if $g_2 \notin \{g_1\}$.*

(iii) *If* $g \in \Sigma$, *then* $g \in N^0$ *where* N^0 *is the polar of* N *in* R^*.

(iv) *The quotient subspaces* $T_g/\{g\}$, *for* $g \in \Sigma$, *are linearly independent in the quotient space* $R^*/R^0_{(1)}$.

Proof. Let (f_α) be a directed set of elements in T_g converging to $f \in R^*$ in $\sigma(R^*, R)$-topology. Since, by Lemma 3.2(iii), A_r is continuous in $\sigma(R^*, R)$-topology, we have that $A_r f_\alpha \to A_r f$. But, by (14),

$$A_r f_\alpha = g(r) f_\alpha - f_\alpha(r) g$$

converges to $g(r)f - f(r)g$. Hence $A_r f = g(r)f - f(r)g$, so that $f \in T_g$.

If $f \in T_g \cap R^0_{(1)}$, then, by (11) and by (14), for every $r, r_1 \in R$

$$(A_r f)(r_1) = f([r, r_1]) = 0 = g(r) f(r_1) - f(r) g(r_1),$$

since $[r, r_1] \in R_{(1)}$. Hence Ker f = Ker g and therefore $f = tg$ where t is complex. Thus (i) is proved.

If $f \in T_{g_1} \cap T_{g_2}$, then for every $r \in R$

$$A_r f = g_1(r) f - f(r) g_1 = g_2(r) f - f(r) g_2.$$

Hence $g(r)f - f(r)g = 0$ where $g = g_1 - g_2$. Hence Ker f = Ker g and therefore $f = tg \in R^0_{(1)}$ where t is complex. By (i), there exist complex λ_1 and λ_2 such that $f = \lambda_1 g_1 = \lambda_2 g_2$. Hence $g_2 = (\lambda_1/\lambda_2) g_1$. Thus if $g_2 \notin \{g_1\}$, then $T_{g_1} \cap T_{g_2} = 0$. If $g_2 = \lambda g_1$, then $T_{g_1} \cap T_{g_2} = \{g_1\}$ and (ii) is proved.

Since $R_{(1)} \subseteq N$, we have that $N^0 \subseteq R^0_{(1)}$. Now suppose that $g \in R^0_{(1)}$ but $g \notin N^0$. Then there exists $n \in N$ such that $g(n) \neq 0$. By (14), for every $f \in T_g$ and for every $r \in R_{(1)}$ we have

$$(A_r f)(n) = f([r, n]) = 0 = g(r) f(n) - f(r) g(n) = -f(r) g(n).$$

Hence $f(r) = 0$ and $f \in R^0_{(1)}$. By (i), $f = tg$. Hence $T_g = \{g\}$ and $g \notin \Sigma$ so (iii) is proved.

Let $f \in T_g$, for $g \in \Sigma$, and let $\tilde{f}$ be its image in the quotient space $R^*/R^0_{(1)}$. Then, by (14), for every $r \in R$ we have that $(\widetilde{A_r f}) = g(r) \tilde{f}$ and the rest of the proof of (iv) is obvious.

Let $g \in R^0_{(1)}$. Put

$$T_g^\perp = T \setminus T_g = \bigcup_{g' \in R^0_{(1)} \setminus g} T_{g'}.$$

Let $(T_g^\perp)^0$ be the polar of $T_g^\perp$ in R. Put $R^g_{(1)} = (T_g^\perp)^0$. By $[T_g^\perp]_\sigma$ we shall denote the $\sigma(R^*, R)$-closed span of $T_g^\perp$ in R^*.

Theorem 3.5.

(i) *A solvable normed Lie algebra* R *belongs to* $\mathfrak{X}$ *if and only if* $[T]_\sigma = R^*$.

(ii) *Let* $R \in \mathfrak{X}$. *The following conditions are equivalent:*

(a) *there exists a closed commutative subalgebra G in R such that $G \cap R_{(1)}=0$ and that linear combinations of elements from G and $R_{(1)}$ are dense in R,*

(b) *there exists a $\sigma(R^*,R)$-closed linear subspace S in R^* such that $S \cap R^0_{(1)}=0$ and that $S \cap T_g$ has codimension 1 in T_g for every $g \in \Sigma$.*

(iii) *$R^g_{(1)} \neq 0$ if and only if $[T_g^{\perp}]_\sigma \neq R^*$; $R^g_{(1)}$ is a closed ideal in $R_{(1)}$ such that for every $r' \in R^g_{(1)}$ and for every $r \in R$*

$$[r,r']=g(r)r'.$$

$$R^{g_1}_{(1)} \cap R^{g_2}_{(1)}=0 \text{ if } g_1 \neq g_2, \text{ and if } g \notin \Sigma, \text{ then } R^g_{(1)}=0.$$

Proof. We shall consider R^* in $\sigma(R^*,R)$-topology. Then R is the dual space of R^*. Let $R \in \mathfrak{X}$. By definition we have that $R^0_{(1)} \subseteq T$. Let T^0 be the polar of T in R which consists of all $r \in R$ such that $f(r)=0$, for all $f \in T$. It follows from Lemma 3.3 that, if $r \in R_{(1)}$, then there exist $g \in R^0_{(1)}$ and $f \in T_g$ such that $f(r) \neq 0$. Hence $T^0 \cap R_{(1)}=0$. Now let $r \notin R_{(1)}$. Then there exists $g \in R^0_{(1)}$ such that $g(r) \neq 0$. Hence $r \notin T^0$. Thus $T^0=0$. If T^{00} is the bipolar of T in R^*, then $T^{00}=R^*$. But $T^{00}=[T]_\sigma$. Hence $[T]_\sigma=R^*$.

Now suppose that R is solvable and that $[T]_\sigma=R^*$. Let $f \in T_g$. If $r,r' \in \operatorname{Ker} f$ then, by (14),

$$f([r,r'])=g(r)f(r')-f(r)g(r')=0.$$

Hence $[r,r'] \in \operatorname{Ker} f$. Thus we obtain that $\operatorname{Ker} f$ is a subalgebra and hence $\operatorname{Ker} f \in S(R)$. Let $r \in \bigcap_{f \in T} \operatorname{Ker} f$. Since $[T]_\sigma=R^*$, we obtain that $f(r)=0$ for every $f \in R^*$. Hence $r=0$. Therefore

$$\bigcap_{M \in S(R)} M \subseteq \bigcap_{f \in T} \operatorname{Ker} f=0.$$

Hence $R \in \mathfrak{X}$ and (i) is proved.

Now let us prove that (b) follows from (a). Put $S=G^0$. Then S is a $\sigma(R^*,R)$-closed linear subspace in R^*. If $f \in S \cap R^0_{(1)}$, then $f|_G=0$ and $f|_{R_{(1)}}=0$. Hence $f=0$. Thus $S \cap R^0_{(1)}=0$. If $g \in \Sigma$, then $g \in R^0_{(1)}$ and hence $g \notin S$. Now let $f \in T_g$. Since $g \notin S$, there exists $r \in G$ such that $g(r) \neq 0$. Then, by (14),

$$g(r)f=f(r)g+A_rf. \tag{15}$$

But, by (11), $(A_rf)(r')=f([r,r'])=0$ for every $r' \in G$, since G is commutative. Hence $A_rf \in S$. Since $g(r) \neq 0$, it follows from (15) that $f \equiv tg \bmod S$ where $t=f(r)/g(r)$. Hence $S \cap T_g$ has codimension 1 in T_g.

Now let us prove that (a) follows from (b). Let S^0 be the polar of S in R. Put $G=S^0$. Then G is a linear subspace in R closed in the norm topology. Let $r_1,r_2 \in G$. Put $r_-=[r_1,r_2]$. Then $r_- \in R_{(1)}$. If $r_- \neq 0$, then, by Lemma 3.3, there exist functionals $g \in R^0_{(1)}$ and $f \in T_g$ such that $f(r_-) \neq 0$. Since $S \cap T_g$ has codimension 1 in T_g and since $g \notin S$, there exists a complex t such that $f_1=f-tg \in S \cap T_g$. Then $f_1(r_1)=f_1(r_2)=0$ and

$$f_1(r_-)=f(r_-)-tg(r_-)=f(r_-) \neq 0. \tag{16}$$

But, by (11) and by (14),

$$(A_{r_1}f_1)(r_2)=f_1([r_1,r_2])=f_1(r_-)=g(r_1)f_1(r_2)-f_1(r_1)g(r_2)=0$$

which contradicts (16). Hence $r_-=0$ and G is commutative.

Now let $r\in G\cap R_{(1)}$. If $r\neq 0$, then repeating the argument which preceded (16) we obtain that there exist functionals $g\in R^0_{(1)}$ and $f_1\in S\cap T_g$ such that $f_1(r)\neq 0$. But this contradicts the fact that $r\in G$ and that $f_1\in S=G^0$. Hence $G\cap R_{(1)}=0$.

Let L be the closed linear span of G and $R_{(1)}$, and let L^0 be the polar of L in R^*. Then $L^0=G^0\cap R^0_{(1)}$. Since G is the polar of S in R, we have that G^0 is the bipolar of S in R^* and hence $G^0=[S]_\sigma=S$. Therefore $L^0=S\cap R^0_{(1)}=0$. Hence $L=R$ which concludes the proof of (ii).

Let $(R^g_{(1)})^0$ be the polar of $R^g_{(1)}$ in R^* and let $(T^\perp_g)^{00}$ be the bipolar of $T^\perp_g$ in R^*. Then

$$(R^g_{(1)})^0=(T^\perp_g)^{00}=[T^\perp_g]_\sigma.$$

If $R^g_{(1)}=0$, then $(R^g_{(1)})^0=R^*=[T^\perp_g]_\sigma$. If, on the other hand, $[T^\perp_g]_\sigma=R^*$, then $(R^g_{(1)})^0=R^*$ and hence $R^g_{(1)}=0$.

We have that $R^0_{(1)}\subset T$. Since $T_g\cap R^0_{(1)}=\{g\}$, we have that

$$R^0_{(1)}\backslash\{g\}\subset T\backslash T_g=T^\perp_g. \tag{17}$$

Hence $R^g_{(1)}=(T^\perp_g)^0\subset(R^0_{(1)}\backslash\{g\})^0$. But since the closure of $R^0_{(1)}\backslash\{g\}$ in the norm topology is $R^0_{(1)}$, we obtain that

$$(R^0_{(1)}\backslash\{g\})^0=(R^0_{(1)})^0=R_{(1)}.$$

Hence $R^g_{(1)}\subseteq R_{(1)}$.

Now let $R^g_{(1)}\neq 0$, let $r'\in R^g_{(1)}$ and let $r\in R$. Put

$$r_1=[r,r']-g(r)r'.$$

Then $r_1\in R_{(1)}$ and hence $g'(r_1)=0$ for every $g'\in R^0_{(1)}$. For every functional $f\in T_g$, by (14),

$$f(r_1)=f([r,r'])-g(r)f(r')=-f(r)g(r')=0,$$

since $g(r')=0$. Let $f\in T_{g'}$, where $g'\in R^0_{(1)}$ and $g'\neq g$, and let $f\neq tg'$. Then $f\in T^\perp_g$ and hence $f(r')=0$. Therefore, by (14),

$$f(r_1)=f([r,r'])-g(r)f(r')=f([r,r'])=g'(r)f(r')-f(r)g'(r')=0,$$

since $g'(r')=0$. Hence $f(r_1)=0$ for every $f\in T$. Therefore, by (i), $r_1=0$ and

$$[r,r']=g(r)r'. \tag{18}$$

If $g_1\neq g_2$, then it follows from (18) that $R^{g_1}_{(1)}\cap R^{g_2}_{(1)}=0$.

If $g \notin \Sigma$, then $T_g = \{g\}$. Since the closure of $R^0_{(1)} \backslash \{g\}$ in the $\sigma(R^*, R)$-topology is $R^0_{(1)}$, we have, by (17), that

$$T_g = \{g\} \subseteq R^0_{(1)} = [R^0_{(1)} \backslash \{g\}]_\sigma \subseteq [T_g^\perp]_\sigma.$$

Since $T_g \subseteq [T_g^\perp]_\sigma$, we get that $[T_g^\perp]_\sigma = [T]_\sigma$. Hence, by (i), $[T_g^\perp]_\sigma = R^*$ and therefore $R^g_{(1)} = 0$ which concludes the proof of the theorem.

The case when $\dim R < \infty$ was considered in **[4]**. It was proved there that R is the direct sum of G and $R_{(1)}$, and that $R_{(1)}$ is the direct sum of $R^{g_i}_{(1)}$, where $\Sigma = \{g_i\}_{i=1}^n$ is a finite set. We shall consider the case when $\dim R = \infty$ and Σ is a finite set later on but now we shall consider an example when $\dim R = \infty$ and Σ is an infinite set. We shall show that, although $\dim(R/R_{(1)}) = 2$ in the example, *there does not exist a commutative subalgebra G such that $G \cap R_{(1)} = 0$ and that linear combinations of elements from G and $R_{(1)}$ are dense in R*. We shall also prove *that $R^g_{(1)} = 0$ for all $g \in \Sigma$*.

Example. Let R be a Hilbert space with a basis $(e_i)_{i=-1}^\infty$, let N be the subspace generated by $(e_i)_{i=1}^\infty$ and let A be the bounded operator on R such that

$$Ae_{-1} = Ae_0 = 0 \quad \text{and} \quad Ae_i = a_i e_i + e_{i+1} \quad \text{for} \quad 1 \leqq i, \tag{19}$$

where a_i are complex numbers such that $a_i \neq a_j$, $a_i \neq 0$ and $\sup_i |a_i| < \frac{1}{2}$. Put

$$[x, y] = 0, \text{ for } x, y \in N;\ [e_0, e_0] = [e_{-1}, e_{-1}] = 0;$$

$$[e_0, e_i] = Ae_i,\ [e_{-1}, e_i] = A^2 e_i, \quad \text{for} \quad 1 \leqq i, \tag{20}$$

and $[e_{-1}, e_0] = e_1$.

It is easy to check that R is a Lie algebra and that

$$[x, y] = (x_{-1} y_0 - y_{-1} x_0) e_1 + x_{-1} A^2 y - y_{-1} A^2 x + x_0 A y - y_0 A x, \tag{21}$$

for $x = \Sigma_{i=-1}^\infty x_i e_i$ and $y = \sum_{i=-1}^\infty y_i e_i$. Then

$$\|[x, y]\| \leqq 2(\|e_1\| + \|A^2\| + \|A\|) \|x\| \, \|y\|.$$

Hence R is a normed Lie algebra. By (20), N is a commutative ideal in R and $R_{(1)} \subseteq N$. Thus R is solvable, N is the nil-radical of R and $R_{(2)} = 0$. If $R_{(1)} \neq N$, then there exists an element $Z = \sum_{i=1}^\infty Z_i e_i$ in N such that for every $i \geqq 0$

$$(Z, [e_0, e_i]) = (Z, [e_{-1}, e_i]) = 0.$$

Since $[e_0, e_i] = a_i e_i + e_{i+1}$, for $i \geqq 1$, we obtain that $Z_i \bar{a}_i + Z_{i+1} = 0$. Since $[e_{-1}, e_0] = e_1$, we obtain that $Z_1 = 0$ and hence all $Z_i = 0$, for $i \geqq 1$. Thus $Z = 0$ and therefore $R_{(1)} = N$.

For every $f \in R^*$ there exists an element y_f in R such that $f(x) = (x, y_f)$ for every

$x \in R$. For $r \in R$ put $f_r = A_r f$. Then

$$(A_r f)(x) = f([r, x]) = ([r, x],\ y_f) = (x, y_{f_r}).$$

From (21) it follows that

$$([r, x], y_f) = (r_{-1}x_0 - x_{-1}r_0)(e_1, y_f) + r_{-1}(A^2x, y_f)$$
$$- x_{-1}(A^2r, y_f) + r_0(Ax, y_f) - x_0(Ar, y_f).$$

Therefore from the two preceding formulae we obtain that

$$y_{f_r} = (\bar{r}_{-1}(A^*)^2 + \bar{r}_0 A^*)y_f - (y_f, r_0e_1 + A^2r)e_{-1} + (y_f, r_{-1}e_1 - Ar)e_0. \tag{22}$$

For every $g \in R^0_{(1)}$ we have that $y_g = \mu e_{-1} + \lambda e_0$. If $f \in T_g$, then, by (14), we have that

$$y_{f_r} = \overline{g(r)}y_f - \overline{f(r)}y_g = (y_g, r)y_f - (y_f, r)y_g. \tag{23}$$

Let $r \in N$. Then $r_{-1} = r_0 = 0$. From (22) and from (23) we get that

$$y_{f_r} = -(y_f,\ A^2r)e_{-1} - (y_f, Ar)e_0 = -(y_f, r)(\mu e_{-1} + \lambda e_0),$$

since $(y_g, r) = 0$. Hence

$$(y_f, Ar) = \lambda(y_f, r) \quad \text{and} \quad (y_f, A^2r) = \mu(y_f, r). \tag{24}$$

Let $y_f = y_{-1}e_{-1} + y_0e_0 + \hat{y}_f$ where $\hat{y}_f \in N$. Since N is invariant under A, we obtain from (24) that for every $r \in N$

$$(\hat{y}_f, A_r) = \lambda(\hat{y}_f, r) \quad \text{and} \quad (\hat{y}_f, A^2r) = \mu(\hat{y}_f, r).$$

Since N is invariant under A^*, we get that $A^*\hat{y}_f = \lambda\hat{y}_f$ and that $(A^*)^2\hat{y}_f = \mu\hat{y}_f$. Hence $\mu = \lambda^2$.

It follows from (19) that

$$A^*e_1 = \bar{a}_1e_1 \quad \text{and} \quad A^*e_i = \bar{a}_ie_i + e_{i-1}, \quad \text{for} \quad i \geqq 2.$$

If $\hat{y}_f = \sum_{i=1}^{\infty} y_ie_i$, then, since $A^*\hat{y}_f = \lambda\hat{y}_f$, we get that

$$(\bar{a}_i - \lambda)y_i + y_{i+1} = 0,$$

for $i \geqq 1$. Hence we obtain that for $i \geqq 2$

$$y_i = y_1 \prod_{j=1}^{i-1} (\lambda - \bar{a}_j). \tag{25}$$

Now let $r_- \neq 0$ and $r_0 \neq 0$ in (22) and in (23). Then after some calculations we obtain from (22) and from (23) that

$$\lambda(\lambda y_0 - y_{-1}) = (\hat{y}_f, e_1) = y_1. \tag{26}$$

But the element y_f, of which coordinates y_i satisfy (25) and (26), belongs to R if and only if

$$\sum_{i=2}^{\infty} |y_i|^2 = |y_1|^2 \sum_{i=2}^{\infty} \left(\prod_{j=1}^{i-1} |\lambda - \bar{a}_j|^2 \right) < \infty. \tag{27}$$

From all these considerations it follows that

(i) $\Sigma = \{g(\lambda) \in R^0_{(1)}: y_{g(\lambda)} = \lambda^2 e_{-1} + \lambda e_0$, where $\lambda \neq 0$ and satisfies (27)$\}$,

(ii) any functional f such that $y_f = \sum_{i=-1}^{\infty} y_i e_i$, where y_i satisfy (25) and (26), belongs to $T_{g(\lambda)}$.

It follows from (i) and (ii) that $\dim T_{g(\lambda)} = 2$. Since $\sup_i |a_i| \leqq \frac{1}{2}$, then (27) uniformly converges for all $|\lambda| \leqq q < \frac{1}{2}$. Now suppose that there exists a $\sigma(R^*, R)$-closed subspace S in R^* such that $S \cap R^0_{(1)} = 0$ and that $S \cap T_{g(\lambda)}$ has codimension 1 in $T_{g(\lambda)}$ for every $g(\lambda) \in \Sigma$. Then S is a Hilbert subspace in R^* and, for every $g(\lambda) \in \Sigma$, there exists a unique element $y(\lambda) = \sum_{i=-1}^{\infty} y_i(\lambda) e_i$ such that $y(\lambda) \in S \cap T_{g(\lambda)}$, that $y_1(\lambda) = 1$ and that $y_i(\lambda)$ satisfy (25) and (26). Then, by (26),

$$1/\lambda = \lambda y_0(\lambda) - y_{-1}(\lambda). \tag{28}$$

Now let $S^\perp$ be the subspace orthogonal to S. Since $S \cap R^0_{(1)} = 0$, it is easy to see that $\dim S^\perp \geqq 2$. Suppose that $S^\perp \cap N \neq 0$ and let $Z \in S^\perp \cap N$. For every $\lambda = \bar{a}_j$ the series (27) converges and it follows from (25) that the coordinates $y_i(\bar{a}_j)$ of the corresponding elements $y(\bar{a}_j)$ satisfy the following conditions:

$$y_i(\bar{a}_j) \neq 0, \quad \text{if} \quad 1 \leqq i \leqq j, \quad \text{and} \quad y_i(\bar{a}_j) = 0, \quad \text{if} \quad j < i. \tag{29}$$

Since $(Z, y(\bar{a}_j)) = 0$ for every $\bar{a}_j$, we obtain easily that $Z = 0$. Hence $S^\perp \cap N = 0$. Then there exist elements Z^1 and Z^2 in $S^\perp$ such that

$$Z^1 = e_{-1} + \sum_{i=1}^{\infty} Z_i^1 e_i, \; Z^2 = e_0 + \sum_{i=1}^{\infty} Z_i^2 e_i.$$

Since $(y(\lambda), Z^K) = 0$ for $K = 1, 2$, we get that

$$y_{-1}(\lambda) = -\sum_{i=1}^{\infty} \bar{Z}_i^1 y_i(\lambda), \; y_0(\lambda) = -\sum_{i=1}^{\infty} \bar{Z}_i^2 y_i(\lambda)$$

for all $y(\lambda) \in S$. By (28),

$$|1/\lambda| = \left| \sum_{i=1}^{\infty} (\bar{Z}_i^1 - \lambda \bar{Z}_i^2) y_i(\lambda) \right| \leqq \left(\sum_{i=1}^{\infty} |\bar{Z}_i^1 - \lambda \bar{Z}_i^2|^2 \right)^{1/2} \left(\sum_{i=1}^{\infty} |y_i(\lambda)|^2 \right)^{1/2}.$$

Earlier we observed that for all $|\lambda|\leqq q<\frac{1}{2}$ (27) converges uniformly. Hence the expression on the right-hand side of the inequality above is bounded for all $|\lambda|<q$. But $1/\lambda\to\infty$. This contradiction shows that S does not exist. Hence, by Theorem 3.5(ii), there does not exist a commutative subalgebra G such that $G\cap R_{(1)}=0$ and that linear combinations of elements from G and $R_{(1)}$ are dense in R.

Now we shall prove that $R_{(1)}^g=0$ for all $g\in\Sigma$. Let $g(\lambda_0)\in\Sigma$. By definition,

$$T_{g(\lambda_0)}^{\perp}=\bigcup_{\substack{g(\lambda)\in\Sigma\\ g(\lambda)\neq g(\lambda_0)}} T_g.$$

It follows from (17) that $R_{(1)}^0\subset[T_{g(\lambda_0)}^{\perp}]_\sigma$. Hence $[T_{g(\lambda_0)}^{\perp}]_\sigma$ contains e_{-1}, e_0 and all $y\in T_{g(\lambda)}$, for $\lambda\neq\lambda_0$. Suppose that $[T_{g(\lambda_0)}^{\perp}]_\sigma\neq R^*$. Then there exists $Z\in R^*$ which is orthogonal to $[T_{g(\lambda_0)}^{\perp}]_\sigma$. Since e_{-1} and e_0 belong to $[T_{g(\lambda_0)}^{\perp}]_\sigma$, we have that $Z_{-1}=Z_0=0$, so that $Z\in N$.

Let $\lambda_0\neq\bar{a}_j$. Then, since all $g(\bar{a}_j)\in\Sigma$, we get that $(Z,y(\bar{a}_j))=0$ for every a_j. Using (29) as above we obtain that $Z=0$ and, hence, that $[T_{g(\lambda_0)}^{\perp}]_\sigma=R^*$. Hence, by Theorem 3.5(iii), $R_{(1)}^{g(\lambda_0)}=0$.

Now let $\lambda_0=\bar{a}_j$. Then $(Z,y(\bar{a}_i))=0$ for every $a_i\neq a_j$. Using (29) we obtained by induction that

$$Z_i=0,\quad\text{for}\quad i=1,\dots,j-1;\quad\text{and that}\quad Z_i=Z_j\prod_{K=j+1}^{i}(a_j-a_K)^{-1},\quad\text{for}\quad i\geqq j+1.$$

Taking into account that $\sup_i|a_i|\leqq\frac{1}{2}$ we get that $|a_j-a_k|\leqq 1$ and therefore $|Z_i|\geqq|Z_j|$. Hence the element Z does not belong to N. Therefore $[T_{g(\bar{a}_j)}^{\perp}]_\sigma=R^*$ and, by Theorem 3.5(iii), $R_{(1)}^{g(\bar{a}_j)}=0$.

Thus in the example $R_{(1)}^g=0$, for every $g\in\Sigma$, and, although $\dim(R/R_{(1)})=2$, Σ is infinite as was shown in (i). In the theorem below we shall consider the case when Σ is finite.

Theorem 3.6. *Let $R\in\mathfrak{X}$ and let $\Sigma=\{g_i\}_{i=1}^n$ be a finite set. Then*

(i) *there exists a finite-dimensional commutative subalgebra Γ in R such that $\dim\Gamma\leqq n$ and that R is the direct sum of Γ and the nil-radical N;*

(ii) *N is the direct sum of $R_{(1)}$ and the centre Z, and $R_{(1)}$ is the direct sum of $R_{(1)}^{g_i}$, for $i=1,\dots,n$.*

Proof. Let S_{g_i}, for $i=1,\dots,n$, be $\sigma(R^*,R)$-closed subspaces in T_{g_i} of codimension 1 such that $T_{g_i}=S_{g_i}\dot{+}\{g_i\}$. First we shall prove that if a directed set $f^{(\alpha)}+g^{(\alpha)}$, where $g^{(\alpha)}\in R_{(1)}^0$, $f^{(\alpha)}=\sum_{i=1}^n f_i^{(\alpha)}$ and $f_i^{(\alpha)}\in S_{g_i}$, converges to an element from R^* in $\sigma(R^*,R)$-topology, then the directed set $g^{(\alpha)}$ and all directed sets $(f_i^{(\alpha)})_{i=1}^n$ converge to some elements from R^*.

Suppose that there exist directed sets $f^{(\alpha)}+g^{(\alpha)}$ which converge to elements from R^* but such that at least one of the corresponding directed sets $f_i^{(\alpha)}$ does not converge. For every such set let $p(f^{(\alpha)},g^{(\alpha)})$ be the number of the sets $f_i^{(\alpha)}$ which do not converge and let p be the smallest of all $p(f^{(\alpha)},g^{(\alpha)})$. Then $1\leqq p\leqq n$. Suppose that $p>1$. Let us choose one of the directed sets $f^{(\alpha)}+g^{(\alpha)}$ which converges to h with exactly p sets $(f_{i_j}^{(\alpha)})_{j=1}^{p}$ which do not converge. Then for every $r\in R$, by Lemma 3.2 and by (14),

$$A_r(f^{(\alpha)}+g^{(\alpha)})=A_rf^{(\alpha)}=\sum_{j=1}^{p}(g_{i_j}(r)f_{i_j}^{(\alpha)}-f_{i_j}^{(\alpha)}(r)g_{i_j})$$

converges to A_rh. Hence the directed set

$$g_{i_p}(r)(f^{(\alpha)}+g^{(\alpha)})-A_r(f^{(\alpha)}+g^{(\alpha)})=g_{i_p}(r)g^{(\alpha)}+\sum_{j=1}^{p}f_{i_j}^{(\alpha)}(r)g_{i_j}$$

$$+\sum_{j=1}^{p-1}(g_{i_p}(r)-g_{i_j}(r))f_{i_j}^{(\alpha)}$$

converges to $g_{i_p}(r)h-A_rh$. Put $\tilde{f}^{(\alpha)}=\sum_{j=1}^{p-1}\tilde{f}_{i_j}^{(\alpha)}$ and

$$\tilde{g}^{(\alpha)}=g_{i_p}(r)g^{(\alpha)}+\sum_{j=1}^{p}f_{i_j}^{(\alpha)}(r)g_{i_j},$$

where $\tilde{f}_{i_j}^{(\alpha)}=(g_{i_p}(r)-g_{i_j}(r))f_{i_j}^{(\alpha)}$. Then $\tilde{g}^{(\alpha)}\in R_{(1)}^{0}$ and the directed set $\tilde{f}^{(\alpha)}+\tilde{g}^{(\alpha)}$ converges to $g_{i_p}(r)h-A_rh$. Since all functionals $(g_{i_j})_{j=1}^{p}$ are different, we can choose such r that $g_{i_p}(r)-g_{i_1}(r)\neq 0$. Then at least the directed set $\tilde{f}_{i_1}^{(\alpha)}$ does not converge. Hence $1\leqq p(\tilde{f}^{(\alpha)},\tilde{g}^{(\alpha)})\leqq p-1$ which contradicts the assumption that $p>1$ is the smallest of such numbers.

Now suppose that $p=1$. Then there exist directed sets $f_i^{(\alpha)}\in S_{g_i}$ and $g^{(\alpha)}\in R_{(1)}^{0}$ such that the directed set $f_i^{(\alpha)}+g^{(\alpha)}$ converges to an element $h\in R^*$ and that the directed set $f_i^{(\alpha)}$ does not converge. Since S_{g_i} is $\sigma(R^*,R)$-closed in R^* and since $g_i\notin S_{g_i}$, there exists $r\in R$ such that $g_i(r)=1$ and that $f(r)=0$ for all $f\in S_{g_i}$. Then, since all $f_i^{(\alpha)}\in S_{g_i}$, we obtain from Lemma 3.2 and from (14) that the directed set

$$A_r(f_i^{(\alpha)}+g^{(\alpha)})=A_rf_i^{(\alpha)}=g_i(r)f_i^{(\alpha)}-f_i^{(\alpha)}(r)g_i=f_i^{(\alpha)}$$

converges to A_rh. This contradiction shows that $p\neq 1$.

Thus from all these considerations we obtain that, if a directed set $\sum_{i=1}^{n}f_i^{(\alpha)}+g^{(\alpha)}$, where $f_i^{(\alpha)}\in S_{g_i}$ and $g^{(\alpha)}\in R_{(1)}^{0}$, converges to an element h in R^*, then all the directed sets $f_i^{(\alpha)}$ converge to elements $h_i\in S_{g_i}$ and, hence, the directed set $g^{(\alpha)}$ converges to an element g in $R_{(1)}^{0}$.

From this fact, from Lemma 3.4 and from Theorem 3.5(i) it follows that R^* is the direct sum of $R_{(1)}^{0}$ and S_{g_i}, for $i=1,\dots,n$.

Put $S=S_{g_1}\dotplus\cdots\dotplus S_{g_n}$. Then S is $\sigma(R^*,R)$-closed, $S\cap R_{(1)}^{0}=0$ and $S\cap T_{g_i}=S_{g_i}$ has codimension 1 in T_{g_i}. Hence, by Theorem 3.5(ii), $G=S^0$ is a commutative subalgebra of R such that $G\cap R_{(1)}=0$ and that linear combinations of elements from G and $R_{(1)}$ are dense in R.

For every $i=1,\dots,n$ we have that

$$T^{\perp}_{g_i} = R^0_{(1)} \backslash \{g_i\} \dot{+} \sum_{\substack{k=1\\k\neq i}}^{n} \dot{+} S_{g_k}.$$

Hence

$$[T^{\perp}_{g_i}]_\sigma = R^0_{(1)} \dot{+} \sum_{\substack{k=1\\k\neq i}}^{n} \dot{+} S_{g_k} \neq R^*. \tag{30}$$

Hence, by Theorem 3.5(iii), $R^{g_i}_{(1)} \neq 0$. Let L be the closed linear span of all $R^{g_i}_{(1)}$, for $i=1,\ldots,n$, and let L^0 be its polar in R^*. Since $R^{g_i}_{(1)} = (T^{\perp}_{g_i})^0$, we have that

$$L^0 = \bigcap_{i=1}^{n} (R^{g_i}_{(1)})^0 = \bigcap_{i=1}^{n} (T^{\perp}_{g_i})^{00} = \bigcap_{i=1}^{n} [T^{\perp}_{g_i}]_\sigma.$$

It follows from (30) that $L^0 = R^0_{(1)}$. Hence $L = R_{(1)}$. Thus $R_{(1)}$ is the closed linear span of $R^{g_i}_{(1)}$, for $i=1,\ldots,n$.

Now suppose that there exist sequences $r^{(k)} = \sum_{i=1}^n r_i^{(k)} + s^{(k)}$, where $r_i^{(k)} \in R^{g_i}_{(1)}$ and $s^{(k)} \in G$, which converge to elements from R but some of the sequences $r_i^{(k)}$ do not converge. For every such sequence let $p(r^{(k)})$ be the number of the sequences $r_i^{(k)}$ which do not converge and let p be the smallest of all $p(r^{(k)})$.

Suppose $p > 1$. Then there exists a sequence $r^{(k)} = \sum_{j=1}^p r_{i_j}^{(k)} + s^{(k)}$, where $r_{i_j}^{(k)} \in R^{g_{i_j}}_{(1)}$ and $s^{(k)} \in G$, which converges to an element r and none of the sequences $r_{i_j}^{(k)}$, for $j=1,\ldots,p$, converge. Then, by (18), for every $r' \in G$ the sequence

$$[r', r^{(k)}] = \sum_{j=1}^{p} g_{i_j}(r') r_{i_j}^{(k)}$$

converges to $[r', r] \in R_{(1)}$. Hence the sequence

$$\tilde{r}^{(k)} = [r', r^{(k)}] - g_{i_p}(r') r^{(k)} = \sum_{j=1}^{p-1} (g_{i_j}(r') - g_{i_p}(r')) r_{i_j}^{(k)} - g_{i_p}(r') s^{(k)}$$

converges to $[r', r] - g_{i_p}(r') r$. Since all functionals g_{i_j} are different, there exists $r' \in G$ such that at least $g_{i_1}(r') - g_{i_p}(r') \neq 0$. Hence $1 \leqq p(\tilde{r}^{(k)}) \leqq p-1$ which contradicts the assumption that $p > 1$ is the smallest of such numbers.

Let $p=1$ and let a sequence $r^{(k)} = r_i^{(k)} + s^{(k)}$, where $r_i^{(k)} \in R^{g_i}_{(1)}$ and $s^{(k)} \in G$, converges to $r \in R$ and let the sequence $r_i^{(k)}$ not converge. Then, by (18), for every $r' \in G$ the sequence

$$[r', r^{(k)}] = [r', r_i^{(k)}] = g_i(r') r_i^{(k)}$$

converges to $[r', r]$. Choosing r' such that $g_i(r') \neq 0$ we get that $r_i^{(k)}$ converges which contradicts the assumption that $r_i^{(k)}$ does not converge.

Therefore we obtain that, if a sequence $r^{(k)} = \sum_{i=1}^n r_i^{(k)} + s^{(k)}$, where $r_i^{(k)} \in R^{g_i}_{(1)}$ and $s^{(k)} \in G$, converges, then all sequences $r_i^{(k)}$ converge to elements in $R^{g_i}_{(1)}$ and, hence, $s^{(k)}$ converges to an element in G. Hence R is the direct sum of $R_{(1)}$ and G, and $R_{(1)}$ is the direct sum of $R^{g_i}_{(1)}$, for $i=1,\ldots,n$.

Now let $Z=(\bigcap_{i=1}^{n}\operatorname{Ker} g_i)\cap G$. If $r=\sum_{i=1}^{n} r_i$, where $r_i\in R_{(1)}^{g_i}$, then for every $z\in Z$, by (18),

$$[z,r]=\sum_{i=1}^{n} g_i(z)r_i=0.$$

Since $Z\subset G$ and since G is commutative, we obtain that Z is the centre of R. Z is closed and has finite codimension in G. Therefore there exists a finite commutative subalgebra Γ in G such that $G=\Gamma\dot{+}Z$ and that $\dim\Gamma\leqq n$. It is easy to see that $Z\dot{+}R_{(1)}$ is the nil-radical in R which concludes the proof of the theorem.

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