Canad. Math. Bull. Vol. **57** (3), 2014 pp. 640–647
http://dx.doi.org/10.4153/CMB-2013-031-0

Equilateral Sets and a Schütte Theorem for the 4-norm

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Abstract. A well-known theorem of Schütte (1963) gives a sharp lower bound for the ratio of the maximum and minimum distances between $n+2$ points in n-dimensional Euclidean space. In this note we adapt Bárány's elegant proof (1994) of this theorem to the space ℓ_4^n. This gives a new proof that the largest cardinality of an equilateral set in ℓ_4^n is $n+1$ and gives a constructive bound for an interval $(4-\varepsilon_n, 4+\varepsilon_n)$ of values of p close to 4 for which it is known that the largest cardinality of an equilateral set in ℓ_p^n is $n+1$.

1 Introduction

A subset S of a normed space X with norm $\|\cdot\|$ is called *equilateral* if for some $\lambda > 0$, $\|\boldsymbol{x}-\boldsymbol{y}\| = \lambda$ for all distinct $\boldsymbol{x}, \boldsymbol{y} \in S$. Denote the largest cardinality of an equilateral set in a finite-dimensional normed space X by $e(X)$.

For $p \geq 1$ define the p-norm of a vector $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ as

$$\|\boldsymbol{x}\|_p = \|(x_1, \dots, x_n)\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}.$$

When dealing with a sequence $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m \in \mathbb{R}^n$ of vectors, we denote the coordinates of $\boldsymbol{x}_i$ as $(x_{i,1}, \dots, x_{i,n})$. Denote the normed space $\mathbb{R}^n$ with norm $\|\cdot\|_p$ by ℓ_p^n. It is not difficult to find examples of equilateral sets showing that $e(\ell_p^n) \geq n+1$. It is a simple exercise in linear algebra to show that $e(\ell_2^n) \leq n+1$. Kusner [4] asks if the same is true for ℓ_p^n, where $p > 1$. For the current best upper bounds on $e(\ell_p^n)$, see [1]. We next mention only the results that decide various cases of Kusner's question. A compactness argument gives for each $n \in \mathbb{N}$ the existence of $\varepsilon_n > 0$ such that $p \in (2-\varepsilon_n, 2+\varepsilon_n)$ implies $e(\ell_p^n) = n+1$. However, this argument gives no information on ε_n. As observed by C. Smyth (unpublished manuscript; see also [8]), the following theorem of Schütte [6] can be used to give an explicit lower bound to ε_n in terms of n.

Theorem 1.1 (Schütte [6]) *Let S be a set of at least $n+2$ points in ℓ_2^n. Then*

$$\frac{\max_{\boldsymbol{x},\boldsymbol{y}\in S} \|\boldsymbol{x}-\boldsymbol{y}\|_2}{\min_{\boldsymbol{x},\boldsymbol{y}\in S, \boldsymbol{x}\neq\boldsymbol{y}} \|\boldsymbol{x}-\boldsymbol{y}\|_2} \geq \begin{cases} \left(1+\frac{2}{n}\right)^{1/2} & \text{if } n \text{ is even,} \\ \left(1+\frac{2}{n-(n+2)^{-1}}\right)^{1/2} & \text{if } n \text{ is odd.} \end{cases}$$

The lower bounds in this theorem are sharp.

Received by the editors April 26, 2013; revised August 2, 2013.
Published electronically December 4, 2013.
AMS subject classification: **46B20**, 52A21, 52C17.

Corollary 1.2 (Smyth) *If*

$$|p-2| < \frac{2\log(1+2/n)}{\log(n+2)} = \frac{4(1+o(1))}{n\log n},$$

then the largest cardinality of an equilateral set in ℓ_p^n is $e(\ell_p^n) = n+1$.

The dependence of $\varepsilon_n = \frac{4(1+o(1))}{n\log n}$ on n is necessary, since $e(\ell_p^n) > n+1$ if $1 \le p < 2 - \frac{1+o(1)}{(\ln 2)n}$ (see [9]). (These are the only known cases where the answer to Kusner's question is negative.)

There is also a linear algebra proof in [9] that $e(\ell_4^n) = n+1$. As in the case of $p = 2$, compactness gives an ineffective $\varepsilon_n > 0$ such that if $p \in (4-\varepsilon_n, 4+\varepsilon_n)$, then $e(\ell_p^n) = n+1$. The question arises whether Schütte's theorem can be adapted to ℓ_4^n, so that a conclusion similar to Corollary 1.2 can be made for p close to 4. Proofs of Schütte's theorem have been given by Schütte [6], Schoenberg [5], Seidel [7], and Bárány [2]. It is the purpose of this note to show that Bárány's simple and elegant proof of Schütte's theorem can indeed be adapted.

Theorem 1.3 *Let S be a set of at least $n+2$ points in ℓ_4^n. Then*

$$\frac{\max_{\boldsymbol{x},\boldsymbol{y}\in S}\|\boldsymbol{x}-\boldsymbol{y}\|_4}{\min_{\boldsymbol{x},\boldsymbol{y}\in S,\boldsymbol{x}\neq\boldsymbol{y}}\|\boldsymbol{x}-\boldsymbol{y}\|_4} \ge \begin{cases} \left(1+\frac{2}{n}\right)^{1/4} & \text{if } n \text{ is even,} \\ \left(1+\frac{2}{n-(n+2)^{-1}}\right)^{1/4} & \text{if } n \text{ is odd.}\end{cases}$$

Corollary 1.4 *If*

$$|p-4| < \frac{4\log(1+2/n)}{\log(n+2)} = \frac{8(1+o(1))}{n\log n},$$

then the largest cardinality of an equilateral set in ℓ_p^n is $e(\ell_p^n) = n+1$.

We do not know whether the lower bounds in Theorem 1.3 are sharp. The following is the best upper bound that we can show.

Proposition 1.5 *There exists a set S of $n+2$ points in ℓ_4^n such that*

$$\frac{\max_{\boldsymbol{x},\boldsymbol{y}\in S}\|\boldsymbol{x}-\boldsymbol{y}\|_4}{\min_{\boldsymbol{x},\boldsymbol{y}\in S,\boldsymbol{x}\neq\boldsymbol{y}}\|\boldsymbol{x}-\boldsymbol{y}\|_4} = 1+\sqrt{\frac{2}{n}}+O(n^{-3/4}).$$

Unfortunately, this bound is far from the lower bound of $1+\frac{1}{2n}+O(n^{-2})$ given by Theorem 1.3.

2 Proofs

Proof of Theorem 1.3 Consider any $\boldsymbol{x}_1,\dots,\boldsymbol{x}_{n+2} \in \mathbb{R}^n$ and let

$$\mu = \min_{i\neq j}\|\boldsymbol{x}_i-\boldsymbol{x}_j\|_4 \quad \text{and} \quad M = \max_{i,j}\|\boldsymbol{x}_i-\boldsymbol{x}_j\|_4.$$

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By Radon's theorem [3] there is a partition $A \cup B$ of $\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_{n+2}\}$ such that the convex hulls of A and B intersect. Without loss of generality we may translate the points so that $\boldsymbol{o}$ lies in both convex hulls. Write $A = \{\boldsymbol{a}_1, \dots, \boldsymbol{a}_K\}$ and $B = \{\boldsymbol{b}_1, \dots, \boldsymbol{b}_L\}$, where $K + L = n + 2$ and $K, L \geq 1$. Then there exist $\alpha_1, \dots, \alpha_K, \beta_1, \dots, \beta_L \geq 0$ such that

$$\begin{aligned} \sum_{i=1}^{K} \alpha_i = 1, \quad \sum_{i=1}^{K} \alpha_i \boldsymbol{a}_i = \boldsymbol{o}, \\ \sum_{j=1}^{L} \beta_j = 1, \quad \sum_{j=1}^{L} \beta_j \boldsymbol{b}_j = \boldsymbol{o}. \end{aligned} \tag{2.1}$$

Also, for all $i \in [K]$ and $j \in [L]$,

$$\|\boldsymbol{a}_i - \boldsymbol{a}_j\|_4^4 \leq M^4 \quad \text{whenever } i \neq j, \tag{2.2}$$

$$\|\boldsymbol{b}_i - \boldsymbol{b}_j\|_4^4 \leq M^4 \quad \text{whenever } i \neq j, \tag{2.3}$$

$$\|\boldsymbol{a}_i - \boldsymbol{b}_j\|_4^4 \geq \mu^4. \tag{2.4}$$

Apply the operation $\sum_{i=1}^{K} \alpha_i \sum_{\substack{j=1 \\ j \neq i}}^{K} \alpha_j$ to both sides of inequality (2.2):

$$\begin{aligned}
&\left(1 - \sum_{i=1}^{K} \alpha_i^2\right) M^4 \\
&= \sum_{i=1}^{K} \alpha_i (1 - \alpha_i) M^4 = \sum_{i=1}^{K} \alpha_i \sum_{\substack{j=1 \\ j \neq i}}^{K} \alpha_j M^4 \\
&\geq \sum_{i=1}^{K} \alpha_i \sum_{j=1}^{K} \alpha_j \sum_{m=1}^{n} (a_{i,m} - a_{j,m})^4 \\
&= \sum_{m=1}^{n} \sum_{i=1}^{K} \sum_{j=1}^{K} \alpha_i \alpha_j (a_{i,m}^4 - 4a_{i,m}^3 a_{j,m} + 6a_{i,m}^2 a_{j,m}^2 - 4a_{i,m} a_{j,m}^3 + a_{j,m}^4) \\
&= \sum_{m=1}^{n} \sum_{i=1}^{K} \alpha_i a_{i,m}^4 - 4 \sum_{m=1}^{n} \left(\sum_{i=1}^{K} \alpha_i a_{i,m}^3 \right) \left(\sum_{j=1}^{K} \alpha_j a_{j,m} \right) \\
&\quad + 6 \sum_{m=1}^{n} \left(\sum_{i=1}^{K} \alpha_i a_{i,m}^2 \right) \left(\sum_{j=1}^{K} \alpha_j a_{j,m}^2 \right) - 4 \sum_{m=1}^{n} \left(\sum_{i=1}^{K} \alpha_i a_{i,m} \right) \left(\sum_{j=1}^{K} \alpha_j a_{j,m}^3 \right) \\
&\quad + \sum_{m=1}^{n} \sum_{j=1}^{K} \alpha_j a_{j,m}^4,
\end{aligned}$$

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which, by (2.1), simplifies to

$$\left(1-\sum_{i=1}^{K}\alpha_i^2\right)M^4 \geq 2\sum_{m=1}^{n}\sum_{i=1}^{K}\alpha_i a_{i,m}^4 + 6\sum_{m=1}^{n}\left(\sum_{i=1}^{K}\alpha_i a_{i,m}^2\right)^2. \tag{2.5}$$

Similarly, if we apply $\sum_{j=1}^{L}\beta_j\sum_{\substack{i=1\\ i\neq j}}^{L}\beta_i$ to (2.3), we obtain

$$\left(1-\sum_{j=1}^{L}\beta_j^2\right)M^4 \geq 2\sum_{m=1}^{n}\sum_{j=1}^{L}\beta_j b_{j,m}^4 + 6\sum_{m=1}^{n}\left(\sum_{j=1}^{L}\beta_j b_{j,m}^2\right)^2. \tag{2.6}$$

Next apply $\sum_{i=1}^{K}\alpha_i\sum_{j=1}^{L}\beta_j$ to (2.4):

$$\begin{aligned}
\mu^4 &= \sum_{i=1}^{K}\alpha_i\sum_{j=1}^{L}\beta_j\mu^4 \leq \sum_{i=1}^{K}\alpha_i\sum_{j=1}^{L}\beta_j\sum_{m=1}^{n}(a_{i,m}-b_{j,m})^4\\
&= \sum_{m=1}^{n}\sum_{i=1}^{K}\sum_{j=1}^{L}\alpha_i\beta_j(a_{i,m}^4-4a_{i,m}^3b_{j,m}+6a_{i,m}^2b_{j,m}^2-4a_{i,m}b_{j,m}^3+b_{j,m}^4)\\
&= \sum_{m=1}^{n}\left(\sum_{i=1}^{K}\alpha_i a_{i,m}^4\right)\left(\sum_{j=1}^{L}\beta_j\right) - 4\sum_{m=1}^{n}\left(\sum_{i=1}^{K}\alpha_i a_{i,m}^3\right)\left(\sum_{j=1}^{L}\beta_j b_{j,m}\right)\\
&\quad + 6\sum_{m=1}^{n}\left(\sum_{i=1}^{K}\alpha_i a_{i,m}^2\right)\left(\sum_{j=1}^{L}\beta_j b_{j,m}^2\right) - 4\sum_{m=1}^{n}\left(\sum_{i=1}^{K}\alpha_i a_{i,m}\right)\left(\sum_{j=1}^{L}\beta_j b_{j,m}^3\right)\\
&\quad + \sum_{m=1}^{n}\left(\sum_{i=1}^{K}\alpha_i\right)\left(\sum_{j=1}^{L}\beta_j b_{j,m}^4\right)\\
&\overset{(2.1)}{=} \sum_{m=1}^{n}\sum_{i=1}^{K}\alpha_i a_{i,m}^4 + 6\sum_{m=1}^{n}\left(\sum_{i=1}^{K}\alpha_i a_{i,m}^2\right)\left(\sum_{j=1}^{L}\beta_j b_{j,m}^2\right) + \sum_{m=1}^{n}\sum_{j=1}^{L}\beta_j b_{j,m}^4,
\end{aligned}$$

that is,

$$\sum_{m=1}^{n}\sum_{i=1}^{K}\alpha_i a_{i,m}^4 + \sum_{m=1}^{n}\sum_{j=1}^{L}\beta_j b_{j,m}^4 \geq \mu^4 - 6\sum_{m=1}^{n}\left(\sum_{i=1}^{K}\alpha_i a_{i,m}^2\right)\left(\sum_{j=1}^{L}\beta_j b_{j,m}^2\right). \tag{2.7}$$

Add (2.5) and (2.6) together:

$$\begin{aligned}
&\left(2-\sum_{i=1}^{K}\alpha_i^2-\sum_{j=1}^{L}\beta_j^2\right)M^4\\
&\geq 2\sum_{m=1}^{n}\sum_{i=1}^{K}\alpha_i a_{i,m}^4 + 2\sum_{m=1}^{n}\sum_{j=1}^{L}\beta_j b_{j,m}^4 + 6\sum_{m=1}^{n}\left(\sum_{i=1}^{K}\alpha_i a_{i,m}^2\right)^2 + 6\sum_{m=1}^{n}\left(\sum_{j=1}^{L}\beta_j b_{j,m}^2\right)^2
\end{aligned}$$

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$$\begin{aligned}
&\overset{(2.7)}{\geq} 2\mu^4 - 12\sum_{m=1}^{n}\Big(\sum_{i=1}^{K}\alpha_i a_{i,m}^2\Big)\Big(\sum_{j=1}^{L}\beta_j b_{j,m}^2\Big)\\
&\quad + 6\sum_{m=1}^{n}\Big(\sum_{i=1}^{K}\alpha_i a_{i,m}^2\Big)^2 + 6\sum_{m=1}^{n}\Big(\sum_{j=1}^{L}\beta_j b_{j,m}^2\Big)^2\\
&= 2\mu^4 + 6\sum_{m=1}^{n}\Big(\Big(\sum_{i=1}^{K}\alpha_i a_{i,m}^2\Big)^2 - 2\Big(\sum_{i=1}^{K}\alpha_i a_{i,m}^2\Big)\Big(\sum_{j=1}^{L}\beta_j b_{j,m}^2\Big) + \Big(\sum_{j=1}^{L}\beta_j b_{j,m}^2\Big)^2\Big)\\
&= 2\mu^4 + 6\sum_{m=1}^{n}\Big(\sum_{i=1}^{K}\alpha_i a_{i,m}^2 - \sum_{j=1}^{L}\beta_j b_{j,m}^2\Big)^2\\
&\geq 2\mu^4.
\end{aligned}$$

Therefore,

$$\frac{M^4}{\mu^4} \geq \frac{2}{2 - \sum_{i=1}^{K}\alpha_i^2 - \sum_{j=1}^{L}\beta_j^2}. \tag{2.8}$$

By (2.1) and the Cauchy–Schwarz inequality, $\sum_{i=1}^{K}\alpha_i^2 \geq 1/K$ and $\sum_{j=1}^{L}\beta_j^2 \geq 1/L$. Therefore,

$$\sum_{i=1}^{K}\alpha_i^2 + \sum_{j=1}^{L}\beta_j^2 \geq \frac{1}{K} + \frac{1}{L} \geq \begin{cases} \frac{2}{n+2} + \frac{2}{n+2} & \text{if } n \text{ is even,}\\ \frac{2}{n+1} + \frac{2}{n+3} & \text{if } n \text{ is odd.}\end{cases}$$

Substitute this estimate into (2.8) to obtain

$$\frac{M^4}{\mu^4} \geq \begin{cases} 1 + \frac{2}{n} & \text{if } n \text{ is even,}\\ 1 + \frac{2}{n-(n+2)^{-1}} & \text{if } n \text{ is odd,}\end{cases}$$

which finishes the proof. ∎

Proof of Corollary 1.4 It is well known and easy to see that for any $\boldsymbol{x} \in \mathbb{R}^n$, if $1 \leq p \leq 4$, then $\|\boldsymbol{x}\|_4 \leq \|\boldsymbol{x}\|_p \leq n^{1/p-1/4}\|\boldsymbol{x}\|_4$, and if $4 \leq p < \infty$, then $\|\boldsymbol{x}\|_p \leq \|\boldsymbol{x}\|_4 \leq n^{1/4-1/p}\|\boldsymbol{x}\|_p$. Suppose that there exists an equilateral set S of $n+2$ points in ℓ_p^n. Then

$$\frac{\max_{\boldsymbol{x},\boldsymbol{y}\in S}\|\boldsymbol{x}-\boldsymbol{y}\|_4}{\min_{\boldsymbol{x},\boldsymbol{y}\in S,\boldsymbol{x}\neq\boldsymbol{y}}\|\boldsymbol{x}-\boldsymbol{y}\|_4} \leq n^{|1/4-1/p|}.$$

Combine this inequality with Theorem 1.3 to obtain $1 + \frac{2}{n} \leq n^{|1-4/p|}$. A calculation then shows that

$$|p-4| \geq \frac{4\log(1+2/n)}{\log(n+2)} = \frac{8}{n\log n}\big(1 + O(n^{-1})\big). \qquad ∎$$

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Proof of Proposition 1.5 Let $k \in \mathbb{N}$, $x, y \in \mathbb{R}$, and

$$\boldsymbol{a} := (1+x, x, x, \dots, x) \in \ell_4^k \quad \text{and} \quad \boldsymbol{b} := (y, y, \dots, y) \in \ell_4^k.$$

We would like to choose x and y such that $\|\boldsymbol{a}\|_4 = \|\boldsymbol{b}\|_4$ and $\|\boldsymbol{a}-\boldsymbol{b}\|_4 = 2^{1/4}$. This is equivalent to the following two simultaneous equations:

$$\begin{aligned} (1+x)^4 + (k-1)x^4 &= ky^4 \\ (1+x-y)^4 + (k-1)(x-y)^4 &= 2. \end{aligned} \tag{2.9}$$

We postpone the proof of the following lemma.

Lemma 2.1 *For each $k \in \mathbb{N}$ the system (2.9) has a unique solution (x_k, y_k) satisfying $y_k > 0$. Asymptotically, as $k \to \infty$ we have*

$$x_k = -k^{-1/2} + k^{-3/4} + O(k^{-1}) \quad \textit{and} \quad y_k = k^{-1/4} - k^{-3/4} + O(k^{-1}).$$

Using the solution $(x, y) = (x_k, y_k)$ from the lemma, we obtain

$$\|\boldsymbol{a}\|_4 = \|\boldsymbol{b}\|_4 = k^{1/4} y = 1 - k^{-1/2} + O(k^{-3/4}).$$

Write $\boldsymbol{a}_1, \dots, \boldsymbol{a}_k$ for the k permutations of $\boldsymbol{a}$ and set $\boldsymbol{a}_{k+1} = \boldsymbol{b}$. Then (2.9) gives that $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_{k+1}\}$ is equilateral in ℓ_4^k. Finally, let $n = 2k$. Then in the set

$$S = \{ (\boldsymbol{a}_i, \boldsymbol{o}) \mid i = 1, 2, \dots, k+1 \} \cup \{ (\boldsymbol{o}, \boldsymbol{a}_i) \mid i = 1, 2, \dots, k+1 \}$$

of $n+2$ points in ℓ_4^n the only nonzero distances are $2^{1/4}$ and $2^{1/4}\|\boldsymbol{a}\|_4$. Therefore,

$$\frac{\max_{\boldsymbol{x},\boldsymbol{y} \in S} \|\boldsymbol{x}-\boldsymbol{y}\|_4}{\min_{\boldsymbol{x},\boldsymbol{y} \in S, \boldsymbol{x} \neq \boldsymbol{y}} \|\boldsymbol{x}-\boldsymbol{y}\|_4} = \frac{1}{\|\boldsymbol{a}\|_4} = 1 + \sqrt{\frac{2}{n}} + O(n^{-3/4}).$$

The case where $n = 2k+1$ is odd is handled by using the points $\boldsymbol{a}_1, \dots, \boldsymbol{a}_{k+1} \in \ell_4^k$ as constructed above and the analogous construction of $k+2$ points $\boldsymbol{a}_1', \dots, \boldsymbol{a}_{k+2}' \in \ell_4^{k+1}$ satisfying $\|\boldsymbol{a}_i' - \boldsymbol{a}_j'\|_4 = 2^{1/4}$ and $\|\boldsymbol{a}_i'\|_4 = 1 - (k+1)^{-1/2} + O(k^{-1})$. Then the nonzero distances between points in

$$S = \{ (\boldsymbol{a}_i, \boldsymbol{o}) \mid i = 1, 2, \dots, k+1 \} \cup \{ (\boldsymbol{o}, \boldsymbol{a}_i') \mid i = 1, 2, \dots, k+2 \}$$

are $2^{1/4}$ and $(\|a_i\|_4^4 + \|a_j'\|_4^4)^{1/4}$, giving the same asymptotics as before. ∎

Sketch of proof for Lemma 2.1 For $t \in \mathbb{R}$ let

$$f(t) = \Big(\frac{(1+t)^4 + (k-1)t^4}{k} \Big)^{1/4} = k^{-1/4} \|(1, 0, \dots, 0) + t(1, 1, \dots, 1)\|_4.$$

Then (2.9) is equivalent to $f(x) = |y|$ and $f(x-y) = (2/k)^{1/4}$. Since $\|\cdot\|_4$ is a strictly convex norm, f is strictly convex. Since $f(0) = k^{-1/4}$ and $\lim_{t \to \pm\infty} f(t) = \infty$, it

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follows that there is a unique $\alpha_k < 0$ and a unique $\beta_k > 0$ such that $f(\alpha_k) = f(\beta_k) = (2/k)^{1/4}$. Thus, $x - y \in \{\alpha_k, \beta_k\}$. It also follows that f is strictly decreasing on $(-\infty, \alpha_k)$. It is immediate from the definition that f is strictly increasing on $(0, \infty)$. Since $f(-k^{-1/4}) < (2/k)^{1/4} < f(k^{-1/4})$, it follows that $\alpha_k < -k^{-1/4}$ and $\beta_k < k^{-1/4}$.

By strict convexity of $\|\cdot\|_4$, f also satisfies the strict Lipschitz condition

$$|f(t+h) - f(t)| < h \quad \text{for all } t, h \in \mathbb{R} \text{ with } h > 0.$$

It follows that $t \mapsto f(t) - t$ is strictly decreasing and $t \mapsto f(t) + t$ is strictly increasing. Since $\lim_{t\to\infty}(f(t) - t) = 1/k$ and $\lim_{t\to-\infty}(f(t) + t) = -1/k$, it follows that $f(t) > t + 1/k$, and for each $r > 1/k$ there is a unique t such that $f(t) - t = r$; also $f(t) > -t - 1/k$, and for each $r > -1/k$ there is a unique t such that $f(t) + t = r$.

We now consider the two cases $x - y = \alpha_k$ and $x - y = \beta_k$.

Case I. If $x - y = \alpha_k$, then $f(x) = |y| = |x - \alpha_k|$. Since $f(x) > -x - 1/k \geq -x - k^{-1/4} > -x + \alpha_k$, necessarily $y = x - \alpha_k > 0$ and $f(x) - x = -\alpha_k$. Since $-\alpha_k > k^{-1/4} \geq 1/k$, there is a unique x_k such that $f(x_k) - x_k = -\alpha_k$, and since $f(0) - 0 = k^{-1/4} < -\alpha_k$, it satisfies $x_k < 0$. Setting $y_k = x_k - \alpha_k$, we obtain that (2.9) has exactly one solution (x_k, y_k) such that $x_k - y_k = \alpha_k$, and it satisfies $x_k < 0 < y_k$.

Case II. If $x - y = \beta_k$, then we similarly obtain a unique solution (x, y), this time satisfying $x < 0$ and $y < 0$.

Therefore, (2.9) has exactly two solutions, one with $y > 0$ and one with $y < 0$. Next we approximate the solution (x_k, y_k) of Case I.

From $f(\alpha_k) = (2/k)^{1/4}$, it follows that

$$(1 + \alpha_k)^4 + (k-1)\alpha_k^4 = 2, \tag{2.10}$$

which shows first that $\alpha_k = O(k^{-1/4})$ as $k \to \infty$, and then, since $\alpha_k < 0$, that $\alpha_k = -k^{-1/4} + O(k^{-1/2})$. We can rewrite (2.10) as

$$\begin{aligned} \alpha_k &= -k^{-1/4}(1 - 4\alpha_k - 6\alpha_k^2 - 4\alpha_k^3)^{1/4} \\ &= -k^{-1/4}\big(1 - \alpha_k - 3\alpha_k^2 - 9\alpha_k^3 + O(k^{-1})\big), \end{aligned} \tag{2.11}$$

where we have used the Taylor expansion $(1+x)^{1/4} = 1 + \frac{1}{4}x - \frac{3}{32}x^2 + \frac{7}{128}x^3 + O(x^4)$. Substitute the estimate $\alpha_k = -k^{-1/4} + O(k^{-1/2})$ into the right-hand side of (2.11) to obtain the improved estimate $\alpha_k = -k^{-1/4} - k^{-1/2} + O(k^{-3/4})$, and again, to obtain

$$\alpha_k = -k^{-1/4} - k^{-1/2} + 2k^{-3/4} + O(k^{-1}).$$

Since

$$f(-k^{-1/2}) + k^{-1/2} = k^{-1/4} + k^{-1/2} - k^{-3/4} + O(k^{-1}) > -\alpha_k$$

for sufficiently large k, and $f(x_k) - x_k = -\alpha_k$, it follows that $x_k > -k^{-1/2}$ for large k, that is, $x_k = O(k^{-1/2})$. It follows that

$$f(x_k) - x_k = k^{-1/4}\big(1 + x_k + O(k^{-1})\big) - x_k.$$

Set this equal to $-\alpha_k$ and solve for x_k to obtain $x_k = -k^{-1/2} + k^{-3/4} + O(k^{-1})$ and $y_k = x_k - \alpha_k = k^{-1/4} - k^{-3/4} + O(k^{-1})$. ∎

Acknowledgment We thank the referee for helpful remarks that led to an improved paper.

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