NUMERICAL SIMULATION OF THE ROTATIONAL MOTION OF THE EARTH AND MOON

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Abstract. Dynamics of the rotational motion of the Earth and Moon is investigated numerically. Very convenient Rodrigues-Hamilton parameters are used for high-precision numerical integration of the rotational motion equations in the post-newtonian approximation over a 400 yr time interval. The results of the numerical solution of the problem are compared with the contemporary analytical theories of the Earth's and Moon's rotation. The analytical theory of the Earth's rotation is composed of the precession theory (Lieske *et al.*, 1977), nutation theory (Souchay and Kinoshita, 1996) and geodesic nutation solution (Fukushima, 1991). The analytical theory of the Moon's rotation consists of the so-called Cassini relations and the analytical solutions of the lunar physical libration problem (Moons, 1982), (Moons, 1984), (Pešek, 1982). The comparisons reveal residuals both of periodic and systematic character. All the secular and periodic terms representing the behavior of the residuals are interpreted as corrections to the mentioned analytical theories. In particular, the secular rate of the luni-solar inclination of the ecliptic to the equator J2000.0 ($-0\overset{''}{.}027$, with a mean square error $0\overset{''}{.}000005$) is very close to its theoretical value (Williams, 1994).

1. Statement of the Problem

The Earth and Moon, on a whole, in their physical properties are close to a rigid body and therefore the most essential features of their rotational motions can be reproduced well enough by the rotation of a rigid body. The rotational motions of the Earth and Moon are the results of the gravitational interaction of their bodies with the perturbing bodies (the Sun, Moon, and planets) in the post-newtonian approximation. The orbital motions of the disturbing bodies are defined by the DE200/LE200 ephemeris. The reference frame of the problems is based on the fixed ecliptic J2000.0. The problem of the Earth rotation is solved separately from the lunar rotation problem.

I. M. Wytrzyszczak, J. H. Lieske and R. A. Feldman (eds.),
Dynamics and Astrometry of Natural and Artificial Celestial Bodies, 275, 1997.

https://doi.org/10.1017/S0252921100046674 Published online by Cambridge University Press

2. Mathematical Model of the Problem

The rotation of a celestial body about its center of mass is described as the rotation of its principal axes of inertia with respect to the non-rotating body-centric coordinate system. As the variables of the problem, four Rodrigues-Hamilton parameters are taken. They are defined by the Euler angles ψ, θ, ϕ:

$$\lambda_0 = \cos\frac{\theta}{2}\cos\frac{\psi+\phi}{2}, \quad \lambda_1 = \sin\frac{\theta}{2}\cos\frac{\psi-\phi}{2},$$
$$\lambda_2 = \sin\frac{\theta}{2}\sin\frac{\psi-\phi}{2}, \quad \lambda_3 = \cos\frac{\theta}{2}\sin\frac{\psi+\phi}{2}$$

The Rodrigues-Hamilton parameters are bounded variables which is very important for the numerical solution of the problem. These variables determine the orientation of the principal axes of inertia with respect to the fixed ecliptic and the fixed vernal equinox at the epoch J2000.0. The differential equations of the problem are deduced from the Lagrange differential equations of the second kind:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\lambda}_i} - \frac{\partial L}{\partial \lambda_i} = 0, \quad i = 0, 1, 2, 3.$$

Here and subsequently, a dot over the letters means a differentiation with respect to the barycentric dynamical time. The Lagrange function has the form, $L = T + U + \Delta L$, where the kinetic energy of the rotational motion is $T = \frac{1}{2}(A\omega_1^2 + B\omega_2^2 + C\omega_3^2)$. In the case of the Earth rotation the equatorial moments of inertia A and B are equal. Projections of the angular velocity vector upon the axes of the principal moments of inertia are expressed via the Rodrigues-Hamilton parameters and their derivatives as follows:

$$\begin{aligned}
\omega_1 &= 2(-\lambda_1\dot{\lambda}_0 + \lambda_0\dot{\lambda}_1 + \lambda_3\dot{\lambda}_2 - \lambda_2\dot{\lambda}_3) \\
\omega_2 &= 2(-\lambda_2\dot{\lambda}_0 - \lambda_3\dot{\lambda}_1 + \lambda_0\dot{\lambda}_2 + \lambda_1\dot{\lambda}_3) \\
\omega_3 &= 2(-\lambda_3\dot{\lambda}_0 + \lambda_2\dot{\lambda}_1 - \lambda_1\dot{\lambda}_2 + \lambda_0\dot{\lambda}_3)
\end{aligned}$$

The force function of the gravitational interaction of the Earth with the disturbing bodies is expressed as:

$$\begin{aligned}
U &= \sum_{j\neq\oplus}\frac{GM_\oplus M_j}{r_{j\oplus}}\left\{1 - \left(\frac{a_\oplus}{r_{j\oplus}}\right)^2 J_2\left[\frac{3}{2}\left(\frac{z_{j\oplus}}{r_{j\oplus}}\right)^2 - \frac{1}{2}\right] - \right. \\
&\quad - \left(\frac{a_\oplus}{r_{j\oplus}}\right)^3 J_3\left[\frac{5}{2}\left(\frac{z_{j\oplus}}{r_{j\oplus}}\right)^3 - \frac{3}{2}\left(\frac{z_{j\oplus}}{r_{j\oplus}}\right)\right] - \\
&\quad \left. - \left(\frac{a_\oplus}{r_{j\oplus}}\right)^4 J_4\left[\frac{35}{8}\left(\frac{z_{j\oplus}}{r_{j\oplus}}\right)^4 - \frac{15}{4}\left(\frac{z_{j\oplus}}{r_{j\oplus}}\right)^2 + \frac{3}{8}\right]\right\},
\end{aligned}$$

where $r_{j\oplus} = \sqrt{x_{j\oplus}^2 + y_{j\oplus}^2 + z_{j\oplus}^2}$ and $x_{j\oplus}, y_{j\oplus}, z_{j\oplus}$ are the geocentric equatorial coordinates of the disturbing body. Then the transformation matrix has a form:

$$\begin{pmatrix} \lambda_0^2+\lambda_1^2-\lambda_2^2-\lambda_3^2 & 2(\lambda_0\lambda_3+\lambda_1\lambda_2) & 2(\lambda_1\lambda_3-\lambda_0\lambda_2) \\ 2(\lambda_1\lambda_2-\lambda_0\lambda_3) & \lambda_0^2-\lambda_1^2+\lambda_2^2-\lambda_3^2 & 2(\lambda_0\lambda_1+\lambda_2\lambda_3) \\ 2(\lambda_0\lambda_2+\lambda_1\lambda_3) & 2(\lambda_2\lambda_3-\lambda_0\lambda_1) & \lambda_0^2-\lambda_1^2-\lambda_2^2+\lambda_3^2 \end{pmatrix}$$

The expression for the force function of the gravitational attraction of the Moon by the disturbing bodies contains all the harmonics up to the 4^{th} order as well as the mixed harmonics, describing gravitational interaction of the lunar and terrestrial bodies.

The additional part of the Lagrange function generating geodetic perturbations in the Earth's rotation is represented by the following expression:

$$\begin{aligned}\Delta L = \sum_{j\neq\oplus}\frac{GM_j}{c^2}\{&\frac{1}{r_{j\oplus}^3}\overline{H}\cdot[\frac{3}{2}(\overline{r}_{j\oplus}\times\overline{v}_\oplus)-2(\overline{r}_{j\oplus}\times\overline{v}_j)]-\\ &-\frac{3}{2r_{j\oplus}^3}\{(C-B)\omega_1(z_{j\oplus}\dot{y}_{j\oplus}+y_{j\oplus}\dot{z}_{j\oplus})+(A-C)\omega_2(x_{j\oplus}\dot{z}_{j\oplus}+z_{j\oplus}\dot{x}_{j\oplus})+\\ &\quad+(B-A)\omega_3(y_{j\oplus}\dot{x}_{j\oplus}+x_{j\oplus}\dot{y}_{j\oplus})+\\ &+\frac{\overline{r}_{j\oplus}\cdot\overline{v}_j}{r_{j\oplus}^2}[(C-B)\omega_1 z_{j\oplus}y_{j\oplus}+(A-C)\omega_2 x_{j\oplus}z_{j\oplus}+(B-A)\omega_3 x_{j\oplus}y_{j\oplus}]\}\}.\end{aligned}$$

Here, the angular momentum vector is $\overline{H} = A\omega\overline{i}_1 + A\omega\overline{i}_2 + C\omega\overline{i}_3$, while $\overline{v}_j$ is the vector of the barycentric velocity of the disturbing body determined in the equatorial coordinate system.

Now, the differential equations of the rotation in the Rodrigues-Hamilton parameters have form,

$$\left.\begin{aligned}\ddot{\lambda}_0 &= -\frac{1}{2}\{\frac{1}{2}\omega^2\lambda_0+\dot{\omega}_1\lambda_1+\dot{\omega}_2\lambda_2+\dot{\omega}_3\lambda_3\}\\ \ddot{\lambda}_1 &= -\frac{1}{2}\{\frac{1}{2}\omega^2\lambda_1-\dot{\omega}_1\lambda_0+\dot{\omega}_2\lambda_3-\dot{\omega}_3\lambda_2\}\\ \ddot{\lambda}_2 &= -\frac{1}{2}\{\frac{1}{2}\omega^2\lambda_2-\dot{\omega}_1\lambda_3-\dot{\omega}_2\lambda_0+\dot{\omega}_3\lambda_1\}\\ \ddot{\lambda}_3 &= -\frac{1}{2}\{\frac{1}{2}\omega^2\lambda_3+\dot{\omega}_1\lambda_2-\dot{\omega}_2\lambda_1-\dot{\omega}_3\lambda_0\}\end{aligned}\right\}, \tag{1}$$

where $\omega^2 = \omega_1^2+\omega_2^2+\omega_3^2$ or $\omega^2 = 4(\dot{\lambda}_0^2+\dot{\lambda}_1^2+\dot{\lambda}_2^2+\dot{\lambda}_3^2)$.

The first derivatives of the angular velocity components are expressed by means of the Euler dynamical equations,

$$\dot{\omega}_1 = \frac{B-C}{A}\omega_2\omega_3-\frac{1}{2A}\left\{\lambda_1\frac{\partial}{\partial\lambda_0}-\lambda_0\frac{\partial}{\partial\lambda_1}-\lambda_3\frac{\partial}{\partial\lambda_2}+\lambda_2\frac{\partial}{\partial\lambda_3}\right\}(U+\Delta L)$$

$$\dot{\omega}_2 = \frac{C-A}{B}\omega_3\omega_1-\frac{1}{2B}\left\{\lambda_2\frac{\partial}{\partial\lambda_0}+\lambda_3\frac{\partial}{\partial\lambda_1}-\lambda_0\frac{\partial}{\partial\lambda_2}-\lambda_1\frac{\partial}{\partial\lambda_3}\right\}(U+\Delta L)$$

$$\dot{\omega}_3 = \frac{A-B}{C}\omega_1\omega_2 - \frac{1}{2C}\left\{\lambda_3\frac{\partial}{\partial\lambda_0} - \lambda_2\frac{\partial}{\partial\lambda_1} + \lambda_1\frac{\partial}{\partial\lambda_2} - \lambda_0\frac{\partial}{\partial\lambda_3}\right\}(U + \Delta L)$$

The independent variable of system (1) is the barycentric dynamical time (TDB). For the problem of the Earth rotation the numerical values of the constants entering system (1) are taken from the system of the astronomical constants of the ephemeris DE200/LE200. For the lunar rotation problem the LURE2 system of astronomical constants is used.

3. Initial Conditions of the Problem

The initial conditions of the numerical solution of system (1) are taken from the semi-analytical theories of the problems. The semi-analytical theory of the Earth rotation is composed of:

a) semi-analytical precession theory (Lieske *et al.*, 1977),
b) mean sidereal time expression,
c) semi-analytical nutation theory SKRE96 (Souchay, Kinoshita, 1996),
d) geodesic nutation solutions (Fukushima, 1991).

The semi-analytical theory of the nutation of the Earth figure equator SKRE96 contains harmonics with amplitudes not less than 0.0001 mas. When compiling the semi-analytical theory of the Earth rotation the harmonics with periods less than 400 yr are retained in SKRE96. The semi-analytical theory of the lunar rotation consists of:

a) so-called Cassini relations,
b) semi-analytical theory of the lunar physical libration (Moons, 1982), (Moons, 1984), (Pešek, 1982).

The amplitudes of the harmonics in the semi-analytical theory of the physical libration of the Moon do not exceed $0''.001$. The power polynomials for the mean longitudes of the Moon and the ascending node of the lunar orbit entering the Cassini relations are taken from (Simon *et al.*, 1993).

4. Method of Numerical Integration

For the solution of system (1) the high-precision numerical integration method (Belikov, 1990) with a number of modifications (Eroshkin *et al.*, 1993) was applied. For the problem of the Earth rotation the numerical integration was performed with a constant step size of one day and a 16^{th} degree Chebyshev polynomial approximating the right-hand sides of the differential equations. In the case of the lunar rotation the size of the step equals 24 days when choosing a 24^{th} degree of the approximating polynomials. The integration of the problems is carried out over the time interval 1750 – 2169 from the initial epoch JD 2440400.5. The initial conditions of the numerical integration are taken from the semi-analytical theories reduced to the fixed ecliptic J2000.0.

5. Results

A comparison of the results of the numerical integration in Euler angles with the semi-analytical theories revealed discrepancies, both of systematic and periodic character. In the case of the Earth rotation problem the discrepancies were analyzed by means of the least squares method. The systematic trends in the luni-solar precession and inclination were approximated by third order power polynomials in time. A number of periodic harmonics were also evaluated. Then these corrections were added to the semi-analytical theory and the process of the numerical integration was repeated anew. After several iterations the following corrections to the expressions of the luni-solar precession and inclination of the precessional theory (Lieske *et al.*, 1977) were found,

$$\begin{aligned} d\psi &= -0''.0051_{\pm 4''\cdot 10^{-6}} - 0''.0144_{\pm 2''\cdot 10^{-6}} T + \\ &\quad + 0''.0004_{\pm 2''\cdot 10^{-6}} T^2 + 0''.0001_{\pm 5''\cdot 10^{-7}} T^3 \\ d\omega &= -0''.0089_{\pm 2''\cdot 10^{-6}} - 0''.0271_{\pm 1''\cdot 10^{-6}} T + \\ &\quad + 0''.00004_{\pm 1''\cdot 10^{-7}} T^2 + 0''.00002_{\pm 7''\cdot 10^{-7}} T^3, \end{aligned}$$

and a number of correcting harmonics to the KSRE96 nutation theory,

$$\begin{aligned} \Delta\psi = & \quad 0''.000238\cos(\Omega) + 0''.000073\cos(-3L_V + 5L_E + 2p_A) + \\ & - 0''.001126\sin(\Omega) + 0''.000322\sin(F - D + \Omega - 8L_V + 12L_E) + \\ & + 0''.000041\cos(2L_J + 2p_A) - 0''.000096\sin(2F - 2D + 2\Omega) \end{aligned}$$

$$\Delta\varepsilon = \quad 0''.000594\cos(\Omega) + 0''.000040\cos(2F - 2D + 2\Omega)$$

where Ω is the mean node longitude of the lunar orbit, l_M is the mean anomaly of the Moon, F is the mean argument of the Moon's latitude, l_S is the mean anomaly of the Sun, D is the difference of the mean longitudes of the Moon and the Sun, L_V, L_E, L_J are the mean longitudes of the Venus, Earth, Jupiter, respectively and p_A is the general precession in longitude.

The secular trend was discovered when comparing the values of the proper rotation angle and the apparent sidereal time. There is no explanation of its appearance. Left part of the Figure 1 depicts the results of comparing the "corrected" semi-analytical theory with results of the numerical integration. The results of the comparison of the semi-analytical theory of the lunar rotation with the numerical integration are presented on the right side of the Figure 1. The residuals do not reveal secular trends. The most essential harmonics of the residuals in the inclination (ϱ) and the node ($I\sigma$) are periodic harmonics with the period of the lunar node revolution and a time dependent amplitude. Probably the presence of these harmonics in the residuals can be explained by the absence of terms of such kind in the semi-analytical theory of the lunar rotation. The behavior of the residuals in the

https://doi.org/10.1017/S0252921100046674 Published online by Cambridge University Press

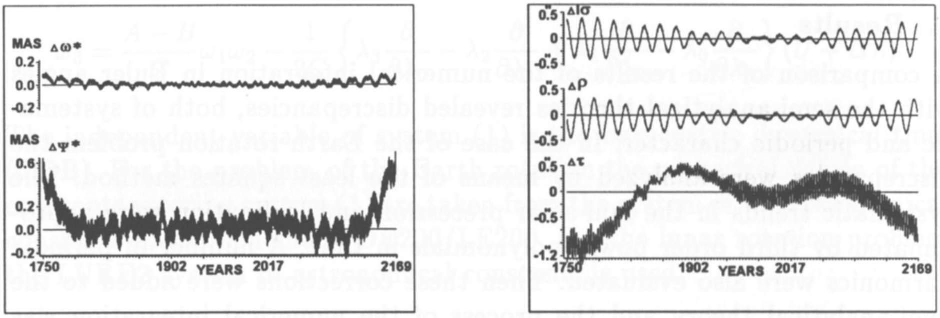

Figure 1. Comparison of the numerical solution of the Earth's (*left figure*) and Moon's (*right figure*) rotation problem with the semi-analytic one at the fixed ecliptic J2000.0.

longitude (τ) is more complicated. It can be described by the superposition of the harmonics representing both the forced physical libration and the fictive free physical libration. In the present case the initial conditions are defined by the not precise enough semi-analytical theory. It seems that for a better agreement between the numerical solutions and semi-analytical theories it is necessary to improve the precessional quantities determining the motion of the ecliptic of date.

Acknowledgements. The research was carried out in the Institute of Theoretical Astronomy, under a financial support of the Russian Foundation for Basic Research, grant No 95-02-04304-a.

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