THE ITERATED EQUATION OF GENERALIZED AXIALLY SYMMETRIC POTENTIAL THEORY, II

GENERAL SOLUTIONS OF WEINSTEIN'S TYPE

J. C. BURNS

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1. Introduction

In the first paper of this series [1], which will be designated I, particular solutions of various kinds have been found for the iterated equation of generalized axially symmetric potential theory (GASPT) which, in the notation defined in I, is

$$L_k^n(f) = 0, \tag{1}$$

where the operator is defined by

$$L_k(f) \equiv \partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 + k y^{-1} \partial f/\partial y.$$

Among these solutions is a family of the type $y^s f_t$ where f_t is a solution of the equation $L_t(f) = 0$.

Weinstein [2] has shown that a general solution of (1) can be formed by taking a linear combination of n solutions of this type (in all of which $s = 0$):

$$f_k + f_{k-2} + \cdots + f_{k-2(n-1)}. \tag{2}$$

This means that every function of the form (2) is a solution of (1) and that every solution of (1) can be expressed in the form (2). Payne [3] has followed Weinstein and produced two further general solutions of (1) which are also linear combinations of n terms of the form $y^s f_t$. The first, like Weinstein's, is valid for all k:

$$f_k + y^2 f_{k+2} + \cdots + y^{2(n-2)} f_{k+2(n-2)} + y^{2(n-1)} f_{k+2(n-1)}. \tag{3}$$

The second, however, requires that $k \neq 1-2i$ for any integer i such that $0 \leqq i \leqq n-2$:

$$f_{k-2} + y^2 f_k + \cdots + y^{2(n-2)} f_{k+2(n-3)} + y^{2(n-1)} f_{k+2(n-1)}. \tag{4}$$

In I, the set of all solutions of (1) of the form $y^s f_t$ has been found and in the present paper a criterion is obtained for deciding whether a given selection of n of these solutions combine linearly to produce a general solution of (1).

2. Solutions of $L_k^n(f) = 0$ of the form $y^s f_t$

It has been shown in theorem 4.5 of I that all solutions of (1) of the form $y^s f_t$ are given by

$$a_{\alpha\beta} = y^{2\alpha} f_{k+2\alpha-2\beta}, \quad A_{\alpha\beta} = y^{1-k+2\beta} f_{2-k+2\beta-2\alpha}, \tag{5}$$

where α, β are non-negative integers such that $0 \leqq \alpha+\beta \leqq n-1$. It is convenient to think of these two classes of solutions as represented by the points of triangular arrays in the $\alpha-\beta$ plane as shown in figure 1(a) for the case $n = 4$.

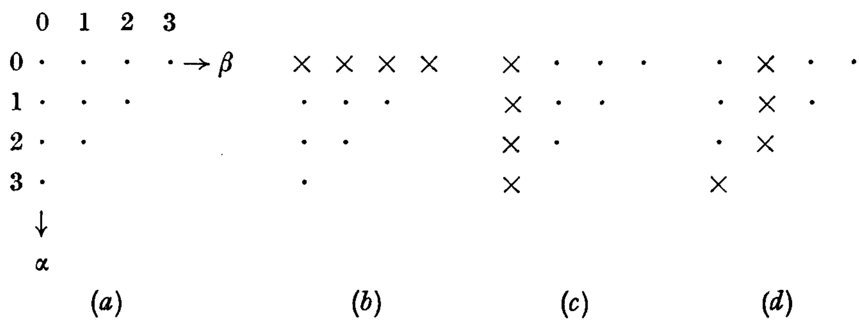

Figure 1

The terms of Weinstein's solution (2) and Payne's two solutions (3) and (4) are all of the form $a_{\alpha\beta}$ and are represented by sets of points such as those shown in figures 1 (b), (c), (d) respectively.

The notation $\Delta_m(\gamma, \delta)$ will be used to denote the triangular array of points which represent terms $a_{\alpha\beta}$ (or $A_{\alpha\beta}$) for which $\alpha = \gamma+\xi$, $\beta = \delta+\eta$, where ξ, η are non-negative integers such that $0 \leqq \xi+\eta \leqq m-1$. The full class of solutions $a_{\alpha\beta}$ of the equation $L_k^n(f) = 0$ is thus represented by the set of points $\Delta_n(0, 0)$.

There are $\frac{1}{2}n(n+1)$ solutions of each of the types $a_{\alpha\beta}$, $A_{\alpha\beta}$ and these are all distinct unless k is an odd integer of the form $1 \pm 2\gamma$ where γ is an integer in the range $0 \leqq \gamma \leqq n-1$. When $k = 1+2\gamma$, each solution $a_{\xi,\gamma+\eta}$ which is represented by a point in the array $\Delta_{n-\gamma}(0, \gamma)$ is identical with the solution $A_{\eta,\gamma+\xi}$ which is also represented by a point lying in this array. A similar result holds when $k = 1-2\gamma$, with the points this time lying in the array $\Delta_{n-\gamma}(\gamma, 0)$. In particular, it will be seen that when $\gamma = 0$ so that $k = 1$, $a_{\alpha\beta} = A_{\alpha\beta}$ for all α, β.

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For the case $k = 0$, when the equation becomes the polyharmonic equation, the solutions are all distinct but for other cases of interest (eg. when $n = 2$ and $k = -1$) some of the $A_{\alpha\beta}$ may be identical with some of the $a_{\alpha\beta}$. However, it will appear shortly that this is not an important restriction.

Since solutions of the equation $L_k^m(f) = 0$ where $m < n$ will also satisfy the equation $L_k^n(f) = 0$, all solutions of $L_k^m(f) = 0$ of the form $y^s f_t$ must be found among the two classes of solutions $a_{\alpha\beta}$, $A_{\alpha\beta}$ of (1). It is clear that those members of each of these classes which are represented by points in the array $\Delta_m(0, 0)$ are solutions of $L_k^n(f) = 0$ for all $n \geqq m$.

3. Weinstein's general solution of $L_k^n(f) = 0$

Weinstein's general solution (2) of the equation $L_k^n(f) = 0$ is given in the notation introduced in (5) by

$$a_{00}+a_{01}+a_{02}+\cdots+a_{0,n-1}. \tag{6}$$

Weinstein proves that (6) is a general solution by mathematical induction, making use of the following results which, when they are used later, will be referred to as Weinstein's lemmas.

LEMMA 3.1 $\mathscr{D}(f_{k-2}) \leftrightarrow f_k.$

(The notation is that introduced in section 4 of I: the lemma states that for any function f_{k-2}, $\mathscr{D}f_{k-2} \equiv y^{-1}\partial f_{k-2}/\partial y$ can be expressed in the form f_k and, conversely, any function f_k can be expressed in the form $\mathscr{D}(f_{k-2})$. The two expressions $\mathscr{D}(f_{k-2})$ and f_k are said to be *equivalent*.)

Equation (18) of I shows that $L_k\mathscr{D}(f_{k-2}) = 0$ so that $\mathscr{D}(f_{k-2}) \rightarrow f_k$ for any f_{k-2}. The converse is not so easily proved and is one of the main results of Weinstein's paper [2].

LEMMA 3.2 *Provided* $k \neq l$, $L_k(f_l) \leftrightarrow \mathscr{D}(f_l)$.

This result is an immediate consequence of equation (20) of I.

Weinstein's solution will be taken as the starting point for the discussion of general solutions of the equation $L_k^n(f) = 0$ which consist of the sum of n terms chosen from the classes $a_{\alpha\beta}$, $A_{\alpha\beta}$ given by (5). Such a sum of n terms will be a general solution if it can be shown to be equivalent to Weinstein's general solution.

A set of n terms of the form $a_{\alpha\beta}$ or $A_{\alpha\beta}$ whose sum is a general solution of (1) will be referred to as a *solution set*. Any two sets of terms of this form will be said to be *equivalent* if their sums are equivalent; in particular, any two solution sets of the same equation $L_k^n(f) = 0$ are equivalent.

4. Relations between solutions $a_{\alpha\beta}$, $A_{\alpha\beta}$

Before sets of n terms $a_{\alpha\beta}, A_{\alpha\beta}$ can be compared with Weinstein's solution set, it is necessary to consider a number of relations between the terms.

4.1 Standard solutions.

THEOREM. *For any* α, β, $a_{\alpha\beta} \leftrightarrow A_{\alpha\beta}$.

This requires that, for any α, β,

$$y^{2\alpha} f_{k+2\alpha-2\beta} \leftrightarrow y^{1-k+2\beta} f_{2-k+2\beta-2\alpha},$$

a result which is given immediately by Weinstein's correspondence principle (equation (31) of I) which states that $f_k \leftrightarrow y^{1-k} f_{2-k}$ for any k.

The equivalence of $a_{\alpha\beta}$ and $A_{\alpha\beta}$ means that only sets of terms of the form $a_{\alpha\beta}$ need be considered as possible solution sets. Such solution sets, of which Weinstein's is one, will be called *standard sets*; the corresponding general solutions will be called *standard solutions*; and the set of points in the array Δ_n which represent the terms of a standard set will be called a *standard set* of points. In particular, the points representing the terms of Weinstein's solution will be called a *Weinstein set*.

It is clear that each standard solution will give rise to a family of general solutions as some or all of the $a_{\alpha\beta}$ are replaced by the corresponding $A_{\alpha\beta}$. The number of general solutions obtained in this way will depend on whether or not all the $A_{\alpha\beta}$ are distinct from the $a_{\alpha\beta}$. Thus when k is not of the form $1 \pm 2\gamma$ where γ is an integer in $0 \leqq \gamma \leqq n-1$, so that all the $A_{\alpha\beta}$ are distinct from the $a_{\alpha\beta}$, a standard solution will be representative of a family of 2^n general solutions; at the other extreme, when $k = 1$, all these solutions will be identical and the standard solution will represent a family consisting of just one member. It is the equivalence of $a_{\alpha\beta}$ and $A_{\alpha\beta}$ which makes the possible identity of some of these solutions for certain values of k unimportant.

4.2 Reflection principle.

THEOREM. *If a set of n terms $a_{\alpha\beta}$ is a standard set, then so also is the set of the n corresponding terms $a_{\beta\alpha}$.*

Let $f_k^{(n)}$ denote any solution of (1). Then, since the n terms $a_{\alpha\beta} \equiv y^{2\alpha} f_{k+2\alpha-2\beta}$ form a solution set of (1),

$$f_k^{(n)} \leftrightarrow \sum y^{2\alpha} f_{k+2\alpha-2\beta} \tag{7}$$

where the sum is taken over the n terms of the solution set. Changing k to $2-k$ shows that if $f_{2-k}^{(n)}$ is any solution of $L_{2-k}^n(f) = 0$, then

$$f_{2-k}^{(n)} \leftrightarrow \sum y^{2\alpha} f_{2-k+2\alpha-2\beta}. \tag{8}$$

The generalized Weinstein correspondence principle (Theorem 4.8 of I) states that $f_k^{(n)} \leftrightarrow y^{1-k} f_{2-k}^{(n)}$. From (8) it follows that

$$f_k^{(n)} \leftrightarrow \sum y^{1-k+2\alpha} f_{2-k+2\alpha-2\beta} \equiv \sum A_{\beta\alpha}.$$

This means that the set of n terms $A_{\beta\alpha}$ is a solution set of equation (1); but theorem 4.1 shows that the standard set equivalent to the set $\{A_{\beta\alpha}\}$ is the set $\{a_{\beta\alpha}\}$ and this proves the theorem.

The two sets $\{a_{\alpha\beta}\}$ and $\{a_{\beta\alpha}\}$ are represented by standard sets of points in the array $\Delta_n(0, 0)$ which are mirror images in the leading diagonal so the relation proved in this theorem may be called the *reflection principle.*

Reflection of Weinstein's set in $\Delta_n(0, 0)$ shows that the set of n points in the first column is equivalent to Weinstein's set and so also forms a standard set. (See figure 1 (b), (c).)

The general solution of (1) found in this way is

$$a_{00}+a_{10}+a_{20}+ \cdots +a_{n-1,0}, \tag{9}$$

which, in the original notation, is that given by (3), the first of Payne's solutions. The points of the column which represent the terms of Payne's solution will be called the *Payne set* of Δ_n (see figure 1(c)).

Since the equivalence of Weinstein's and Payne's solutions holds for all integral values of $n \geqq 1$, it follows that the Weinstein and Payne sets are equivalent for any array $\Delta_m(0, 0)$ included within $\Delta_n(0, 0)$.

4.3 Translation principle.

Theorem. *If two sets of m terms $\{a_{ij}\}$ and $\{a_{pq}\}$ are equivalent, then so are the two sets of terms $\{a_{i+\alpha, j+\beta}\}$ and $\{a_{p+\alpha, q+\beta}\}$ (for any integers α, β provided that all the terms are represented by points in the array Δ_n).*

The equivalence of the sets $\{a_{ij}\}$, $\{a_{pq}\}$ is expressed by the relation

$$\sum y^{2i} f_{k+2i-2j} \leftrightarrow \sum y^{2p} f_{k+2p-2q},$$

where the sum is in each case over the members of the set. Changing k to $k+2\alpha-2\beta$ and multiplying both sides of the resulting relation by $y^{2\alpha}$ gives

$$\sum y^{2(i+\alpha)} f_{k+2(i+\alpha)-2(j+\beta)} \leftrightarrow \sum y^{2(p+\alpha)} f_{k+2(p+\alpha)-2(q+\beta)},$$

which proves the equivalence of $\{a_{i+\alpha, j+\beta}\}$ and $\{a_{p+\alpha, q+\beta}\}$.

The theorem shows that if any two equivalent sets of m points, denoted say by $A(0, 0)$ and $B(0, 0)$, in the array $\Delta_m(0, 0)$, are translated through the displacement (α, β) then they become equivalent sets of points which may be denoted by $A(\alpha, \beta)$ and $B(\alpha, \beta)$ and lie in the array $\Delta_m(\alpha, \beta)$. If the original sets $A(0, 0)$ and $B(0, 0)$ are standard sets of $\Delta_m(0, 0)$, the new sets $A(\alpha, \beta)$, $B(\alpha, \beta)$ will be called standard sets of $\Delta_m(\alpha, \beta)$. For example, the Weinstein and Payne sets in $\Delta_3(0, 0)$ (see figure 2), which are equivalent become, on translation through the displacement (1, 2), the corresponding sets in $\Delta_3(1, 2)$ which are therefore equivalent and standard sets.

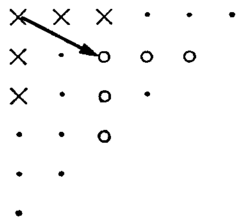


Figure 2

This example is an illustration of what may be called the extended reflection principle: if a set of m points $A(\alpha, \beta)$ is a standard set in $\Delta_m(\alpha, \beta)$ and so is obtained by translation from a set of points $A(0, 0)$ which is a standard set in $\Delta_m(0, 0)$, then the set $A(\alpha, \beta)$ is equivalent to the set $B(\alpha, \beta)$ obtained by reflecting the set $A(\alpha, \beta)$ in the leading diagonal of $\Delta_m(\alpha, \beta)$.

4.4 Weinstein-Almansi relation.

Weinstein [2] quotes Almansi's general solution of the polyharmonic equation $L_0^n(f) = 0$ as

$$f_{0,0}+yf_{0,1}+y^2f_{0,2}+\cdots y^{n-1}f_{0,n-1}, \tag{10}$$

where the functions $f_{0,i}$ are arbitrary solutions of the equation $L_0(f) = 0$ i.e. arbitrary harmonic functions; and he proves the equivalence of this solution and his own solution of the polyharmonic equation, obtained from (2) by putting $k = 0$:

$$f_0+f_{-2}+f_{-4}+\cdots+f_{-2(n-1)}.$$

Almansi's solution (10) of the polyharmonic equation is obtained by putting $k = 0$ in the expression

$$f_{k,0}+y^{1-k}f_{-k,1}+y^2f_{k,2}+y^{3-k}f_{-k,3}+y^4f_{k,4}+\cdots \text{ (to } n \text{ terms)}, \tag{11}$$

where the functions $f_{k,i}$ and $f_{-k,i}$ satisfy the equations $L_k(f) = 0$ and $L_{-k}(f) = 0$ respectively. In the notation of (5), (11) becomes

$$a_{00}+A_{10}+a_{11}+A_{21}+a_{22}+\cdots \text{ (to } n \text{ terms)},$$

so that this sum of n terms is a member of the family characterised by the sum of the n terms

$$a_{00}+a_{10}+a_{11}+a_{21}+a_{22}+\cdots \text{ (to } n \text{ terms)}. \tag{12}$$

If (11) is to be a general solution of (1), then (12) must be a standard solution of (1). The set of terms making up (12) will be called Almansi's set and the points representing these terms make up a set to be called Almansi's set of points. These form a staircase as shown in figure 3.

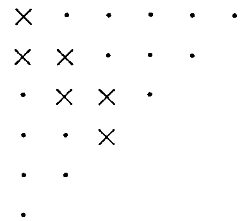
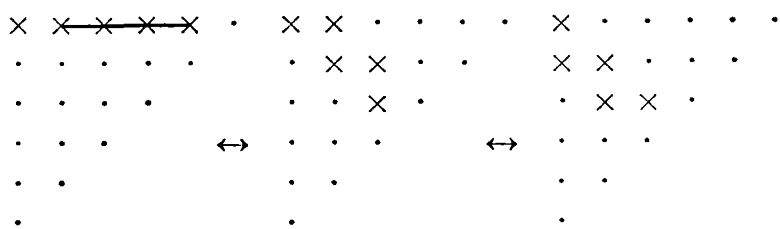


Figure 3

THEOREM. *Almansi's set of points in $\Delta_n(0, 0)$ is a standard set.*

The theorem is trivial when $n = 1$; and, when $n = 2$, Almansi's solution is identical with Payne's as is seen by comparing (9) and (12). The theorem is now proved by induction. It is assumed that the Weinstein and Almansi sets of points in the array $\Delta_{m-1}(0, 0)$ are equivalent. The translation principle shows that these sets are also equivalent in any array $\Delta_{m-1}(\alpha, \beta)$.



(*a*) (*b*) (*c*)

Figure 4

Consider the Weinstein set of m points from $\Delta_m(0, 0)$ in two parts as indicated in figure 4(*a*), the first point and the remaining $(m-1)$ points being taken separately. The latter points form the Weinstein set of the array $\Delta_m(0, 1)$ and so, from the inductive hypothesis, are equivalent to the Almansi set of $\Delta_m(0, 1)$. The original set of m points is thus transformed into a new standard set, as shown in figure 4(*b*), which by the reflection principle for $\Delta_m(0, 0)$ are equivalent to the set of points shown in figure 4(*c*). These form the Almansi set for $\Delta_m(0, 0)$ which is the result required to set up the inductive proof.

This equivalence relation between the Weinstein and Almansi sets for any array will be referred to as a Weinstein-Almansi relation. The

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theorem shows that the expression (11) is a general solution of (1) and provides an alternative proof of the equivalence of Weinstein's and Almansi's solutions for the polyharmonic equation.

4.5 Payne relations.

All the results of section 4 so far have been valid for all values of k. Some further results are now obtained which are subject to some restriction on k and for this purpose a preliminary result is needed.

Payne's lemma. *Provided* $k \neq 1$, $f_k \to f_{k-2}+y^2 f_{k+2}$.

For $k \neq 1$, f_k can be written as

$$f_k = \frac{1}{k-1}\left[y\frac{\partial f_k}{\partial y} + (k-1)f_k\right] - \frac{y^2}{k-1}\mathscr{D}f_k.$$

The result follows from Weinstein's first lemma (3.1) and the case $n = 1$ of another result given by Weinstein [4]: for any integer $n \geqq 0$,

$$f_k \leftrightarrow y^{1-k}\mathscr{D}^n(y^{k+2n-1}f_{k+2n}).$$

(The relation is not reversible: it is *not* the case that $f_{k-2}+y^2 f_{k+2} \to f_k$ because, in replacing the terms f_{k-2} and $y^2 f_{k+2}$ by expressions involving f_k, there can be no guarantee that the arbitrarily chosen f_{k-2} and f_{k+2} will give rise to the same function f_k as is necessary if the resulting expression is to reduce simply to f_k.)

This result is called Payne's Lemma because it was suggested by Payne's proof [3] that the expression given in (4) is equivalent to Payne's general solution (3) of equation (1) and so is itself a general solution of (1). The lemma is now used to prove a more general theorem which includes this result.

The terms making up Payne's general solution (3) will now be called Payne's first set to distinguish them from Payne's second set, the terms which make up the expression (4). The sets of points representing these terms in the array $\Delta_n(0, 0)$ (see figure 1 (c), (d)) will be called Payne's first and second sets respectively and the corresponding sets of points for any array $\Delta_m(\alpha, \beta)$ are obtained by translation.

Theorem. *Payne's first and second sets of points for the array* $\Delta_m(\alpha, \beta)$ *are equivalent, provided* $k \neq 1-2\alpha+2\beta-2i$ *where* $i = 0, 1, 2, \cdots (m-2)$.

The theorem is proved first for $n = 2$ and for $\alpha = \beta = 0$. If $y^2 f_{k+2}$, where f_{k+2} is arbitrary, is added to both sides of the relation of Payne's lemma, then, provided $k \neq 1$,

$$f_k+y^2 f_{k+2} \to f_{k-2}+y^2 f_{k+2}, \tag{13}$$

because the sum of any two functions f_{k+2} can be expressed simply as f_{k+2}.

On the other hand, (5) shows that f_{k-2} is a solution of $L_k^2(f) = 0$ so that Payne's first solution (3) gives

$$f_{k-2} \to f_k + y^2 f_{k+2}.$$

Adding $y^2 f_{k+2}$ to both sides of this relation gives

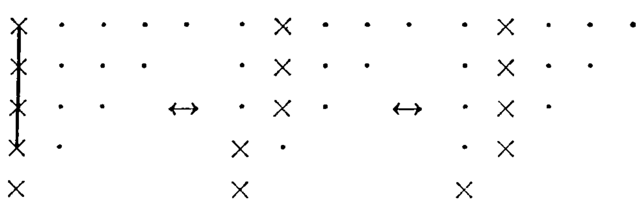
$$f_{k-2} + y^2 f_{k+2} \to f_k + y^2 f_{k+2}. \tag{14}$$

Relations (14) and (15) together show that, provided $k \neq 1$,

$$f_k + y^2 f_{k+2} \leftrightarrow f_{k-2} + y^2 f_{k+2},$$

so that the two Payne sets are equivalent for $\Delta_2(0, 0)$. The translation principle is now used to derive the corresponding result for $\Delta_2(\alpha, \beta)$ and since this involves changing k to $k+2\alpha-2\beta$, the result is that, provided $k \neq 1-2\alpha+2\beta$, the two Payne sets are equivalent for the array $\Delta_2(\alpha, \beta)$. The theorem is thus true for $n = 2$ and the general result is proved by induction.

Assume that the Payne sets for any array $\Delta_{m-1}(\alpha, \beta)$ are equivalent provided $k \neq 1-2\alpha+2\beta-2i$ for $i = 0, 1, 2, \cdots (m-3)$. Consider the first Payne set for the array $\Delta_m(\alpha, \beta)$ in two parts as indicated in figure 5(a),



(a) (b) (c)

Figure 5

the first $(m-1)$ points and the last point being considered separately. The first $(m-1)$ points form the first Payne set for $\Delta_{m-1}(\alpha, \beta)$ and so, from the inductive hypothesis, are equivalent to the second Payne set for this array. The original set of m points is thus transformed into a new set, shown in figure 5(b), which is equivalent to the original set, provided $k \neq 1-2i$ where $i = 0, 1, 2, \cdots (m-3)$. In this new set there are two points remaining in the first column of $\Delta_m(\alpha, \beta)$ and these form the first Payne set of the array $\Delta_2(\alpha+m-2, \beta)$. Since the theorem has already been proved for an array of this size, these two points can be replaced by the second Payne set of this array provided $k \neq 1-2\alpha+2\beta-(m-2)$. This leads to the set of points shown in figure 5(c), the second Payne set for $\Delta_m(\alpha, \beta)$ and the equivalence of the two Payne sets is established for $k \neq 1-2\alpha+2\beta-2i$ where $i = 0, 1, 2, \cdots (m-2)$. The inductive proof can now be completed.

Payne's relation can be applied successively to produce a series of standard sets of points for $\Delta_m(\alpha, \beta)$, starting with Payne's first set, as indicated in figure 6.

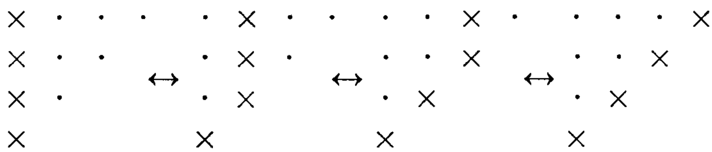


Figure 6

These successive transformations are valid provided $k \neq 1-2\alpha+2\beta-2i$ where i is an integer which for the first transformation lies in $0 \leqq i \leqq m-2$, for the second in $1 \leqq i \leqq m-2$, for the third in $2 \leqq i \leqq m-2$ and so on. The first condition includes all the others so all of the sets are standard sets provided this condition holds.

5. General solutions of $L_k^n(f) = 0$

The main purpose of this paper is to decide which sets of n terms chosen from the $\frac{1}{2}n(n+1)$ terms $a_{\alpha\beta}$ give standard solutions of equation (1). This is done by examining the geometrical patterns of sets of n points chosen from the array $\Delta_n(0, 0)$ and deciding whether these are standard sets. A criterion will be given in terms of an operation ω_n defined as follows: the operation ω_n, applied to an array Δ_n and a set of n points chosen from it, consists of the removal from Δ_n of the n points which lie along one of its three sides in such a way that just one of the chosen points is removed. If it is possible to apply the operation ω_n, an array Δ_{n-1} remains which contains $(n-1)$ of the chosen points and it may or may not be possible to apply the operation ω_{n-1}.

The required criterion states: provided $k \neq 1-2i$ where $i = 0, 1, 2, \cdots (n-2)$, a set of n points chosen from the array $\Delta_n(0, 0)$ form a standard set if it is possible to apply in succession the operations $\omega_n, \omega_{n-1}, \cdots, \omega_2$. (This series of operations will be denoted by Ω_n.)

The criterion is a special case of the following theorem which, being more general, is rather more easily proved.

THEOREM. *Provided $k \neq 1-2\alpha+2\beta-2i$ where $i = 0, 1, 2, \cdots (m-2)$, a set of m points chosen from the array $\Delta_m(\alpha, \beta)$ form a standard set if it is possible to apply the series of operations Ω_m.*

For $m = 2$, there are three possible choices of two points from $\Delta_2(\alpha, \beta)$ and, as shown in figure 7, the operation ω_2 can be applied to each of them. (a) is Weinstein's set, (b) is Payne's first set and (c) is Payne's

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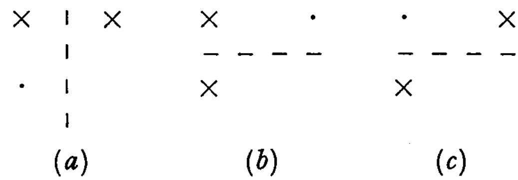

Figure 7

second set. (*a*) and (*b*) are known to be standard sets for all k while (*c*) has been proved to be a standard set for $k \neq 1-2\alpha+2\beta$ (section 4.5). The theorem is thus true for $m = 2$ and the general result is proved by induction.

On the assumption that the theorem is true for any array $\Delta_{m-1}(\alpha, \beta)$, it must be proved that the set of m chosen points form a standard set in $\Delta_m(\alpha, \beta)$ if the operations Ω_m can be applied. These operations are applied in two phases, ω_m first and then the remaining operations Ω_{m-1}, and there are three cases to consider according as the array which remains after the operation of ω_m is $\Delta_{m-1}(\alpha, \beta)$, $\Delta_{m-1}(\alpha+1, \beta)$ or $\Delta_{m-1}(\alpha, \beta+1)$.

(i) If the chosen point removed by ω_m represents a term of the form $a_{\alpha+r,\beta+m-1-r}$, the remaining $(m-1)$ chosen points lie in $\Delta_{m-1}(\alpha, \beta)$. Consider the cases $0 \leqq r \leqq [\frac{1}{2}(m+1)]$ as shown in figure 8 (for the case $m = 5$).

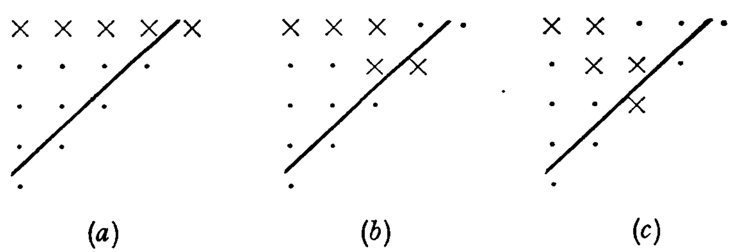

Figure 8

The $(m-1)$ chosen points which remain in $\Delta_{m-1}(\alpha, \beta)$ are such that the operations Ω_{m-1} can be applied to them and so, by the inductive hypothesis, provided $k \neq 1-2\alpha+2\beta-2i$ where $i = 0, 1, 2, \cdots (m-3)$, they form a standard set. They are therefore equivalent to any other standard set and in particular to Weinstein's set, as in figure 8(*a*), or to standard sets which can be derived from Weinstein's set by the application of the Weinstein-Almansi relation, as in figure 8(*b*) and (*c*). These standard sets in $\Delta_{m-1}(\alpha, \beta)$ are constructed so that they combine with the chosen point which has been excluded by ω_m to produce in each case a set of m points in $\Delta_m(\alpha, \beta)$ which is a standard set because it is Weinstein's set for this array or can be transformed to this set by the use of a Weinstein-Almansi relation. This shows

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that the original set of m chosen points is equivalent to a standard set and so is itself a standard set. The inductive proof can now be completed for this case.

If r is such that $[\frac{1}{2}(n+1)] < r \leqq m-1$, the set of m points is first reflected in the leading diagonal to obtain a set of the type just considered, this new set is shown to be a standard set and the extended reflection principle then shows that the original set is a standard set.

It will be noted that only Weinstein-Almansi relations have been used so far and that these introduce no restrictions on k. Hence if all the operations $\omega_m, \omega_{m-1}, \cdots, \omega_2$ are of the kind considered in this section of the proof, the theorem can be proved without restrictions on k.

(ii) In the second case, the chosen point removed by ω_m represents a term of the form $a_{\alpha,\beta+r}$ where $0 \leqq r \leqq (m-1)$ and the remaining $(m-1)$ chosen points lie in $\Delta_{m-1}(\alpha+1, \beta)$. The operations Ω_{m-1} can be applied to these points, so, by the inductive hypothesis, provided $k \neq 1-2\alpha+2\beta-2i$ where $i = 1, 2, \cdots (m-2)$, these points form a standard set in $\Delta_{m-1}(\alpha+1,\beta)$. They are therefore equivalent to any other standard set and in particular to Payne's first set or to one of the sets which were shown in section 4.5 to be equivalent to it (these sets are shown in figure 6). These sets are all standard sets under the restriction already imposed on k. The appropriate sets are shown in figure 9 for the case $m = 4$ and are constructed so that

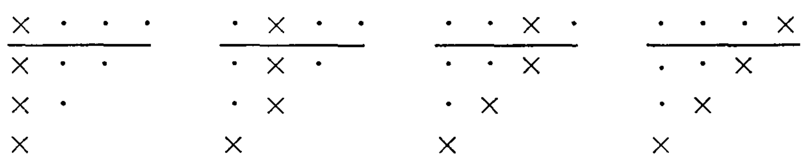

Figure 9

they combine with the point excluded by ω_m to give sets of m points in $\Delta_m(\alpha, \beta)$ which are standard sets, again because they are sets which have been shown to be equivalent to Payne's first set, provided that $k \neq 1-2\alpha+2\beta-2i$ where $i = 0, 1, 2, \cdots (m-2)$. Thus the original set of m points is a standard set provided this condition is satisfied. The inductive proof can now be completed.

(iii) The third case, when ω_m removes a point which represents a term of the form $a_{\alpha+r,\beta}$ can be reduced to the previous case by the use of the extended reflection principle.

The three cases having been considered, the theorem is proved. The restrictions imposed on k are strong enough to allow the use of any Payne relation which may be needed. In particular cases, a set of points in $\Delta_m(\alpha, \beta)$ to which the operations Ω_m can be applied may be a standard set for values of k not included in the statement of the theorem. The extreme

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case when no Payne relations are needed and there are no restrictions on k has already been noted.

It is natural to ask how many of the possible sets of n points chosen from the array $\Delta_n(0, 0)$ satisfy the criterion and so form standard sets (under appropriate conditions on k), thus leading to general solutions of the equation $L_k^n(f) = 0$. It appears not to be a trivial matter to find this number for general n. When $n = 2$, it has already been noted that all three possible sets of two points chosen from $\Delta_2(0, 0)$ are standard sets; when $n = 3$, 16 of the 20 possible sets of 3 points chosen from $\Delta_3(0, 0)$ satisfy the criterion; and when $n = 4$, 119 of the 210 possible sets of 4 points chosen from $\Delta_4(0, 0)$ satisfy the criterion and so are standard sets.

What can be said about the sets of points which do not satisfy the criterion? The four sets for the case $n = 3$ which do not satisfy the criterion are shown in figure 10.

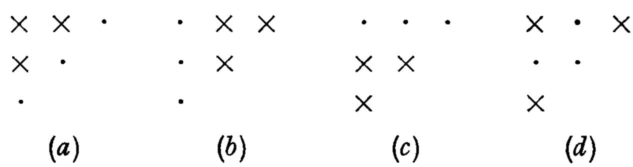

Figure 10

The sets shown in figures 10(a), (b), (c) are not standard sets of $\Delta_3(0, 0)$. Since, in each case, the three points lie in an array Δ_2, any two of them form a standard set in this array. This means that one of the terms represented by the points can be expressed as a combination of the other two so that there are in effect only two independent terms which clearly do not form a general solution of $L_k^n(f) = 0$. The set shown in figure 10(d) cannot be dismissed so easily and it remains an open question whether or not this forms a standard set.

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The Australian National University
Canberra, A.C.T.

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