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Ground State and Multiple Solutions for Kirchhoff Type Equations With Critical Exponent

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Abstract. In this paper, we consider the following critical Kirchhoff type equation:

$$\begin{cases} -(a+b\int_\Omega |\nabla u|^2)\Delta u = Q(x)|u|^4 u + \lambda |u|^{q-1}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

By using variational methods that are constrained to the Nehari manifold, we prove that the above equation has a ground state solution for the case when $3 < q < 5$. The relation between the number of maxima of Q and the number of positive solutions for the problem is also investigated.

1 Introduction and Main Results

Consider the following Kirchhoff type equation:

$$\begin{cases} -(a+b\int_\Omega |\nabla u|^2)\Delta u = Q(x)|u|^4 u + \lambda |u|^{q-1}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $a, b > 0$, $3 < q < 5$, $Q \in C(\overline{\Omega}, (0, +\infty))$, $\lambda > 0$ is a parameter, and $\Omega \subset \mathbb{R}^3$ is an open bounded domain with C^1-smooth boundary. Problem (1.1) has been widely investigated in the literature over the past several decades, especially on the existence and multiplicity of positive solutions, ground states, and sign-changing solutions (see [2, 4, 9, 13, 15, 27, 29, 33, 36, 40] and the references therein).

This problem is related to the stationary analogue of the following wave equation proposed by Kirchhoff [18],

$$u_{tt} - \Big(a + b\int_\Omega |\nabla u|^2 \mathrm{d}x\Big)\Delta u = f(x,u), \tag{1.2}$$

where u denotes the displacement, $f(x,u)$ the external force, and a the initial tension, while b is related to the intrinsic properties of the string, such as Young's modulus. Because of the presence of the nonlocal term $(\int_\Omega |\nabla u|^2 \mathrm{d}x)\Delta u$, equation (1.2) is not a pointwise identity, which provokes some mathematical difficulties and makes the study of such a class of problems particularly interesting. Nonlocal effect also finds its applications in modelling suspension bridges [1] and describing the growth and

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movement of a particular species in biological systems [10]. For more mathematical and physical background of problem (1.1), see [5, 11, 18] and the references therein.

When $b = 0$, equation (1.1) is reduced to the well-known scalar Schrödinger equation

$$\begin{cases} -\Delta u = Q(x)u^{2^*-1} + \lambda u^q, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, and

$$2^* = \begin{cases} \dfrac{2N}{N-2} & \text{if } N \geq 3, \\ +\infty & \text{if } N = 1, 2. \end{cases}$$

Let us recall some results. Brézis and Nirenberg studied (1.3) with $Q(x) \equiv 1$, and obtained some classic results [7]. When $N = 3$, if $1 < q \leq 3$, they obtained that (1.3) has a solution for $\lambda > 0$ large enough; if $3 < q < 2^* - 1 = 5$, (1.3) has a solution for every $\lambda > 0$; if $q = 1$ and Ω is a ball, (1.3) has a solution if and only if $\lambda \in (\frac{\lambda_1}{4}, \lambda_1)$, where λ_1 is the first eigenvalue of $-\Delta$ with the Dirichlet condition on Ω. Ambrosetti, Brévis, and Cerami [4] considered the case $0 < q < 1$ and showed that there exists a $\lambda_0 > 0$ such that (1.3) admits two positive solutions for $\lambda \in (0, \lambda_0)$. When $q = Q(x) \equiv 1$ and $N \geq 7$, infinitely many nontrivial solutions was proved by Devillanova and Solimini [12] for $\lambda > 0$. Later, this result was generalized to p-Laplacian by Cao, Peng, and Yan [9] with the aid of the local Pohozaev identity. Recently, Liao, Liu, Zhang, and Tang studied (1.3) and obtained the existence and multiplicity of positive solutions via the Nehari manifold method [21]. For more related results, see [6, 8, 32, 35, 39] and the references therein.

When $b \neq 0$, a vast literature on the study of the existence and multiplicity of solutions for Kirchhoff type equations via variational methods has grown since Lions introduced an abstract framework for this problem [23].

Replacing the right-hand side of equation (1.1) by $f(x,u)$, we are led to following equation:

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2 \mathrm{d}x\right)\Delta u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

When the primitive of f is subcritical, Corrêa [11] obtained positive solutions for both $N = 1$ and $N \geq 2$ via fixed point theorems; He and Zou [15] established infinitely many solutions under the classical (AR) condition or the following Nehari type condition:

$$t \mapsto \frac{f(x,t)}{|t|^3} \text{ is increasing on } (-\infty, 0) \cup (0, \infty). \tag{Ne}$$

See also [16] where multiplicity results were obtained when f has an oscillating behavior. Perera and Zhang [27] considered the asymptotically 4-linear case provided that f satisfies

$$\lim_{t\to 0} \frac{f(x,t)}{at} = \lambda, \quad \lim_{t\to\infty} \frac{f(x,t)}{bt^3} = \mu \text{ uniformly in } x.$$

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Using the Yang index and critical group, they showed that if $\lambda \in (\lambda_l, \lambda_{l+1})$ and $\mu \in (\mu_m, \mu_{m+1})$ with $l \neq m$, then (1.4) has a nontrivial solution, where $0 < \lambda_1 < \lambda_2 \le \cdots$ is the sequence of all eigenvalues of $-\Delta$ in $H_0^1(\Omega)$ and $0 < \mu_1 \le \mu_2 \le \cdots$ is the sequence of all eigenvalues of the problem

$$\begin{cases} -\left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)\Delta u = \mu u^3, & \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{cases}$$

We also considered the asymptotically 4-linear case and proved the existence of ground state solution for (1.4) by improving the Nehari manifold method [28]. More precisely, a homeomorphism between a subset of the unit sphere and the Nehari manifold (defined later by (2.12)) is constructed, from which a minimizing sequence on the Nehari manifold can be found naturally. For the case of 4-sublinear or 4-superlinear, positive, negative, and sign-changing solutions were all obtained by Zhang and Perera [40] by using invariant sets of descent flow. See also [26] for similar results. Recently, Chen, Kuo, and Wu [10] investigated (1.4) for the case that

$$f(x,t) = \lambda h(x)|t|^{q-2}t + g(x)|t|^{p-2}t, \quad 1 < q < 2 < p < 2^*,$$

where $h, g \in C(\overline{\Omega})$ satisfy $h^+ := \max\{h, 0\} \not\equiv 0$ and $g^+ := \max\{g, 0\} \not\equiv 0$. By using the Nehari manifold and fibering map methods, some existence results were obtained for cases $p > 4$, $p = 4$, and $p < 4$, respectively. For autonomous nonlinearity f satisfying (Ne), least energy sign-changing solutions to problem (1.4) were established recently by Shuai [31] with the aid of the quantitative deformation lemma and degree theory. Later, Tang and Cheng [34] improved this result by using the non-Nehari manifold method introduced in [32].

When the primitive of f is of critical growth having the following form

$$f(x,t) = t^5 + \lambda g(x,t) \quad \text{and} \quad 0 < \rho \int_0^t g(x,s)\mathrm{d}s \le g(x,t)t \text{ with } 4 < \rho < 6, \tag{1.5}$$

Alves, Corrêa, and Ma [2] showed that there exists a $\lambda_* > 0$ such that problem (1.4) has a positive solution of mountain pass type for all $\lambda \ge \lambda_*$. Later, Hamydy, Massar, and Tsouli [14] extended this result to a p-Kirchhoff type problem. Under assumption (1.5) with $5 < \rho < 6$ and $\lambda = 1$, Xie, Wu, and Tang [36] proved that (1.4) has two distinct solutions. Figueiredo [13] improved the result of [2] by relaxing ρ to the interval $(2, 6)$. Moreover, by applying a truncation argument used in [3], they showed that there exists a $\lambda_* > 0$ such that (1.4) has a positive solution for all $\lambda \ge \lambda_*$. In a recent paper Lei et al. [19] considered the case that $g(x,t) \equiv t^q$ with $0 < q < 1$ in (1.5). Assuming that $b > 0$ is sufficiently small and using the concentration compactness argument, they showed that there exists a $\lambda_* > 0$ such that (1.4) has two positive solutions for $0 < \lambda < \lambda_*$. For similar Kirchhoff type problems in the whole space, see [17, 20, 25, 37] and the references therein.

Motivated by above works, we continue to study (1.1), that is,

$$f(x,t) = Q(x)|t|^4 t + \lambda |t|^{q-1} t,$$

for all $3 < q < 5$ in (1.4). When $Q \not\equiv$ const, problem (1.1) is more delicate. The main difficulty lies in the analysis of a (PS) sequence, that is, a sequence $\{u_n\} \in H_0^1(\Omega)$ such that $\{I_\lambda(u_n)\}$ is bounded and $I'_\lambda(u_n) \to 0$, where I_λ is the corresponding energy functional of (1.1). In order to obtain a positive solution, $\lambda > 0$ needs to be large

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enough in [2, 13]. Note that there is no restriction on λ in [7] when $3 < q < 5$. It is natural to ask whether a similar result of Brézis and Nirenberg [7] holds for (1.1). In this paper we not only give a positive answer, but also obtain the multiplicity result of (1.1) for $\lambda > 0$ small.

For positive Q, we introduce the following assumptions.

(Q1) there exists $a^0 \in \Omega$ such that $Q(a^0) = Q_M = \max_{x\in\overline{\Omega}} Q(x)$ and

$$|Q(x) - Q(a^0)| = o(|x - a^0|^{\frac{5-q}{2}}), \quad \text{as } x \to a^0.$$

(Q2) there exist k points $a^1, a^2, \dots, a^k$ in Ω such that $Q(a^i)$ are strict local maxima satisfying $Q(a^i) = Q_M = \max_{x\in\overline{\Omega}} Q(x)$ and

$$|Q(x) - Q(a^i)| = o(|x - a^i|^{\frac{5-q}{2}}), \quad \text{as } x \to a^i,$$

for every $i = 1, 2, \dots, k$.

Our main results are as follows.

Theorem 1.1 *Let* (Q1) *be satisfied. Then problem* (1.1) *has a positive ground state solution for all* $\lambda > 0$.

Theorem 1.2 *Let* (Q2) *be satisfied. Then there exists* $\widetilde{\lambda} > 0$ *such that problem* (1.1) *has at least k positive solutions for all* $0 < \lambda < \widetilde{\lambda}$.

Conditions like or similar to (Q1) and (Q2) are commonly used in the study of critical problems (see[8, 21, 22] and the references therein).

Remark 1.3 Theorems 1.1 and 1.2 improve and extend related results of [2, 13, 19, 36] in the sense that not only the ground state solution of (1.1) is proved, but also the restriction that λ should be large enough in [2, 13] is removed. Moreover, we also obtain the multiplicity of positive solutions of (1.1) for $\lambda > 0$ small.

To complete this section, we sketch our proof. By using the mountain pass theorem, we can find a $(\mathrm{PS})_{c_\lambda}$ sequence, where the mountain pass level c_λ is defined later by (2.19). So it is crucial to find a proper energy level c^* below which the (PS) condition can be verified. This can be done by some delicate analysis with the aid of (Q1). To show the multiplicity of positive solutions of (1.1), we minimize the energy functional corresponding to (1.1) on some submanifold of the Nehari manifold. Define the minimal energy by $c_{\lambda,i}$ as in (3.3); the main difficulty is to find and estimate the $(\mathrm{PS})_{c_{\lambda,i}}$ sequence. We solve the problem by using some techniques and the Nehari manifold method.

This paper is organized as follows. In Section 2, we prove the existence of a ground state solution of (1.1) via the mountain pass theorem. By considering a barycenter function restricted to the Nehari manifold, we investigate the multiplicity of positive solutions of (1.1) in Section 3. Some remarks will be given in Section 4.

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2 Proof of Theorem 1.1

The natural space for the Kirchhoff type equation (1.1) is the Sobolev space $H_0^1(\Omega)$ equipped with the inner product $(u,v)=\int_\Omega \nabla u\cdot\nabla v\mathrm{d}x$, for all $u,\ v\in H_0^1(\Omega)$, and the norm $\|u\|=(u,u)^{1/2}$. It is well known that $H_0^1(\Omega)$ is continuously embedded in $L^s(\Omega)$ for $s\in[1,6]$, and compactly for $s\in[1,6)$ (see [38, Theorem 1.9]). Then there exists a $\gamma_s>0$ such that

$$|u|_s\le\gamma_s\|u\|,\quad\forall\ u\in H_0^1(\Omega),1\le s\le 6, \tag{2.1}$$

where $|\cdot|_s$ denotes the usual $L^s(\Omega)$ norm. Solutions of (1.1) are critical points of the functional

$$I_\lambda(u)=\frac{a}{2}\|u\|^2+\frac{b}{4}\|u\|^4-\frac{1}{6}\int_\Omega Q(x)|u|^6\mathrm{d}x-\frac{\lambda}{q+1}\int_\Omega|u|^{q+1}\mathrm{d}x,\quad u\in H_0^1(\Omega).$$

By virtue of (2.1), I_λ is of class C^1 and

$$\langle I'_\lambda(u),v\rangle=(a+b\|u\|^2)(u,v)-\int_\Omega Q(x)|u|^4uv\mathrm{d}x-\lambda\int_\Omega|u|^{q-1}uv\mathrm{d}x,\quad u,v\in H_0^1(\Omega).$$

Let S be the best Sobolev constant (see [38, Proposition 1.43]):

$$S=\inf_{u\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}}\frac{\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x}{(\int_{\mathbb{R}^N}|u|^6\mathrm{d}x)^{1/3}}=\inf_{u\in H_0^1(\Omega)\setminus\{0\}}\frac{\int_\Omega|\nabla u|^2\mathrm{d}x}{(\int_\Omega|u|^6\mathrm{d}x)^{1/3}}. \tag{2.2}$$

Recall that S is attained by the function $U_\varepsilon(x)=\frac{(3\varepsilon^2)^{1/4}}{(\varepsilon^2+|x|^2)^{1/2}}$, $\varepsilon>0$. Moreover,

$$\int_{\mathbb{R}^N}|\nabla U_\varepsilon|^2\mathrm{d}x=\int_{\mathbb{R}^N}|U_\varepsilon|^6\mathrm{d}x=S^{\frac{3}{2}}.$$

For $0\le i\le k$, let $\varphi_i\in C_0^\infty(\Omega)$ be a cut-off function such that $0\le\varphi_i\le1$ and $|\nabla\varphi_i|\le C_0$, and for some $\delta_0>0$,

$$\varphi_i(x)=\begin{cases}1 & |x-a^i|\le\frac{\delta_0}{2},\\ 0 & |x-a^i|\ge\delta_0.\end{cases} \tag{2.3}$$

Then we define

$$v_\varepsilon^i(x)=\varphi_i(x)U_\varepsilon(x-a^i),\quad i=1,2,\dots,k. \tag{2.4}$$

Since we mainly consider the case $q\in(3,5)$, we make use of it without mention when no confusion can arise.

Lemma 2.1 *Let* (Q1) *be satisfied. Then for* $\varepsilon>0$ *small,*

$$\sup_{t\ge0}I_\lambda(tv_\varepsilon^0(x))<c^*:=\frac{a}{3}\Big(\frac{bS^3+\sqrt{b^2S^6+4aQ_MS^3}}{2Q_M}\Big)+\frac{b}{12}\Big(\frac{bS^3+\sqrt{b^2S^6+4aQ_MS^3}}{2Q_M}\Big)^2.$$

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Proof From [7, 38], we have following results

$$(2.5)\qquad \begin{aligned} |v_\varepsilon^0|_6^2 &= \Big(\int_{\mathbb{R}^N} |U_\varepsilon|^6 \mathrm{d}x\Big)^{1/3} + O(\varepsilon^3) = S^{1/2} + O(\varepsilon^3),\\ \|v_\varepsilon^0\|^2 &= \int_{\mathbb{R}^N} |\nabla U_\varepsilon|^2 \mathrm{d}x + O(\varepsilon) = S^{3/2} + O(\varepsilon),\end{aligned}$$

and

$$(2.6)\qquad \int_\Omega |v_\varepsilon^0|^s \mathrm{d}x = \begin{cases} O(\varepsilon^{\frac{3}{2}}|\ln\varepsilon|) & s = 3,\\ O(\varepsilon^{\frac{6-s}{2}}) & 3 < s < 6,\\ O(\varepsilon^{\frac{s}{2}}) & 1 \le s < 3.\end{cases}$$

Note that

$$(2.7)\quad I_\lambda(tv_\varepsilon^0) = \frac{a}{2}t^2\|v_\varepsilon^0\|^2 + \frac{b}{4}t^4\|v_\varepsilon^0\|^4 - \frac{t^6}{6}\int_\Omega Q(x)|v_\varepsilon^0|^6\mathrm{d}x - \frac{\lambda}{q+1}t^{q+1}\int_\Omega |v_\varepsilon^0|^{q+1}\mathrm{d}x,$$

and

$$\frac{dI_\lambda(tv_\varepsilon^0)}{dt} = at\|v_\varepsilon^0\|^2 + bt^3\|v_\varepsilon^0\|^4 - t^5\int_\Omega Q(x)|v_\varepsilon^0|^6\mathrm{d}x - \lambda t^q\int_\Omega |v_\varepsilon^0|^{q+1}\mathrm{d}x.$$

From $p \in (3,5)$, we deduce that $I_\lambda(tv_\varepsilon^0) \to 0$ as $t \to 0$ and $I_\lambda(tv_\varepsilon^0) \to -\infty$ as $t \to +\infty$. Moreover, there exists a unique $t_\varepsilon > 0$ such that $I_\lambda(tv_\varepsilon^0)$ achieves its maximum. We claim that there exist two constants $T_1, T_2 > 0$ such that $T_1 < t_\varepsilon < T_2$. In fact, from $\lim_{t\to 0} I_\lambda(tv_\varepsilon^0) = 0$ uniformly for all $\varepsilon > 0$, we choose

$$\xi = \frac{I_\lambda(t_\varepsilon v_\varepsilon^0)}{4} > 0.$$

Then there exists $T_1 > 0$ independent of ε such that $|I_\lambda(T_1v_\varepsilon^0)| = |I_\lambda(T_1v_\varepsilon^0) - I_\lambda(0)| < \xi$. According to the monotonicity of $I_\lambda(tv_\varepsilon^0)$ near $t = 0$, we have $t_\varepsilon > T_1$. Similarly, from $\lim_{t\to+\infty} I_\lambda(tv_\varepsilon^0) = -\infty$ uniformly for all $\varepsilon > 0$, we have $t_\varepsilon < T_2$.

Set

$$(2.8)\qquad B_\varepsilon = \frac{\|v_\varepsilon^0\|^2}{(\int_\Omega Q(x)|v_\varepsilon^0|^6\mathrm{d}x)^{1/3}}$$

and let $h(t) := \frac{a}{2}t^2\|v_\varepsilon^0\|^2 + \frac{b}{4}t^4\|v_\varepsilon^0\|^4 - \frac{t^6}{6}\int_\Omega Q(x)|v_\varepsilon^0|^6\mathrm{d}x$. Then

$$h'(t) = t\Big(a\|v_\varepsilon^0\|^2 + bt^2\|v_\varepsilon^0\|^4 - t^4\int_\Omega Q(x)|v_\varepsilon^0|^6\mathrm{d}x\Big),$$

and $h(t)$ attains its maximum at

$$t_{\max} = \left(\frac{b\|v_\varepsilon^0\|^4 + \sqrt{b^2\|v_\varepsilon^0\|^8 + 4a\|v_\varepsilon^0\|^2\int_\Omega Q(x)|v_\varepsilon^0|^6\mathrm{d}x}}{2\int_\Omega Q(x)|v_\varepsilon^0|^6\mathrm{d}x}\right)^{1/2}.$$

Moreover, $h'(t) > 0$ for all $0 < t < t_{\max}$ and $h'(t) < 0$ for all $t > t_{\max}$. Direct computation implies that

$$
\begin{aligned}
(2.9)\quad h(t_{\max}) &= \frac{a\|v_\varepsilon^0\|^2\left(b\|v_\varepsilon^0\|^4 + \sqrt{b^2\|v_\varepsilon^0\|^8 + 4a\|v_\varepsilon^0\|^2\int_\Omega Q(x)|v_\varepsilon^0|^6\mathrm{d}x}\right)}{6\int_\Omega Q(x)|v_\varepsilon^0|^6\mathrm{d}x} \\
&\quad + \frac{b\|v_\varepsilon^0\|^4(b\|v_\varepsilon^0\|^4 + \sqrt{b^2\|v_\varepsilon^0\|^8 + 4a\|v_\varepsilon^0\|^2\int_\Omega Q(x)|v_\varepsilon^0|^6\mathrm{d}x})^2}{12(2\int_\Omega Q(x)|v_\varepsilon^0|^6\mathrm{d}x)^2} \\
&= \frac{a}{3}\Big(\frac{bB_\varepsilon^3 + \sqrt{b^2B_\varepsilon^6 + 4aB_\varepsilon^3}}{2}\Big) + \frac{b}{12}\Big(\frac{bB_\varepsilon^3 + \sqrt{b^2B_\varepsilon^6 + 4aB_\varepsilon^3}}{2}\Big)^2.
\end{aligned}
$$

Let $\alpha := \frac{5-q}{2}$. Then $\alpha \in (0,1)$ by $q \in (3,5)$. In view of (Q1),

$$|Q(x) - Q(a^0)| = o(|x - a^0|^\alpha).$$

Then for any $\eta > 0$, there exists a $\delta > 0$ such that

$$|Q(x) - Q(a^0)| < \eta|x - a^0|^\alpha, \quad \forall\ 0 < |x - a^0| < \delta. \tag{2.10}$$

We claim that

$$\Big(\int_\Omega Q(x)|v_\varepsilon^0|^6\mathrm{d}x\Big)^{1/3} = Q_M^{\frac{1}{3}}|v_\varepsilon^0|_6^2 + o(\varepsilon^\alpha). \tag{2.11}$$

Indeed, for $0 < \varepsilon < \delta$, it follows from (2.3), (2.4), and (2.10) that

$$
\begin{aligned}
\Big|\int_\Omega Q(x)|v_\varepsilon^0|^6\mathrm{d}x - \int_\Omega Q_M|v_\varepsilon^0|^6\mathrm{d}x\Big| &\le \int_\Omega |Q(x) - Q(a^0)|\,|v_\varepsilon^0|^6\mathrm{d}x \\
&= \int_{\{x\in\Omega:|x-a^0|\le\delta_0\}} |Q(x) - Q(a^0)||v_\varepsilon^0|^6\mathrm{d}x \\
&\le \int_{\{x\in\Omega:|x-a^0|\le\delta\}} \eta|x - a^0|^\alpha \frac{(3\varepsilon^2)^{\frac{3}{2}}}{(\varepsilon^2 + |x - a^0|^2)^3}\mathrm{d}x \\
&\quad + 2Q_M\int_{\{x\in\Omega:\delta\le|x-a^0|\le\delta_0\}} \frac{(3\varepsilon^2)^{3/2}}{(\varepsilon^2 + |x - a^0|^2)^3}\mathrm{d}x \\
&\le C_1\eta\int_0^\delta \frac{\varepsilon^3 r^{2+\alpha}}{(\varepsilon^2 + r^2)^3}\mathrm{d}r + C_1\int_\delta^{\delta_0}\frac{\varepsilon^3 r^2}{(\varepsilon^2 + r^2)^3}\mathrm{d}r \\
&= C_1\eta\varepsilon^\alpha\int_0^{\frac{\delta}{\varepsilon}}\frac{r^{2+\alpha}}{(1 + r^2)^3}\mathrm{d}r + C_1\int_{\frac{\delta}{\varepsilon}}^{\frac{\delta_0}{\varepsilon}}\frac{r^2}{(1 + r^2)^3}\mathrm{d}r \\
&\le C_2\eta\varepsilon^\alpha + C_2\varepsilon^3,
\end{aligned}
$$

where $C_1, C_2 > 0$ are constants. From $\alpha \in (0,1)$, we get

$$\limsup_{\varepsilon\to0}\frac{\big|\int_\Omega Q(x)|v_\varepsilon^0|^6\mathrm{d}x - \int_\Omega Q_M|v_\varepsilon^0|^6\mathrm{d}x\big|}{\varepsilon^\alpha} \le C_2\eta,$$

which implies (2.11) since $\eta > 0$ is arbitrary.

By (2.5), (2.8), and (2.9), we have

$$B_\varepsilon = \frac{S^{3/2} + O(\varepsilon)}{Q_M^{\frac{1}{3}}(S^{1/2} + O(\varepsilon^3)) + o(\varepsilon^\alpha)}$$

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and

$$
\begin{aligned}
h(t_{\max}) &\le \frac{a}{3}\Big(\frac{bS^3+\sqrt{b^2S^6+4aQ_MS^3}}{2Q_M}\Big)+\frac{b}{12}\Big(\frac{bS^3+\sqrt{b^2S^6+4aQ_MS^3}}{2Q_M}\Big)^2+o(\varepsilon^\alpha)\\
&= c^*+o(\varepsilon^\alpha).
\end{aligned}
$$

Then it follows from (2.6), (2.7), and $T_1<t_\varepsilon<T_2$ that

$$
\begin{aligned}
\sup_{t\ge 0} I_\lambda(tv_\varepsilon^0) = I_\lambda(t_\varepsilon v_\varepsilon^0) &\le h(t_{\max})-\frac{\lambda}{q+1}T_1^{q+1}\int_\Omega |v_\varepsilon^0|^{q+1}\mathrm{d}x\\
&\le c^*+o(\varepsilon^\alpha)-C_3\varepsilon^\alpha<c^*,
\end{aligned}
$$

where $\varepsilon>0$ is small enough and C_3 depending on λ, q is a positive constant. This completes the proof. ■

Lemma 2.2 *Let $p\in(3,5)$. The following statements hold.*

(i) *If $\{u_n\}$ is a $(\mathrm{PS})_c$ sequence in $H_0^1(\Omega)$, then $u_n \rightharpoonup u$ for some $u\in H_0^1(\Omega)$ and $I'_\lambda(u)=0$. Moreover, if $u\not\equiv 0$, then $\int_\Omega|\nabla u_n|^2\mathrm{d}x\to\int_\Omega|\nabla u|^2\mathrm{d}x$.*

(ii) *For any $u\in H_0^1(\Omega)\setminus\{0\}$, there exists a unique $t_u>0$ such that $t_uu\in\mathcal{N}_\lambda$, and $I_\lambda(t_uu)=\max_{t\ge0}I_\lambda(tu)$, where the Nehari manifold $\mathcal{N}_\lambda$ is defined by*

$$
\mathcal{N}_\lambda := \{v\in H_0^1(\Omega)\setminus\{0\} : \langle I'_\lambda(v), v\rangle = 0\}. \tag{2.12}
$$

Note that for $p\in(3,5)$, the function

$$
u\longmapsto \frac{\lambda|u|^{q-1}u+Q(x)|u|^4u}{|u|^3}
$$

is increasing on $(-\infty,0)\cup(0,\infty)$. Taking advantage of this fact and using the Nehari manifold method, it is not difficult to verify (i) and (ii). See [20, Lemma 3.2] and [37, Lemma 2.2 (i)].

Lemma 2.3 *Let $c\in(-\infty,c^*)$. Then I_λ satisfies the $(\mathrm{PS})_c$ condition in $H_0^1(\Omega)$ for all $\lambda>0$.*

Proof Let $\{u_n\}\subset H_0^1(\Omega)$ be a $(\mathrm{PS})_c$ sequence satisfying

$$
I_\lambda(u_n)=c+o(1), \quad \text{and} \quad I'_\lambda(u_n)=o(1). \tag{2.13}
$$

Then for n large,

$$
\begin{aligned}
c+1+\|u_n\| &\ge I_\lambda(u_n)-\frac{1}{q+1}\langle I'_\lambda(u_n),u_n\rangle\\
&=\Big(\frac12-\frac{1}{q+1}\Big)a\|u_n\|^2+\Big(\frac14-\frac{1}{q+1}\Big)b\|u_n\|^4\\
&\quad+\Big(\frac{1}{q+1}-\frac16\Big)\int_\Omega Q(x)|u_n|^6\mathrm{d}x,
\end{aligned} \tag{2.14}
$$

which implies that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Passing to a subsequence, we may assume that

$$
u_n\rightharpoonup u \text{ in } H_0^1(\Omega),\quad u_n\to u \text{ in } L^s(\Omega), 1\le s<6, \quad\text{and}\quad u_n\to u \text{ a.e. on } \Omega. \tag{2.15}
$$

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Lemma 2.2 (i) implies that $I'_\lambda(u) = 0$. Set $v_n = u_n - u$. By Brezis–Lieb's lemma, we have

$$\|v_n\|^2 = \|u_n\|^2 - \|u\|^2 + o(1), \tag{2.16}$$
$$\int_\Omega Q(x)|v_n|^6 \mathrm{d}x = \int_\Omega Q(x)|u_n|^6 \mathrm{d}x - \int_\Omega Q(x)|u|^6 \mathrm{d}x + o(1).$$

Then we deduce from (2.13), (2.15), and (2.16) that

(2.17)
$$I_\lambda(u_n) - I_\lambda(u) = \frac{a}{2}\|v_n\|^2 + \frac{b}{4}(\|v_n\|^4 + 2\|v_n\|^2\|u\|^2) - \frac{1}{6}\int_\Omega Q(x)|v_n|^6 \mathrm{d}x + o(1),$$
$$o(1) = \langle I'_\lambda(u_n), u_n\rangle - \langle I'_\lambda(u), u\rangle$$
$$= a\|v_n\|^2 + b(\|v_n\|^4 + 2\|v_n\|^2\|u\|^2) - \int_\Omega Q(x)|v_n|^6 \mathrm{d}x + o(1).$$

We may assume there exist $l_i \geq 0$, $i = 1, 2, 3$ such that

$$a\|v_n\|^2 \to l_1, \quad b(\|v_n\|^4 + 2\|v_n\|^2\|u\|^2) \to l_2, \quad \int_\Omega Q(x)|v_n|^6 \mathrm{d}x \to l_3.$$

Clearly, $l_3 = l_1 + l_2$. Next, we show that $l_1 = 0$.

If $l_1 > 0$, then $l_2, l_3 > 0$. By Sobolev inequality, we have

$$a^3 \int_\Omega Q(x)|v_n|^6 \mathrm{d}x \leq a^3 Q_M \int_\Omega |v_n|^6 \mathrm{d}x \leq a^3 Q_M (S^{-1}\|v_n\|^2)^3 = a^3 Q_M S^{-3}\|v_n\|^6,$$
$$b\Big(\int_\Omega Q(x)|v_n|^6 \mathrm{d}x\Big)^{\frac{2}{3}} \leq b Q_M^{\frac{2}{3}} (S^{-1}\|v_n\|^2)^2 = b Q_M^{\frac{2}{3}} S^{-2}\|v_n\|^4,$$

which implies that $l_1 \geq a Q_M^{-\frac{1}{3}} S (l_1 + l_2)^{\frac{1}{3}}$ and $l_2 \geq b Q_M^{-\frac{2}{3}} S^2 (l_1 + l_2)^{\frac{2}{3}}$. Direct computation shows that

(2.18)
$$l_1 \geq a\Big(\frac{bS^3 + \sqrt{b^2S^6 + 4aQ_M S^3}}{2Q_M}\Big) \quad \text{and} \quad l_2 \geq b\Big(\frac{bS^3 + \sqrt{b^2S^6 + 4aQ_M S^3}}{2Q_M}\Big)^2.$$

Note that $I_\lambda(u) = I_\lambda(u) - \frac{1}{q+1}\langle I'_\lambda(u), u\rangle \geq 0$. Then it follows from (2.17) and (2.18) that

$$c \geq \lim_{n\to\infty}\Big[\frac{a}{3}\|v_n\|^2 + \frac{b}{12}(\|v_n\|^4 + 2\|v_n\|^2\|u\|^2)\Big] = \frac{1}{3}l_1 + \frac{1}{12}l_2 \geq c^*.$$

This contradicts with $c < c^*$. Therefore, $l_1 = 0$, *i.e.*, $\|v_n\| = \|u_n - u\| \to 0$. The proof is completed. ■

By a standard argument we can verify the following result (see [25]).

Lemma 2.4 *I_λ satisfies the mountain pass geometry.*

(i) *There exist $\kappa, \rho > 0$ such that $I_\lambda(u) \geq \kappa$ for $\|u\| = \rho$.*
(ii) *There exists $e \in H_0^1(\Omega)$ with $\|e\| > \rho$ such that $I_\lambda(e) < 0$.*

Define

$$c_\lambda := \inf_{\gamma\in\Gamma_\lambda} \sup_{t\in[0,1]} I_\lambda(\gamma(t)) > 0, \tag{2.19}$$

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where $\Gamma_\lambda = \{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0,\ I_\lambda(\gamma(1)) < 0\}$. Then by the mountain pass theorem without the (PS) condition, there exists a $(PS)_{c_\lambda}$ sequence $\{u_n\} \subset H_0^1(\Omega)$ satisfying

$$I_\lambda(u_n) \to c_\lambda, \quad \text{and} \quad I'_\lambda(u_n) \to 0 \ \text{ in } (H_0^1(\Omega))^{-1}. \tag{2.20}$$

Moreover, for $p \in (3,5)$, as in [30, 38], we can prove

$$c_\lambda = \inf_{u \in \mathcal{N}_\lambda} I_\lambda(u) = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \max_{t \geq 0} I_\lambda(tu) > 0. \tag{2.21}$$

Proof of Theorem 1.1 By Lemma 2.1 we get $c_\lambda \leq \max_{t>0} I_\lambda(tv_\varepsilon^0(x)) < c^*$ for $\varepsilon > 0$ small. Consider the sequence $\{u_n\}$ given by (2.20). Lemma 2.3 implies that there exists $u \in H_0^1(\Omega)$ such that, up to a subsequence, $u_n \to u$ in $H_0^1(\Omega)$. Then $I'_\lambda(u) = 0$ and $I_\lambda(u) = c_\lambda$. By (2.21), $u \not\equiv 0$, and it is a ground state solution of problem (1.1). Since $I_\lambda(u) = I_\lambda(|u|)$ and $\langle I'_\lambda(u), u\rangle = \langle I'_\lambda(|u|), |u|\rangle$, we see that u is a nonnegative solution. By the strong maximum principle, $u(x) > 0$ for $x \in \Omega$. Thus u is a positive ground state solution of (1.1). ∎

3 Proof of Theorem 1.2

Let $q \in (3,5)$. It is easy to verify that I_λ is coercive and bounded from below on $\mathcal{N}_\lambda$ for all $\lambda > 0$. Indeed, for every $u \in \mathcal{N}_\lambda$,

$$\begin{aligned} I_\lambda(u) &= I_\lambda(u) - \frac{1}{q+1}\langle I'_\lambda(u), u\rangle \\ &= \Big(\frac{1}{2} - \frac{1}{q+1}\Big) a\|u\|^2 + \Big(\frac{1}{4} - \frac{1}{q+1}\Big) b\|u\|^4 + \Big(\frac{1}{q+1} - \frac{1}{6}\Big)\int_\Omega Q(x)|u|^6 \mathrm{d}x \\ &\geq \Big(\frac{1}{2} - \frac{1}{q+1}\Big) a\|u\|^2. \end{aligned} \tag{3.1}$$

We minimize the energy functional I_λ on some submanifold of $\mathcal{N}_\lambda$. As in [6, 8], we define a barycenter map $\beta\colon H_0^1(\Omega) \setminus \{0\} \to \mathbb{R}^3$ by

$$\beta(u) = \frac{\int_\Omega x|u|^6 \mathrm{d}x}{\int_\Omega |u|^6 \mathrm{d}x}.$$

Let

$$\begin{aligned} \mathcal{N}_{\lambda,i} &= \{u \in \mathcal{N}_\lambda : |\beta(u) - a^i| < r_0\}, \\ \Theta_{\lambda,i} &= \{u \in \mathcal{N}_\lambda : |\beta(u) - a^i| = r_0\}, \end{aligned} \qquad i = 1, 2, \dots, k,$$

where $r_0 > 0$ such that

$$\overline{B_{r_0}(a^i)} \cap \overline{B_{r_0}(a^j)} = \varnothing \text{ for } i \neq j\ (i,j = 1, 2, \dots, k), \quad \bigcup_{i=1}^k \overline{B_{r_0}(a^i)} \subset \Omega$$

and $\overline{B_{r_0}(a^i)} = \{x \in \mathbb{R}^N : |x - a^i| \leq r_0\}$. By Lemma 2.2 (ii), there exists a unique $t_{v_\varepsilon^i} > 0$ such that $t_{v_\varepsilon^i} v_\varepsilon^i \in \mathcal{N}_\lambda$. Moreover, by the definition of β, one has

$$\beta(t_{v_\varepsilon^i} v_\varepsilon^i) = \frac{\int_\Omega x\varphi_i^6(x)\frac{(3\varepsilon^2)^{3/2}}{(\varepsilon^2+|x-a^i|^2)^3}\mathrm{d}x}{\int_\Omega \varphi_i^6(x)\frac{(3\varepsilon^2)^{3/2}}{(\varepsilon^2+|x-a^i|^2)^3}\mathrm{d}x} = \frac{\int_\Omega (\varepsilon x + a^i)\varphi_i^6(\varepsilon x + a^i)\frac{3^{3/2}}{(1+|x|^2)^3}\mathrm{d}x}{\int_\Omega \varphi_i^6(\varepsilon x + a^i)\frac{3^{3/2}}{(1+|x|^2)^3}\mathrm{d}x} \longrightarrow a^i$$

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as $\varepsilon \to 0$. Then we have actually proved the following result.

Lemma 3.1 *For each $i = 1, 2, \dots, k$, $\beta(t_{v_\varepsilon^i} v_\varepsilon^i) \to a^i$ as $\varepsilon \to 0$. Moreover, there exists $\varepsilon_0 > 0$ such that $\beta(t_{v_\varepsilon^i} v_\varepsilon^i) \in B_{r_0}(a^i)$ for all $0 < \varepsilon < \varepsilon_0$ and $1 \le i \le k$.*

Lemma 3.1 shows that $\mathcal{N}_{\lambda,i} \neq \varnothing$ for all $1 \le i \le k$. By (3.1), the following terms

$$c_{\lambda,i} = \inf_{u \in \mathcal{N}_{\lambda,i}} I_\lambda(u), \quad \widetilde{c_{\lambda,i}} = \inf_{u \in \Theta_{\lambda,i}} I_\lambda(u) \tag{3.3}$$

are well defined. Moreover, it follows from (2.21) that $c_{\lambda,i} \ge c_\lambda > 0$.

Lemma 3.2 *Let (Q2) be satisfied. Then for all $\lambda > 0$ and $1 \le i \le k$,*

$$\sup_{t \ge 0} I_\lambda(t v_\varepsilon^i) < c^*.$$

Lemma 3.2 can be proved similarly to Lemma 2.1 by using v_ε^i and a^i instead of v_ε^0 and a^0, respectively.

Lemma 3.3 *There exists $\widetilde{\lambda} > 0$ such that*

$$\widetilde{c_{\lambda,i}} > c^* \quad \textit{for all } 0 < \lambda < \widetilde{\lambda}, 1 \le i \le k. \tag{3.4}$$

Proof Suppose by contradiction that there exists a positive sequence $\{\lambda_n\}$ such that $\lambda_n \to 0$ and $\widetilde{c_{\lambda_n,i}} \to \widetilde{c} \le c^*$ for some $1 \le i \le k$. Up to a subsequence, we may assume that

$$\widetilde{c} - \frac{1}{n} < \widetilde{c_{\lambda_n,i}} < \widetilde{c} + \frac{1}{n}. \tag{3.5}$$

By the definition of $\widetilde{c_{\lambda_n,i}}$, there exists a sequence $\{u_{n,m}\}_m \subset \Theta_{\lambda_n,i}$ such that for any $n \in \mathbb{N}$, $I_{\lambda_n}(u_{n,m}) \to \widetilde{c_{\lambda_n,i}}$ as $m \to \infty$. Now we choose a subsequence $\{m_n\} \subset \mathbb{N}$ such that $\widetilde{c_{\lambda_n,i}} \le I_{\lambda_n}(u_{n,m_n}) < \widetilde{c_{\lambda_n,i}} + \frac{1}{n}$, $n \in \mathbb{N}$, which, together with (3.5), implies that

$$\widetilde{c} - \frac{1}{n} < I_{\lambda_n}(u_{n,m_n}) < \widetilde{c} + \frac{2}{n}.$$

Let $u_k = u_{k,m_k}$, $k \in \mathbb{N}$. Then, up to a subsequence, we have

$$I_{\lambda_n}(u_n) \to \widetilde{c} \le c^*. \tag{3.6}$$

By a similar argument as in (2.14), we have that $\{u_n\}$ is bounded in $H_0^1(\Omega)$ and

$$a\|u_n\|^2 + b\|u_n\|^4 - \int_\Omega Q(x)|u_n|^6 \mathrm{d}x - \lambda_n \int_\Omega |u_n|^{p+1} \mathrm{d}x = 0. \tag{3.7}$$

Using Sobolev inequalities, we have $\lambda_n \int_\Omega |u_n|^{p+1} \mathrm{d}x = o(1)$ and

$$\|u_n\|^2 \ge C_4, \quad \int_\Omega Q(x)|u_n|^6 \mathrm{d}x \ge C_4$$

for some positive constant C_4. Then, up to a subsequence, we may assume that there exists an $l > 0$ such that

$$\|u_n\|^2 \longrightarrow l \quad \text{and} \quad \lim_{n \to \infty} \int_\Omega Q(x)|u_n|^6 \mathrm{d}x \longrightarrow al + bl^2. \tag{3.8}$$

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By (2.2),

$$al + bl^2 \le \lim_{n\to\infty} \int_\Omega Q_M |u_n|^6 \mathrm{d}x \le Q_M \lim_{n\to\infty} (S^{-1}\|u_n\|^2)^3 = Q_M S^{-3} l^3. \tag{3.9}$$

Direct computation shows that

$$l \ge \frac{bS^3 + \sqrt{b^2S^6 + 4aQ_M S^3}}{2Q_M}.$$

Using (3.6) and (3.7), we are led to

$$\widetilde{c} = \lim_{n\to\infty} I_{\lambda_n}(u_n) = \frac{a}{2}l + \frac{b}{4}l^2 - \frac{1}{6}(al + bl^2) = \frac{a}{3}l + \frac{b}{12}l^2 \ge c^*.$$

Therefore, $\widetilde{c} = c^*$ and all the inequalities in (3.9) must be equalities. We deduce from (3.8) and (3.9) that

$$\lim_{n\to\infty} \int_\Omega Q_M |u_n|^6 \mathrm{d}x = Q_M S^{-3} l^3 = al + bl^2 = \lim_{n\to\infty} \int_\Omega Q(x)|u_n|^6 \mathrm{d}x. \tag{3.10}$$

Set $w_n = \frac{u_n}{|u_n|_6}$. Then $|w_n|_6 = 1$ and (3.10) implies that

$$\lim_{n\to\infty} \|w_n\|^2 = \lim_{n\to\infty} \frac{\|u_n\|^2}{|u_n|_6^2} = \frac{l}{(S^{-3}l^3)^{\frac{1}{3}}} = S.$$

Thus $\{w_n\}$ is a minimizing sequence for S. We apply a result of Lions [24] to conclude that there exist an $x_0 \in \overline{\Omega}$ and a subsequence, still denoted by $\{w_n\}$, such that $|\nabla w_n|^2 \rightharpoonup d\mu = S\delta_{x_0}$, $|w_n|^6 \rightharpoonup d\nu = \delta_{x_0}$, weakly in the sense of measure, where μ, ν are finite measures and δ_{x_0} is a Dirac measure assigned to x_0. On one hand, in view of $u_n \in \Theta_{\lambda_n, i}$, we have

$$\beta(w_n) = \frac{\int_\Omega x|w_n|^6 \mathrm{d}x}{\int_\Omega |w_n|^6 \mathrm{d}x} \to x_0, \quad \text{as } n \longrightarrow \infty,$$

From $|\beta(w_n) - a^i| = r_0$ and $\overline{B_{r_0}(a^i)} \cap \overline{B_{r_0}(a^j)} = \varnothing$ for $i \ne j$, we deduce that $x_0 \ne a^i$ for each $1 \le i \le k$. On the other hand, it follows from (3.10) that

$$Q_M = Q_M \lim_{n\to\infty} \int_\Omega |w_n|^6 \mathrm{d}x = \lim_{n\to\infty} \int_\Omega Q(x)|w_n|^6 \mathrm{d}x = Q(x_0).$$

This is a contradiction. Thus (3.4) holds. ■

Lemma 3.4 *Let $1 \le i \le k$ be fixed. Then for every $u \in \mathcal{N}_{\lambda,i}$, there exists $\tau > 0$ and $\theta\colon B(0;\tau) \subset H_0^1(\Omega) \to \mathbb{R}$ such that $\theta(0) = 1$, and $\theta(v)(u - v) \in \mathcal{N}_{\lambda,i}$ for any $v \in B(0;\tau)$. Moreover,*

$$\langle \theta'(0), \varphi \rangle = \frac{2a(u,\varphi) + 4b\|u\|^2(u,\varphi) - 6\int_\Omega Q(x)|u|^4 u\varphi \mathrm{d}x - \lambda(q+1)\int_\Omega |u|^{q-1}u\varphi \mathrm{d}x}{a(1-q)\|u\|^2 + b(3-q)\|u\|^4 - (5-q)\int_\Omega Q(x)|u|^6 \mathrm{d}x}$$

for any $\varphi \in C_0^\infty(\Omega)$.

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Proof The original idea is contained in [35, 39]. Here we give the proof for completeness. For every $u \in \mathcal{N}_{\lambda,i}$, we define a function $f^u : R \times H_0^1(\Omega) \to \mathbb{R}$ by

$$\begin{aligned} f^u(\theta, v) &= \langle I'_\lambda(\theta(u-v)), \theta(u-v)\rangle \\ &= a\theta^2\|u-v\|^2 + b\theta^4\|u-v\|^4 - \theta^6\int_\Omega Q(x)|u-v|^6 \mathrm{d}x \\ &\quad - \lambda\theta^{q+1}\int_\Omega |u-v|^{q+1}\mathrm{d}x. \end{aligned}$$

Note that $f^u(1,0) = \langle I'_\lambda(u), u\rangle = 0$ and

$$\begin{aligned} \frac{df^u}{d\theta}\Big|_{(\theta,v)=(1,0)} &= 2a\|u\|^2 + 4b\|u\|^4 - 6\int_\Omega Q(x)|u|^6\mathrm{d}x - \lambda(q+1)\int_\Omega |u|^{q+1}\mathrm{d}x \\ &= a(1-q)\|u\|^2 + b(3-q)\|u\|^4 - (5-q)\int_\Omega Q(x)|u|^6\mathrm{d}x < 0. \end{aligned}$$

Applying the implicit function theorem, there exist a $\tau > 0$ and a differential function $\theta : B(0;\tau) \subset H_0^1(\Omega) \to \mathbb{R}$ such that $\theta(0) = 1$, and for any $\varphi \in C_0^\infty(\Omega)$,

$$\begin{aligned} &\langle \theta'(0), \varphi\rangle \\ &= -\Big\langle \frac{(f^u)'_v(\theta(v),v)}{(f^u)'_\theta(\theta(v),v)}\Big|_{v=0}, \varphi\Big\rangle \\ &= \frac{2a(u,\varphi) + 4b\|u\|^2(u,\varphi) - 6\int_\Omega Q(x)|u|^4u\varphi\mathrm{d}x - \lambda(q+1)\int_\Omega |u|^{q-1}u\varphi\mathrm{d}x}{a(1-q)\|u\|^2 + b(3-q)\|u\|^4 - (5-q)\int_\Omega Q(x)|u|^6\mathrm{d}x}, \end{aligned}$$

and $f^u(\theta(v), v) = 0$ for all $v \in B(0;\tau)$, which implies that

$$\langle I_\lambda(\theta(v)(u-v)), \theta(v)(u-v)\rangle = 0 \quad \text{for all } v \in B(0;\tau).$$

Therefore $\theta(v)(u-v) \in \mathcal{N}_\lambda$. By the continuity of β, taking $\tau > 0$ small enough, we have $\beta(\theta(v)(u-v)) \in B(a^i, r_0)$. Thus $\theta(v)(u-v) \in \mathcal{N}_{\lambda,i}$. ■

Proof of Theorem 1.2 By Lemmas 3.1and 3.2, we have

$$0 < c_{\lambda,i} \le I_\lambda(t_{v^i_\varepsilon} v^i_\varepsilon) = \sup_{t\ge 0} I_\lambda(t v^i_\varepsilon) < c^*.$$

It follows from Lemma 3.3 that

$$c_{\lambda,i} < c^* < \widetilde{c_{\lambda,i}} \quad \text{for all } 0 < \lambda < \widetilde{\lambda}. \tag{3.11}$$

This implies that $c_{\lambda,i} = \inf\{I_\lambda(u) : u \in N_{\lambda,i} \cup \Theta_{\lambda,i}\}$, for all $0 < \lambda < \widetilde{\lambda}$. Consider a minimizing sequence $\{w_n^i\} \subset N_{\lambda,i} \cup \Theta_{\lambda,i}$ for I_λ. Since $I_\lambda(|\cdot|) = I_\lambda(\cdot)$, we may assume that $w_n^i(x) \ge 0$ for $x \in \Omega$. By the Ekeland variational principle [38], there exists a sequence $\{u_n^i\} \subset N_{\lambda,i} \cup \Theta_{\lambda,i}$ such that

(3.12)

$$\|u_n^i - w_n^i\| \le \frac{1}{n}, \quad I_\lambda(u_n^i) \le c_{\lambda,i} + \frac{1}{n^2}, \quad I_\lambda(v) \ge I_\lambda(u_n^i) - \frac{1}{n}\|v - u_n^i\|, \quad \forall\, v \in N_{\lambda,i}.$$

We claim that $\{u_n^i\}$ is a $(\mathrm{PS})_{c_{\lambda,i}}$ sequence for I_λ. It is sufficient to show that

$$I'_\lambda(u_n^i) \longrightarrow 0$$

in $(H_0^1(\Omega))^{-1}$ as $n \to \infty$. By (3.11) and (3.12), we have $u_n^i \in N_{\lambda,i}$ for n large. Applying Lemma 3.4, there exist a τ_n^i and a differentiable functional $\theta_n^i : B(0;\tau_n^i) \subset$

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$H_0^1(\Omega) \to \mathbb{R}$ such that $\theta_n^i(0) = 1$, and $\theta_n^i(v)(u_n^i - v) \in \mathcal{N}_{\lambda,i}$ for any $v \in B(0;\tau_n^i)$. Let $\varphi \in H_0^1(\Omega)$ with $\|\varphi\| = 1$ and $s \in (0, \tau_n^i)$. Then

$$v = s\varphi \in B(0;\tau_n^i) \quad \text{and} \quad \theta_n^i(s\varphi)(u_n^i - s\varphi) \in \mathcal{N}_{\lambda,i}.$$

By (3.12) and the mean value theorem, we have

$$\begin{aligned}
\frac{\|\theta_n^i(s\varphi)(u_n^i - s\varphi) - u_n^i\|}{n} &\geq I_\lambda(u_n^i) - I_\lambda(\theta_n^i(s\varphi)(u_n^i - s\varphi)) \\
&= \langle I'_\lambda(t_0 u_n^i + (1-t_0)\theta_n^i(s\varphi)(u_n^i - s\varphi)), u_n^i - \theta_n^i(s\varphi)(u_n^i - s\varphi)\rangle \\
&= \langle I'_\lambda(u_n^i), u_n^i - \theta_n^i(s\varphi)(u_n^i - s\varphi)\rangle + o(\|u_n^i - \theta_n^i(s\varphi)(u_n^i - s\varphi)\|) \\
&= s\theta_n^i(s\varphi)\langle I'_\lambda(u_n^i), \varphi\rangle + (1 - \theta_n^i(s\varphi))\langle I'_\lambda(u_n^i), u_n^i\rangle \\
&\quad + o(\|u_n^i - \theta_n^i(s\varphi)(u_n^i - s\varphi)\|) \\
&= s\theta_n^i(s\varphi)\langle I'_\lambda(u_n^i), \varphi\rangle + o(\|u_n^i - \theta_n^i(s\varphi)(u_n^i - s\varphi)\|),
\end{aligned}$$

where $t_0 \in (0,1)$ and the second equality follows from $\|u_n^i - \theta_n^i(s\varphi)(u_n^i - s\varphi)\| \to 0$ as $s \to 0$. Letting $s \to 0$, we obtain

$$\begin{aligned}
\langle I'_\lambda(u_n^i), \varphi\rangle &\leq \frac{\|u_n^i - \theta_n^i(s\varphi)(u_n^i - s\varphi)\|(\frac{1}{n} + |o(1)|)}{s|\theta_n^i(s\varphi)|} \\
&= \frac{\|u_n^i(\theta_n^i(s\varphi) - \theta_n^i(0)) - s\varphi\theta_n^i(s\varphi)\|(\frac{1}{n} + |o(1)|)}{s|\theta_n^i(s\varphi)|} \\
&\leq \frac{(\|u_n^i\||\theta_n^i(s\varphi) - \theta_n^i(0)| + s|\theta_n^i(s\varphi)|\|\varphi\|)(\frac{1}{n} + |o(1)|)}{s|\theta_n^i(s\varphi)|} \\
&= \Big(1 + \|u_n^i\|\frac{|\theta_n^i(s\varphi) - \theta_n^i(0)|}{s}\Big)\Big(\frac{1}{n} + |o(1)|\Big) \\
&\leq (1 + \|u_n^i\|\|(\theta_n^i)'(0)\|)\Big(\frac{1}{n} + |o(1)|\Big).
\end{aligned}$$

In view of the boundedness of $\{u_n^i\}$ and $\{(\theta_n^i)'(0)\}$ we have that $I'_\lambda(u_n^i) \to 0$ in $(H_0^1(\Omega))^{-1}$ as $n \to \infty$. Hence $\{u_n^i\}$ is a $(\mathrm{PS})_{c_{\lambda,i}}$ sequence for I_λ.

Equation (3.11) implies that $c_{\lambda,i} < c^*$. From Lemma 2.3, there exists $u^i \in H_0^1(\Omega)$ with $u^i \geq 0$ such that, up to a subsequence, $u_n^i \to u^i$ in $H_0^1(\Omega)$ for every $1 \leq i \leq k$. Then $I'_\lambda(u^i) = 0$ and $I_\lambda(u^i) = c_{\lambda,i} > 0$. Thus, u^i is a nontrivial and nonnegative solution of problem (1.1). Using the strong maximum principle, we have $u^i(x) > 0$ in $\Omega, 1 \leq i \leq k$. Therefore, problem (1.1) admits at least k positive solutions u^i $(1 \leq i \leq k)$ for all $0 < \lambda < \widetilde{\lambda}$. ∎

4 Remarks on the Results and Approach

In Theorem 1.1 and Theorem 1.2, positive solutions are obtained via the Nehari manifold method provided $3 < q < 5$, which forces the boundedness of any Palais–Smale sequence of the functional I_λ. It is an intriguing problem to study the case $0 < q \leq 3$. The boundedness of the Palais–Smale sequence becomes a major difficulty in proving the existence of a positive solution in that case. We point out that if the domain is the whole space $\mathbb{R}^3$, He and Li [17] considered Kirchhoff type equation (1.1) with

$Q(x) \equiv 1$ and $1 < q \leq 3$. By using variational methods that are constrained to a Nehari–Pohozaev manifold, they showed the existence of positive solutions of (1.1) and studied the concentration phenomena of the semiclassical solutions. However, the method used there is not applicable to the bounded domain case. New approaches and techniques should be introduced. So, it is very interesting and challenging to investigate further problem (1.1) with $0 < q \leq 3$. This is the work under consideration.

We emphasize that the method used in this paper is still valid for the following Schrödinger–Poisson equation:

$$\begin{cases} -\Delta u + \phi u = Q(x)|u|^4 u + \lambda |u|^{q-1} u, & \text{in } \Omega, \\ -\Delta \phi = u^2, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases}$$

We leave the details to the readers.

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