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ON A PERIODIC MUTUALISM MODEL

YONGKUN LI[1]

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Abstract

Sufficient conditions are obtained for the existence of a globally attracting positive periodic solution of the mutualism model

$$\frac{\mathrm{d}N_1(t)}{\mathrm{d}t} = r_1(t)N_1(t)\left[\frac{K_1(t)+\alpha_1(t)N_2(t-\tau_2(t))}{1+N_2(t-\tau_2(t))} - N_1(t-\sigma_1)\right],$$
$$\frac{\mathrm{d}N_2(t)}{\mathrm{d}t} = r_2(t)N_2(t)\left[\frac{K_2(t)+\alpha_2(t)N_1(t-\tau_1(t))}{1+N_1(t-\tau_1(t))} - N_2(t-\sigma_2)\right],$$

where $r_i, K_i, \alpha_i \in C(R, R^+)$ and $\alpha_i > K_i$, $i = 1, 2$, $\tau_i, \sigma_i \in C(R, R_+)$, $i = 1, 2$ and r_i, $K_i, \alpha_i, \tau_i, \sigma_i$ $(i = 1, 2)$ are functions of period $\omega > 0$.

1. Introduction

Consider the mutualism model

$$\begin{cases} \dfrac{\mathrm{d}N_1(t)}{\mathrm{d}t} = r_1N_1(t)\left[\dfrac{K_1+\alpha_1N_2(t)}{1+N_2(t)} - N_1(t)\right], \\ \dfrac{\mathrm{d}N_2(t)}{\mathrm{d}t} = r_2N_2(t)\left[\dfrac{K_2+\alpha_2N_1(t)}{1+N_1(t)} - N_2(t)\right], \end{cases} \tag{1.1}$$

where $r_i, K_i, \alpha_i \in R^+$ are constants and $\alpha_i > K_i$, $i = 1, 2$. Depending on the nature of K_i $(i = 1, 2)$, system (1.1) can be classified as facultative, obligate or a combination of both. For more details of mutualistic interactions we refer to Vandermeer and Boucher [7], Boucher *et al.* [2], Dean [3], Wolin and Lawlor [8] and Boucher [1]. A

[1]Department of Mathematics, Yunnan University, Kunming, Yunnan 650091, People's Republic of China.

modification of system (1.1) leads to the time-lagged model

$$\begin{cases} \dfrac{\mathrm{d}N_1(t)}{\mathrm{d}t} = r_1N_1(t)\left[\dfrac{K_1+\alpha_1N_2(t-\tau_2)}{1+N_2(t-\tau_2)} - N_1(t)\right], \\ \dfrac{\mathrm{d}N_2(t)}{\mathrm{d}t} = r_2N_2(t)\left[\dfrac{K_2+\alpha_2N_1(t-\tau_1)}{1+N_1(t-\tau_1)} - N_2(t)\right], \end{cases} \tag{1.2}$$

where $\tau_1, \tau_2 \in [0, \infty)$ are constants. In system (1.2) the mutualistic or cooperative effects are not realized instantaneously but take place with time delays. For further ecological applications of system (1.2), we refer to [5] and the references cited therein.

The effects of a periodically varying environment are important for evolutionary theory as the selective forces on systems in a fluctuating environment differ from those in a stable environment. Thus, the assumptions of periodicity of the parameters are a way of incorporating the periodicity of the environment (such as seasonal effects of weather, food supplies, mating habits and so forth). We refer to Pianka [6] for a discussion of the relevance of periodic environments to evolutionary theory. The purpose of this article is to consider the model

$$\begin{cases} \dfrac{\mathrm{d}N_1(t)}{\mathrm{d}t} = r_1(t)N_1(t)\left[\dfrac{K_1(t)+\alpha_1(t)N_2(t-\tau_2(t))}{1+N_2(t-\tau_2(t))} - N_1(t-\sigma_1(t))\right], \\ \dfrac{\mathrm{d}N_2(t)}{\mathrm{d}t} = r_2(t)N_2(t)\left[\dfrac{K_2(t)+\alpha_2(t)N_1(t-\tau_1(t))}{1+N_1(t-\tau_1(t))} - N_2(t-\sigma_2(t))\right], \end{cases} \tag{1.3}$$

together with the initial conditions:

$$\begin{gathered} N_i(t) = \varphi_i(t) \geq 0, \quad t \in [-\tau^*, 0], \quad \varphi_i(0) > 0; \\ \varphi_i \in C([-\tau^*, 0), R_+), \quad i = 1, 2, \end{gathered} \tag{1.4}$$

where $r_i, K_i, \alpha_i \in C(R, R^+)$, $\alpha_i > K_i$, $i = 1, 2$, $\tau_i, \sigma_i \in C(R, R_+)$, $i = 1, 2$, r_i, K_i, α_i, τ_i, σ_i $(i = 1, 2)$ are functions of period $\omega > 0$ and

$$\tau^* = \max_{1\leq i\leq 2}\left\{\max_{t\in[0,\omega]}\tau_i(t), \max_{t\in[0,\omega]}\sigma_i(t)\right\}.$$

In Section 2 we discuss the existence of a positive ω-periodic solution of (1.3)–(1.4), in Section 3 we study the uniqueness and global attractivity of the positive periodic solution of (1.3)–(1.4) and in Section 4 we give an example to illustrate that the conditions of our results can be realized.

2. Existence of a positive periodic solution

In this section we use Mawhin's continuation theorem to show the existence of at least one positive periodic solution of (1.3)–(1.4). To do so, we need to introduce the following notation.

Let X, Y be real Banach spaces, $L : \operatorname{Dom} L \subset X \to Y$ a Fredholm mapping of index zero and $P : X \to X$, $Q : Y \to Y$ continuous projectors such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L$ and $X = \operatorname{Ker} L \bigoplus \operatorname{Ker} P$, $Y = \operatorname{Im} L \bigoplus \operatorname{Im} Q$. Let L_P denote the restriction of L to $\operatorname{Dom} L \bigcap \operatorname{Ker} P$, $K_P : \operatorname{Im} L \to \operatorname{Ker} P \bigcap \operatorname{Dom} L$ the inverse (to L_P) and $J : \operatorname{Im} Q \to \operatorname{Ker} L$ an isomorphism of $\operatorname{Im} Q$ onto $\operatorname{Ker} L$.

For convenience, we introduce Mawhin's continuation theorem [4, page 40] as follows.

LEMMA 2.1. *Let $\Omega \subset X$ be an open bounded set and $N : X \to Y$ be a continuous operator which is L-compact on $\bar{\Omega}$ (that is, $QN : \bar{\Omega} \to Y$ and $K_P(I - Q)N : \bar{\Omega} \to Y$ are compact). Assume*

(a) *for each $\lambda \in (0, 1)$, $x \in \partial\Omega \bigcap \operatorname{Dom} L$, $Lx \neq \lambda N x$;*
(b) *for each $x \in \partial\Omega \bigcap \operatorname{Ker} L$, $QNx \neq 0$, and* $\deg\{JQN, \Omega \bigcap \operatorname{Ker} L, 0\} \neq 0$.

Then $Lx = Nx$ has at least one solution in $\bar{\Omega} \bigcap \operatorname{Dom} L$.

LEMMA 2.2. *Let*

$$f(x, y) = \left(a_1 - \frac{a_1 - b_1}{1 + e^y} - c_1 e^x, a_2 - \frac{a_2 - b_2}{1 + e^x} - c_2 e^y\right)$$

and $\Omega = \{(x, y)^T \in R^2 : |x| + |y| < M\}$, where $M, a_i, b_i, c_i \in R^+$ are constants, $a_i > b_i$, $i = 1, 2$, and $M >$ $\max\{|\ln(a_i/c_i)|, |\ln(b_i/c_i)|, i = 1, 2\}$. Then

$$\deg\{f, \Omega, (0, 0)\} \neq 0.$$

PROOF. Set

$$H(x, y, \mu) = \left(a_1 - \frac{a_1 - b_1}{1 + \mu e^y} - c_1 e^x, a_2 - \frac{a_2 - b_2}{1 + \mu e^x} - c_2 e^y\right), \quad 0 \leq \mu \leq 1.$$

It is then easy to see that, for $(x, y, \mu)^T \in R^2 \times [0, 1]$,

$$a_1 - \frac{a_1 - b_1}{1 + \mu e^y} - c_1 e^x \leq a_1 - c_1 e^x < 0 \quad \text{as} \quad x \geq \frac{M}{2},$$
$$a_2 - \frac{a_2 - b_2}{1 + \mu e^x} - c_1 e^y \leq a_2 - c_2 e^y < 0 \quad \text{as} \quad y \geq \frac{M}{2},$$
$$a_1 - \frac{a_1 - b_1}{1 + \mu e^y} - c_1 e^x \geq b_1 - c_1 e^x > 0 \quad \text{as} \quad x \leq -\frac{M}{2}$$

and

$$a_2 - \frac{a_2 - b_2}{1 + \mu e^x} - c_2 e^y \geq b_2 - c_2 e^y > 0 \quad \text{as} \quad y \leq -\frac{M}{2}.$$

Hence

$$H(x, y, \mu) \neq 0 \quad \text{for} \quad (x, y, \mu) \in \partial\Omega \times [0, 1].$$

It follows from the property of invariance under a homotopy that

$$\deg\{f(x, y), \Omega, (0, 0)\} = \deg\{H(x, y, 0), \Omega, (0, 0)\}.$$

By a straightforward computation, we find

$$\deg\{H(x, y, 0), \Omega, (0, 0)\} = -1 \neq 0.$$

The proof is complete.

We now come to the fundamental theorem of this paper.

THEOREM 2.3. *The initial value problem* (1.3)–(1.4) *has at least one positive ω-periodic solution.*

PROOF. Since solutions of (1.3)–(1.4) remain positive for $t \geq 0$, we can let

$$x(t) = \log[N_1(t)] \quad \text{and} \quad y(t) = \log[N_2(t)] \tag{2.1}$$

and derive that

$$\begin{cases} \dfrac{dx(t)}{dt} = r_1(t)\left[\dfrac{K_1(t) + \alpha_1(t)e^{y(t-\tau_2(t))}}{1 + e^{y(t-\tau_2(t))}} - e^{x(t-\sigma_1(t))}\right], \\ \dfrac{dy(t)}{dt} = r_2(t)\left[\dfrac{K_2(t) + \alpha_2(t)e^{x(t-\tau_1(t))}}{1 + e^{x(t-\tau_1(t))}} - e^{y(t-\sigma_2(t))}\right]. \end{cases} \tag{2.2}$$

Take

$$X = Y = \left\{(x(t), y(t))^T : x(t), y(t) \in C(R, R), x(t+\omega) = x(t), y(t+\omega) = y(t)\right\}$$

and

$$\left\|(x, y)^T\right\| = \max_{0 \leq t \leq \omega} |x(t)| + \max_{0 \leq t \leq \omega} |y(t)|.$$

With this norm, X is a Banach space. Let

$$N\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r_1(t)\left[\dfrac{K_1(t) + \alpha_1(t)e^{y(t-\tau_2(t))}}{1 + e^{y(t-\tau_2(t))}} - e^{x(t-\sigma_1(t))}\right] \\ r_2(t)\left[\dfrac{K_2(t) + \alpha_2(t)e^{x(t-\tau_1(t))}}{1 + e^{x(t-\tau_1(t))}} - e^{y(t-\sigma_2(t))}\right] \end{bmatrix},$$

$$L\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \dfrac{dx(t)}{dt} \\ \dfrac{dy(t)}{dt} \end{bmatrix}, \quad P\begin{bmatrix} x \\ y \end{bmatrix} = Q\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\omega}\displaystyle\int_0^\omega x(t)dt \\ \dfrac{1}{\omega}\displaystyle\int_0^\omega y(t)dt \end{bmatrix}, \quad \begin{bmatrix} x \\ y \end{bmatrix} \in X.$$

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Since $\operatorname{Ker} L = R^2$ and $\operatorname{Im} L$ is closed in X, L is a Fredholm mapping of index zero. Furthermore, we have that N is L-compact on $\bar{\Omega}$ (see [4]), where Ω is any open bounded set in X. Corresponding to the equation $Lx = \lambda Nx$, we have

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = \lambda r_1(t)\left[\dfrac{K_1(t) + \alpha_1(t)e^{y(t-\tau_2(t)))}}{1 + e^{y(t-\tau_2(t))}} - e^{x(t-\sigma_1(t))}\right], \\ \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = \lambda r_2(t)\left[\dfrac{K_2(t) + \alpha_2(t)e^{x(t-\tau_1(t)))}}{1 + e^{x(t-\tau_1(t))}} - e^{y(t-\sigma_2(t))}\right]. \end{cases} \tag{2.3}$$

Assume that $(x(t), y(t))^T \in X$ is a solution of system (2.4) for a certain $\lambda \in (0, 1)$. By integrating (2.3) over $[0, \omega]$, we obtain

$$\int_0^\omega r_1(t)\left[\frac{K_1(t) + \alpha_1(t)e^{y(t-\tau_2(t))}}{1 + e^{y(t-\tau_2(t))}} - e^{x(t-\sigma_1)}\right]\mathrm{d}t = 0 \tag{2.4}$$

and

$$\int_0^\omega r_2(t)\left[\frac{K_2(t) + \alpha_2(t)e^{x(t-\tau_1(t))}}{1 + e^{x(t-\tau_1(t))}} - e^{y(t-\sigma_2)}\right]\mathrm{d}t = 0. \tag{2.5}$$

It is easy to see that we can rewrite (2.4) and (2.5) respectively as

$$\int_0^\omega \frac{r_1(t)(\alpha_1(t) - K_1(t))}{1 + e^{y(t-\tau_2(t))}}\mathrm{d}t + \int_0^\omega r_1(t)e^{x(t-\sigma_1)}\mathrm{d}t = \int_0^\omega r_1(t)\alpha_1(t)\mathrm{d}t \tag{2.6}$$

and

$$\int_0^\omega \frac{r_2(t)(\alpha_2(t) - K_2(t))}{1 + e^{x(t-\tau_1(t))}}\mathrm{d}t + \int_0^\omega r_2(t)e^{y(t-\sigma_2)}\mathrm{d}t = \int_0^\omega r_2(t)\alpha_2(t)\mathrm{d}t. \tag{2.7}$$

Thus from (2.3) and (2.6), it follows that

$$\begin{aligned} \int_0^\omega |x'(t)|\mathrm{d}t &< \lambda \int_0^\omega r_1(t)\left[\frac{K_1(t) + \alpha_1(t)e^{y(t-\tau_2(t))}}{1 + e^{y(t-\tau_2(t))}} + e^{x(t-\sigma_1)}\right]\mathrm{d}t \\ &< \int_0^\omega r_1(t)\alpha_1(t)\mathrm{d}t + \int_0^\omega \frac{r_1(t)(\alpha_1(t) - K_1(t))}{1 + e^{y(t-\tau_2(t))}}\mathrm{d}t + \int_0^\omega r_1(t)e^{x(t-\sigma_1)}\mathrm{d}t \\ &= 2\int_0^\omega r_1(t)\alpha_1(t)\mathrm{d}t \overset{\text{def}}{=} M_1, \end{aligned}$$

that is,

$$\int_0^\omega |x'(t)|\mathrm{d}t < M_1. \tag{2.8}$$

Similarly, by (2.3) and (2.7) we have

$$\int_0^\omega |y'(t)|\mathrm{d}t < 2\int_0^\omega r_2(t)\alpha(t)\mathrm{d}t \overset{\text{def}}{=} M_2. \tag{2.9}$$

Moreover, from (2.6) it follows that

$$\int_0^\omega r_1(t)\alpha_1(t)\mathrm{d}t \geq \int_0^\omega r_1(t)e^{x(t-\sigma_1)}\mathrm{d}t \geq \int_0^\omega r_1(t)K_1(t)\mathrm{d}t,$$

which implies that there exists a point $t_1' \in [0, \omega]$ and a constant $C_1 > 0$ such that

$$|x(t_1' - \sigma_1(t_1'))| < C_1.$$

Suppose that $t_1' - \sigma_1(t_1') = t_1 + n\omega$, $t_1 \in [0, \omega]$ and n is an integer, then

$$|x(t_1)| < C_1. \tag{2.10}$$

Similarly, by (2.7) we can obtain that there exists a point $t_2 \in [0, \omega]$ and a constant $C_2 > 0$ such that

$$|y(t_2)| < C_2. \tag{2.11}$$

Therefore it follows from (2.8)–(2.11) that

$$\max_{t\in[0,\omega]} |x(t)| \le |x(t_1)| + \int_0^\omega |x'(t)|\mathrm{d}t < C_1 + M_1,$$
$$\max_{t\in[0,\omega]} |y(t)| \le |y(t_1)| + \int_0^\omega |y'(t)|\mathrm{d}t < C_2 + M_2.$$

Clearly M_i and C_i ($i = 1, 2$) are independent of λ. Denote $M=M_1+M_2+C_1+C_2+D$, where $D > 0$ is taken sufficiently large such that $M > \max\{|\ln(a_i/c_i)|, |\ln(b_i/c_i)|, i = 1, 2\}$. Now we take $\Omega = \{(x(t), y(t))^T \in X : \|(x, y)^T\| < M\}$. This satisfies condition (a) in Lemma 2.1.

When $(x, y)^T \in \partial\Omega \bigcap \operatorname{Ker} L = \partial\Omega \bigcap R^2$, $(x, y)^T$ is a constant vector in R^2 with $|x| + |y| = M$. Then

$$QN\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \overline{r_1\alpha_1} - \dfrac{\overline{r_1\alpha_1} - \overline{r_1K_1}}{1+e^y} - \bar{r}_1 e^x \\ \overline{r_2\alpha_2} - \dfrac{\overline{r_2\alpha_2} - \overline{r_2K_2}}{1+e^x} - \bar{r}_2 e^y \end{bmatrix} \ne \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where

$$\bar{r}_i = \frac{1}{\omega}\int_0^\omega r_i(t)\,\mathrm{d}t, \quad \overline{r_i\alpha_i} = \frac{1}{\omega}\int_0^\omega r_i(t)\alpha_i(t)\,\mathrm{d}t, \quad \overline{r_iK_i} = \frac{1}{\omega}\int_0^\omega r_i(t)K_i(t)\,\mathrm{d}t,$$

$i = 1, 2$. Furthermore, take $J = I : \operatorname{Im} Q \to \operatorname{Ker} L$, $(x, y)^T \mapsto (x, y)^T$. By Lemma 2.2, we have

$$\deg\left\{J\,QN(x, y)^T, \Omega, (0, 0)\right\} = \deg\left\{QN(x, y)^T, \Omega, (0, 0)\right\} \ne 0.$$

We now know that Ω verifies all the requirements in Lemma 2.1 and thus that (2.2) has at least one ω-periodic solution. By (2.1), we easily see that (1.3) -(1.4) has at least one positive ω-periodic solution. The proof is complete.

3. Uniqueness and global attractivity

We first obtain certain upper and lower estimates for solutions of (1.3)–(1.4). For convenience we introduce the notation:

$$\begin{aligned}
r_{11} &= \max_{t\in[0,\omega]} r_1(t), & r_{21} &= \max_{t\in[0,\omega]} r_2(t),\\
\alpha_{11} &= \max_{t\in[0,\omega]} \alpha_1(t), & \alpha_{21} &= \max_{t\in[0,\omega]} \alpha_2(t),\\
K_{12} &= \min_{t\in[0,\omega]} K_1(t), & K_{22} &= \min_{t\in[0,\omega]} K_2(t),\\
\sigma_{11} &= \max_{t\in[0,\omega]} \sigma_1(t), & \sigma_{21} &= \max_{t\in[0,\omega]} \sigma_2(t).
\end{aligned}$$

LEMMA 3.1. *If $(N_1(t), N_2(t))$ is a solution of the initial value problem* (1.3)–(1.4) *then there exist numbers T_1 and T_2 such that*

$$B_1 \le N_1(t) \le A_1 \quad \textit{for } t \ge T_1$$

and

$$B_2 \le N_2(t) \le A_2 \quad \textit{for } t \ge T_2,$$

in which $A_1 = \alpha_{11}\exp(\alpha_{11}r_{11}\sigma_{11})$, $A_2 = \alpha_{21}\exp(\alpha_{21}r_{21}\sigma_{21})$, $B_1 = K_{12}\exp[r_{11}\sigma_{11}(K_{12} - A_1)]$ and $B_2 = K_{22}\exp[r_{21}\sigma_{21}(K_{22} - A_1)]$.

PROOF. It is easy to see that N_1 and N_2 satisfy

$$\begin{cases}
\dfrac{\mathrm{d}N_1(t)}{\mathrm{d}t} \le r_1(t)N_1(t)[\alpha_{11} - N_1(t-\sigma_1(t))],\\[2mm]
\dfrac{\mathrm{d}N_2(t)}{\mathrm{d}t} \le r_2(t)N_2(t)[\alpha_{21} - N_2(t-\sigma_2(t))].
\end{cases} \tag{3.1}$$

Now either $N_1(t)$ is oscillatory about α_{11} or it is nonoscillatory. In the case where $N_1(t)$ is oscillatory about α_{11}, we let $\{t_n\}$ be the sequence such that $\lim_{n\to\infty} t_n = \infty$ and $\alpha_{11} - N_1(t_n) = 0$. Let $N_1(t_n^*)$ be the local maximum of $N_1(t)$ on (t_n, t_{n+1}). Then

$$0 = N_1'(t_n^*) \le r_1(t_n^*)N_1(t_n^*)[\alpha_{11} - N_1(t_n^* - \sigma_1(t_n^*))].$$

Now $N_1(t_n^* - \sigma_1(t_n^*)) \le \alpha_{11}$; so let ξ be the zero of $\alpha_{11} - N_1(t)$ in $[t_n^* - \sigma_1(t_n^*), t_n^*]$. By integrating (3.1) from ξ to t_n^*, we have

$$\log\frac{N_1(t_n^*)}{N_1(\xi)} \le \int_{\xi}^{t_n^*} \alpha_{11}r_1(t)\mathrm{d}t,$$

or

$$N_1(t_n^*) \leq \alpha_{11} \exp\left[\alpha_{11} \int_{\xi}^{t_n^*} r_1(t)\mathrm{d}t\right] \leq \alpha_{11} \exp(\alpha_{11} r_{11} \sigma_{11}),$$

that is,

$$N_1(t) \leq \alpha_{11} \exp(\alpha_{11} r_{11} \sigma_{11}) = A_1 \quad \text{for} \quad \mathrm{t} \geq \mathrm{t}_1 + 2\sigma_{11}. \tag{3.2}$$

Next suppose that $N_1(t)$ is nonoscillatory about α_{11}. Then it is easy to see that for every $\varepsilon > 0$ there exists a $T_1' = T_1'(\varepsilon)$ such that

$$N_1(t) < \alpha_{11} + \varepsilon \quad \text{for } t > T_1'.$$

This together with (3.2) implies that there exists a T_2' such that

$$N_1(t) \leq A_1 \quad \text{for } t > T_2'.$$

On the other hand, from (1.3) we find

$$\begin{cases} \dfrac{\mathrm{d}N_1(t)}{\mathrm{d}t} \geq r_1(t)N_1(t)[K_{12} - N_1(t - \sigma_1(t))], \\ \dfrac{\mathrm{d}N_2(t)}{\mathrm{d}t} \geq r_2(t)N_2(t)[K_{22} - N_2(t - \sigma_2(t))]. \end{cases} \tag{3.3}$$

Let $N_1(t)$ be an oscillatory solution about K_{12} and let $\{s_n\}$ be a sequence such that $\lim_{n\to\infty} s_n = \infty$ and $N_(s_n) - K_{12} = 0$. Suppose that $N_1(s_n^*)$ is a local minimum of $N_1(t)$ on (s_n, s_{n+1}). Then

$$0 = N_1'(s_n^*) \geq r_1(s_n^*)N_1(s_n^*)[K_{12} - N_1(s_n^* - \sigma_1(s_n^*))].$$

So $K_{12} - N_1(s_n^* - \sigma_1(s_n^*)) \leq 0$, that is, there exists a point $\eta \in [s_n^* - \sigma_1(s_n^*), s_n^*]$ such that $N_1(\eta) = K_{12}$. Note that $K_{12} - A_1 < 0$, then

$$\log \frac{N_1(s_n^*)}{K_{12}} \geq \int_{\eta}^{s_n^*} r_1(t)(K_{12} - A_1)\,\mathrm{d}t \geq r_{11}\sigma_{11}(K_{12} - A_1).$$

Hence

$$N_1(s_n^*) \geq K_{12} \exp[r_{11}\sigma_{11}(K_{12} - A_1)],$$

that is,

$$N_1(t) \geq K_{12} \exp[r_{11}\sigma_{11}(K_{12} - A_1)] = B_1 \quad \text{for } t \geq t_1 + 2\sigma_{11}. \tag{3.4}$$

Next, suppose that $N_1(t)$ is nonoscillatory about K_{12}. One can easily prove in this case that for every positive ε there exists a $T_3' = T_3'(\varepsilon)$ such that

$$N_1(t) > K_{12} - \varepsilon \quad \text{for } t \geq T_3'.$$

This together with (3.4) implies that there exists a T_4' such that

$$N_1(t) \geq B_1 \quad \text{for} \quad t \geq T_4'.$$

Similarly, one can prove that there exists a T_2 such that

$$B_2 \leq N_2(t) \leq A_2 \quad \text{for } t \geq T_2.$$

The proof of Lemma 3.1 is complete.

We will now proceed to derive sufficient conditions under which (1.3)–(1.4) has a unique positive ω-periodic solution $(N_1^*(t), N_2^*(t))$ which globally attracts all other positive solutions of (1.3)–(1.4).

Let $(N_1^*(t), N_2^*(t))$ be a positive ω-periodic solution of (1.3)–(1.4), whose existence is given by Theorem 2.1. We set

$$N_1(t) = N_1^*(t)e^{x(t)} \quad \text{and} \quad N_2(t) = N_2^*(t)e^{y(t)} \tag{3.5}$$

and derive that

$$\begin{cases} \dfrac{dx(t)}{dt} = F[x(t-\sigma_1(t)), y(t-\tau_2(t))] - F(0,0), \\ \dfrac{dy(t)}{dt} = G[x(t-\tau_1(t)), y(t-\sigma_2(t))] - G(0,0), \end{cases} \tag{3.6}$$

where

$$F(u,v) = -N_1^*(t-\sigma_1(t))e^u - \frac{\alpha_1(t) - K_1(t)}{1 + N_2^*(t-\tau_2(t))e^v},$$
$$G(u,v) = -N_2^*(t-\sigma_2(t))e^v - \frac{\alpha_2(t) - K_2(t)}{1 + N_1^*(t-\tau_1(t))e^u}.$$

By the mean value theorem of differential calculus, we can rewrite (3.6) in the form

$$\begin{cases} \dfrac{dx(t)}{dt} = -a_{11}(t)x(t-\sigma_1(t)) + a_{12}(t)y(t-\tau_2(t)), \\ \dfrac{dy(t)}{dt} = a_{21}(t)x(t-\tau_1(t)) - a_{22}(t)y(t-\sigma_2(t)), \end{cases} \tag{3.7}$$

where

$$\begin{aligned} a_{11}(t) &= \eta_1(t), \quad & a_{12}(t) &= \frac{(\alpha_1(t) - K_1(t))\eta_2(t)}{(1+\eta_2(t))^2}, \\ a_{22}(t) &= \eta_3(t), \quad & a_{21}(t) &= \frac{(\alpha_2(t) - K_2(t))\eta_4(t)}{(1+\eta_4(t))^2}, \end{aligned} \tag{3.8}$$

and $\eta_1(t)$ lies between $N_1^*(t-\sigma_1(t))$ and $N_1(t-\sigma_1(t))$, $\eta_2(t)$ lies between $N_2^*(t-\tau_2(t))$ and $N_2(t-\tau_2(t))$, $\eta_3(t)$ lies between $N_2^*(t-\sigma_2(t))$ and $N_2(t-\sigma_2(t))$, and $\eta_4(t)$ lies between $N_1^*(t-\tau_1(t))$ and $N_1(t-\tau_1(t))$. By Lemma 3.1, we can conclude that there exists a number T^* such that for all $t \ge T^*$, we have

$$B_1 \le a_{11}(t) \le A_1, \quad \frac{(\alpha_1(t)-K_1(t))B_2}{(1+A_2)^2} \le a_{12}(t) \le \frac{(\alpha_1(t)-K_1(t))A_2}{(1+B_2)^2} \le C_{12},$$
$$B_2 \le a_{22}(t) \le A_2, \quad \frac{(\alpha_2(t)-K_2(t))B_1}{(1+A_1)^2} \le a_{21}(t) \le \frac{(\alpha_2(t)-K_2(t))A_1}{(1+B_1)^2} \le C_{21},$$

where

$$C_{12} = \frac{(a_{11}-K_{12})A_2}{(1+B_2)^2}, \quad C_{21} = \frac{(a_{21}-K_{22})A_1}{(1+B_1)^2}.$$

With the above preparation we formulate our second fundamental result.

THEOREM 3.2. *Assume that every solution of* (3.7)–(3.8) *satisfies*

$$\lim_{t\to\infty}[x^2(t)+y^2(t)] = 0. \tag{3.9}$$

Then there exists a unique positive ω-periodic solution $(N_1^*(t), N_2^*(t))$ of (1.3)–(1.4) *such that all other positive solutions of* (1.3)–(1.4) *satisfy*

$$\lim_{t\to\infty}\{N_1(t), N_2(t)\} = \{N_1^*(t), N_2^*(t)\}. \tag{3.10}$$

PROOF. The existence of at least one positive ω-periodic solution of (1.3)–(1.4) is a consequence of Theorem 2.1. The uniqueness of the periodic solution will follow from (3.10). But every solution $(x(t), y(t))$ of (3.7)–(3.8) satisfies (3.9), which implies (3.10). This completes the proof.

Next, assume that $\tau_i(t) \equiv \tau_i$, $\sigma_i(t) \equiv \sigma_i$ $(i = 1, 2)$ are constants, and define two numbers μ_1^* and μ_2^* which satisfy

$$\begin{aligned}\mu_1^* = B_1 - [&A_1(A_1\sigma_1 + C_{12}\tau_2) + C_{21}(C_{21}\tau_1 + A_2\sigma_2) + A_1\sigma_1(A_1 + C_{12}) \\ &+ C_{21}\tau_1(C_{21} + A_2)], \\ \mu_2^* = B_2 - [&C_{12}(A_1\sigma_1 + C_{12}\tau_2) + C_{21}(C_{21}\tau_1 + A_2\sigma_2) + C_{12}\tau_2(A_1 + C_{12}) \\ &+ A_2\sigma_2(C_{21} + A_2)].\end{aligned}$$

Then we have the following result.

COROLLARY 3.3. *Assume the following conditions hold*:

(i) $\tau_i(t) \equiv \tau_i, \sigma_i(t) \equiv \sigma_i (i = 1, 2)$ *are constants*;
(ii) $\mu_1^* > 0, \mu_2^* > 0$;

(iii) *the quadratic form*

$$Q(x, y) = [x, y]\begin{bmatrix} B_1 & -(C_{12}+C_{21}) \\ -(C_{12}+C_{21}) & B_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

is nonnegative.

Then (1.3)–(1.4) *has a unique positive ω-periodic solution* $(N_1^*(t), N_2^*(t))$, *and all other positive solutions of* (1.3)–(1.4) *satisfy* (3.10).

PROOF. Entirely similar to the proof of [5, Theorem 4.3.6], one can obtain that every solution of (3.7)–(3.8) satisfies (3.9). Therefore the conclusions of the theorem follow from Theorem 3.1. The proof is complete.

4. An example

Finally, as an application of our main results, we consider the system

$$\begin{cases} \dfrac{dN_1(t)}{dt} = \left(\dfrac{1}{2} + \dfrac{1}{2}\sin^2(t+\phi_1)\right) N_1(t) \\ \qquad \times \left[\dfrac{100+\frac{1}{2}\cos^2 t + (101+\sin^2 t)N_2(t-e^{-100})}{1+N_2(t-e^{-100})} - N_1(t-e^{-100})\right], \\ \dfrac{dN_2(t)}{dt} = \left(\dfrac{1}{2} + \dfrac{1}{2}\cos^2(t+\phi_2)\right) N_2(t) \\ \qquad \times \left[\dfrac{100+\frac{1}{2}\sin^2 t + (101+\cos^2 t)N_1(t-e^{-100})}{1+N_1(t-e^{-100})} - N_2(t-e^{-100})\right], \end{cases} \tag{4.1}$$

together with the initial conditions (1.4), where ϕ_i, $i = 1, 2$, are constants. One can easily verify that (4.1)–(1.4) satisfies all the conditions of Corollary 3.1. Therefore, system (4.1) has a unique positive ω-periodic solution, which attracts all other positive solutions of (4.1)–(1.4).

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