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On the Diameter of Plane Curves

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Abstract. Recently, differential geometric properties of embedded projective varieties have gained increasing interest. In this note, we consider plane algebraic curves equipped with the Fubini–Study metric from $\mathbb{P}_2(\mathbb{C})$ and give an estimate for the diameter in terms of the degree, initiated in a paper by F. A. Bogomolov.

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Recently, differential geometric properties of embedded projective varieties have gained increasing interest. In this note, we consider plane algebraic curves equipped with the Fubini–Study metric from $\mathbb{P}_2(\mathbb{C})$ and give an estimate for the diameter in terms of the degree, initiated in a paper by F. A. Bogomolov [2]. In particular, this paper implied that contrary to a general belief the diameter is not bounded from above. The result was extended by N. A'Campo [1]. The curvature had been explicitly computed by L. Ness [5]. Her results show the existence of areas of negative curvature and that the curvature is not bounded from below in the family of all embedded algebraic curves of a fixed degree. Using curvature when proving an estimate for the diameter requires a careful consideration of these areas. Bogomolov pointed out that the best estimate to expect is logarithmic, since Gromov's Betti number theorem implies a lower estimate for the diameter in the following sense: Under the restriction to curves, whose curvature is bounded from below by a number $-\kappa^2$, the diameter is bounded from below by $C\log(d)/\kappa$, where C denotes a posivite constant. We use rather explicit methods to show the following result

THEOREM 1. *The diameter of a plane algebraic curve $C \subset \mathbb{P}_2(\mathbb{C})$ of degree d, equipped with the Fubini-Study metric is bounded by $(2d^2 - 2d + 1)(4d^2 + 1) \cdot \pi$.*

The theorem has immediate consequences for the diameter of complete intersections in $\mathbb{P}_n$. Our estimate seems to be also of interest in connection with results of Y. Yomdin [6] and M. Briskin–Y. Yomdin [3] in the area of polynomial control problems.

1. Preparations

Our estimates will be based upon projections onto projective lines. Let $(x_0 : x_1 : x_2)$ denote homogeneous coordinates on $\mathbb{P}_2 = \mathbb{P}_2(\mathbb{C})$. For $j = 0, 1, 2$ and $\{j, k, \ell\} = \{0, 1, 2\}$ we consider the lines L_j defined as zero sets $V(x_j)$ and the points P_j defined as zero sets $V(x_k, x_\ell)$. Furthermore, we have the canonical projections $\pi_j\colon \mathbb{P}_2\backslash P_j \to L_j$, defined by omitting the jth coordinate. The projective plane $\mathbb{P}_2$ and the lines L_j resp. are equipped with the Fubini-Study forms $\omega_{\mathbb{P}_2}$ and $\omega_{\mathbb{P}_1}$ resp.

LEMMA 1. *Let* $\gamma\colon [0, 1] \to \mathbb{P}_2\backslash\{P_0, P_1, P_2\}$ *be a curve of class* C^∞. *Then the length is estimated by* $L(\gamma) \leqslant \sum_{j=0}^{2} L(\pi_j \circ \gamma)$, *or equivalently* $\omega_{\mathbb{P}_2} \leqslant \sum_{j=0}^{2} \pi_j^*(\omega_{\mathbb{P}_1})$. *on* $\mathbb{P}_2\backslash \bigcup L_j$.

Proof. With respect to inhomogenous coordinates $(1 : x_1 : x_2)$ we have

$$\mathrm{d}s^2_{\mathbb{P}_2} = \frac{|\mathrm{d}x_1|^2 + |\mathrm{d}x_2|^2 + |x_1\mathrm{d}x_2 - x_2\mathrm{d}x_1|^2}{(1 + |x_1|^2 + |x_2|^2)^2}$$

$$\leqslant \frac{|\mathrm{d}x_1|^2}{(1 + |x|_1^2)^2} + \frac{|\mathrm{d}x_2|^2}{(1 + |x_2|^2)^2} + \frac{|\mathrm{d}(\frac{x_2}{x_1})|^2}{(1 + |\frac{x_2}{x_1}|^2)^2}.$$

Obviously, it is sufficient to show an upper bound in terms of the degree only for a generic class of embedded curves: Let $\mathcal{C}_d$ be the set of all smooth plane curves C of degree d such that

(i) $P_j \notin C$ for $j = 0, 1, 2$;
(ii) $\pi_2|C : C \to L_2$ is a simple branched covering.

We estimate the length of a particular class of real algebraic curves. Let $C \in \mathcal{C}_d$, and let $L_{\mathbb{R}} \subset L_2$ be a closed geodesic. We denote its preimage under $\pi_2|C : C \to L_2$ by $C_{\mathbb{R}}$.

LEMMA 2. *The curves* $\pi_0(C_{\mathbb{R}}) \subset L_0$ *and* $\pi_1(C_{\mathbb{R}}) \subset L_1$ *are real algebraic of degree at most* d^2.

Proof. Let C be the zero set $V(F)$ with $F = F(x_0, x_1, x_2)$ homogeneous and irreducible of degree d. We first show the claim for the $L_{\mathbb{R}} = \overline{\{(t, 1); t \in \mathbb{R}\}} \subset L_2$ and $\pi_0(C_{\mathbb{R}})$ say. Since C is irreducible of degree greater than one, it intersects any fiber of the map π_0 in a discrete set of points. Therefore we can restrict ourselves to an affine set $U = \mathbb{P}_2\backslash L_1 = \pi_0^{-1}(L_0\backslash\{P_2\})$. Then $\pi_0(C_{\mathbb{R}})$ is the closure of $\pi_0(U \cap C_{\mathbb{R}})$. Now $(0 : 1 : x_2) \in \pi_0(U \cap C_{\mathbb{R}})$, if and only if there exists $t \in \mathbb{R}$ such that $F(t, 1, x_2) = 0$. Classical elimination theory yields the following. Denote by $R(x_2, \overline{x_2})$ the resultant of $\mathrm{Re}(F(t, 1, x_2))$ and $\mathrm{Im}(F(t, 1, x_2))$ with respect to t. We use the fact that for any two polynomials $g(y, z)$, $h(y, z)$ of degree m and n resp. the resultant $R_{g,h}(z)$ (where y is eliminated) is a polynomial of degree at most $m \cdot n$. Hence $\pi_0(C_{\mathbb{R}})$ is real algebraic of degree at most d^2.

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Let $L_{\mathbb{R}} \subset L_2$ be an arbitrary geodesic. Then $L_{\mathbb{R}}$ is the closure of the set of all $(a + b \cdot t : c + d \cdot t : 0) \in L_2$; $t \in \mathbb{R}$, where the a, b, c, d determine an element of SU(2). As above the resultant of $\mathrm{Re}(F(a + b \cdot t, c + d \cdot t, x_2))$ and $\mathrm{Im}(F(a + b \cdot t, c + d \cdot t, x_2))$ is a polynomial of degree at most d^2 in x_2 and $\overline{x_2}$.

LEMMA 3. *Let $C_{\mathbb{R}} \subset \mathbb{P}_1(\mathbb{C})$ be a real algebraic curve of degree δ. Then the length of $C_{\mathbb{R}}$ with respect to the Fubini-Study metric of $\mathbb{P}_1(\mathbb{C})$ is at most $2\pi\delta$.*

Proof. We assume that $C_{\mathbb{R}} \subset \mathbb{C} \subset \mathbb{P}_1$ is connected and choose a piecewise smooth parametrization $\gamma\colon [0, 1] \to C_{\mathbb{R}}$; $\gamma(t) = u(t) + iv(t)$. Then

$$L(\gamma) = \int_0^1 \frac{(|u'|^2 + |v'|^2)^{1/2}}{1 + u^2 + v^2}\,\mathrm{d}t \leqslant \int_0^1 \frac{|u'|}{1 + u^2}\,\mathrm{d}t + \int_0^1 \frac{|v'|}{1 + v^2}\,\mathrm{d}t.$$

Since the projections $z \mapsto u$ and $z \mapsto v$, restricted to $C_{\mathbb{R}}$ have at most a number of δ sheets, the above integral is at most $2\delta \cdot \int_{-\infty}^{+\infty} (\mathrm{d}u/1 + u^2) = 2\delta\pi$.

2. Proof of the Theorem

In the sequel, we describe a generic type of branching. Let $C \subset \mathbb{P}_2 \backslash \{P_0, P_1, P_2\}$ and denote by $\pi\colon C \to \mathbb{P}_1$ the restriction of $\pi_2\colon \mathbb{P}_2 \backslash P_2 \to L_2$ to C. Again π must have only simple generic branch points P_j and we impose that

(iii) the images $Q_j = \pi(P_j)$ of the branch points P_j are distinct, where $j = 1, \ldots, b$, with $b = d^2 - d$, and no three of these are contained in a closed geodesic.

Next we choose a point R in $\mathbb{P}_1 \backslash \bigcup_{j<k} L_{jk}$, where L_{jk} is the closed geodesic through Q_j and Q_k. Let S_j be the segment of the real projective line from R to Q_j, $j = 1, \ldots, d$, and $S = \cup S_j$. The complement $\mathbb{P}_1 \backslash S$ is simply connected and $\pi^{-1}(\mathbb{P}_1 \backslash S)$ decomposes into b isomorphic copies E_ν; $\nu = 1, \ldots, d$, where we set $E_\nu = \mathbb{P}_1^{(\nu)} \backslash \cup S_j^{(\nu)}$, with copies $\mathbb{P}_1^{(\nu)}$ and $S_j^{(\nu)}$ resp. of $\mathbb{P}_1$ and S_j resp. Let $R^{(\nu)} \in \mathbb{P}_1^{(\nu)}$ correspond to R.

Any branch point B_j is contained in the closure $\overline{E_\nu}$ of E_ν in C for exactly two values of ν. For all j with $P_j \notin \overline{E_\nu}$ we fill $S_j^{(\nu)} \backslash R^{(\nu)}$ into E_ν and obtain $\tilde{E}_\nu$, which is a copy of $\mathbb{P}_1$ with a certain number of segments emanating from one point removed. We count boundary points of $\tilde{E}_\nu$ twice, except for the endpoints of line segments. The domain $\tilde{E}_\nu$ with boundary added is called $\hat{E}_\nu$. Now C is obtained from $\cup \hat{E}_\nu$ by means of the usual gluing process. There is a natural projection $\rho_\nu\colon \hat{E}_\nu \to \mathbb{P}_1$. We chose arbitrary sheets $\hat{E}_1$ and $\hat{E}_2$ say and points $R_j \in \hat{E}_j$; $j = 1, 2$ with $\rho_j(R_j) = R$; $j = 1, 2$. We want to connect the images of R_1 and R_2 in C by the images of line segments in the boundaries of $\hat{E}$, where sheets are switched at branch points. We give the construction. Let $S_{j,1}^{(\nu)}, S_{j,2}^{(\nu)} \subset \partial \hat{E}_\nu$ correspond to $S_j^{(\nu)} \subset \mathbb{P}_1^{(\nu)}$. We follow one of these segments from R_1 in $\hat{E}_1$ to the adjacent branch point. Either we switch sheets at the branch point, or go back on the opposite edge of the same $\hat{E}_\nu$. We follow the next edge on the present sheet to the next

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branch point and switch again sheets, or not, keeping the orientation, i.e., in a way such that the set $\hat{E}_\nu$ is always ot the same side of the edge. After circulating a certain number of times we arrive at R_2. Now we need to visit any branch point at most once: otherwise we get a closed loop which we can eliminate from our path. Hence the total number of segments does not exceed twice the number of branch points $2b$.

In order to conclude the proof, is is sufficient to show the claim for generic $C \subset \mathbb{P}_2$ with $\pi\colon C \to \mathbb{P}_1$ as above. Let two points of C be given. By a continuity argument one of these can play the role of R_2, whereas the other point is contained in the image of some other sheet $\hat{E}_\nu$ say $\hat{E}_2$ and can be connected with some R_2 located over R_1 on the corresponding boundary component. This amounts to a total of at most $2b + 1$ segments. According to Lemma 2 and 3 the length of each segment is at most $(4d^2 + 1)\pi$, which shows that the diameter is bounded from above by $(2b + 1)(4d^2 + 1) \cdot \pi = (2d^2 - 2d + 1)(4d^2 + 1)\pi$.

References

1. A'Campo, N.: On the shape of projective plane curves, *Math. Res. Lett*. 2 (1995), 537–539.
2. Bogomolov, F. A.: On the diameter of plane algebraic curves, *Math. Res. Lett*. **1** (1994), 95–98.
3. Briskin, M. and Yomdin, Y.: Semialgebraic geometry of polynomial control problems, In: *Computatonal Algebraic Geometry*, Progr. Math. 109, Birkhäuser, Basel, 1993, pp. 21–28.
4. Gromov, M.: Curvature and Betti numbers, *Comment. Math. Helv*. **56** (1981), 179–195.
5. Ness, L.: Curvature on algebraic plane curves I, *Compositio Math*. **35** (1977), 57–63.
6. Yomdin, Y.: The geometry of critical and near-critical values of differential mappings, *Math. Ann*. **264** (1983), 495–515.