J. Aust. Math. Soc. **86** (2009), 145–154
doi:10.1017/S1446788708000529

VECTOR BUNDLES OVER A NONDEGENERATE CONIC

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(Received 12 June 2007; accepted 12 December 2007)

Communicated by J. Du

Abstract

Let k be a field and X a k-form of the projective line. We classify all the isomorphism classes of vector bundles over X.

2000 *Mathematics subject classification*: primary 14H60; secondary 14F05.
Keywords and phrases: conic, vector bundle, Harder–Narasimhan filtration.

1. Introduction

Let k be any field. Let X be an absolutely irreducible regular closed subscheme of the projective plane $\mathbb{P}^2_k$ of dimension one and degree two. Such a scheme is also called a nondegenerate conic. These are precisely the k-forms of the projective line. If X has a k-rational point x_0, then a projection from x_0 gives an isomorphism of X with the projective line $\mathbb{P}^1_k$.

A theorem of Grothendieck says that any vector bundle E of rank r over $\mathbb{P}^1_k$ is isomorphic to a vector bundle of the form $\bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1_k}(d_i)$, where $d_i \in \mathbb{Z}$ and $\mathcal{O}_{\mathbb{P}^1_k}(1)$ denotes the tautological line bundle, and, furthermore, the element $\{d_i\}_{i=1}^r \in \mathbb{Z}^r$ is uniquely determined by E up to a permutation of the index set $\{1, 2, \ldots, r\}$.

Assume now that the conic X does not have any k-rational point. We first show that X admits a unique indecomposable vector bundle of rank two and degree two. This unique vector bundle over X will be denoted by S. We then prove the following result (see Theorem 4.1).

Any vector bundle E over X is isomorphic to a vector bundle of the form

$$\left(\bigoplus_{i=1}^{r_0} (T_X)^{\otimes m_i}\right) \oplus \left(\bigoplus_{i=1}^{r_1} S \otimes (T_X)^{\otimes n_i}\right),$$

The second-named author would like to thank the CNRS for support.

where $n_i, m_i \in \mathbb{Z}$ and T_X is the tangent bundle of X. Furthermore, the two elements $\{m_i\}_{i=1}^{r_0} \in \mathbb{Z}^{r_0}$ and $\{n_i\}_{i=1}^{r_1} \in \mathbb{Z}^{r_1}$ are uniquely determined by E up to permutations of the index sets $\{1, 2, \ldots, r_0\}$ and $\{1, 2, \ldots, r_1\}$ respectively.

In the special case where k is the field of real numbers, the above result was obtained in [2] by a different method.

2. Preliminaries

Fix a field k. By a vector bundle on a scheme Y defined over k we will mean a locally free $\mathcal{O}_Y$-module of finite rank. For any integer $n \geq 1$, the ample generator of the Picard group of the projective space $\mathbb{P}^n_k$ is denoted by $\mathcal{O}_{\mathbb{P}^n_k}(1)$.

The following proposition is due to Grothendieck [3]; see [6, p. 61, Lemma 4.4.1] for a proof.

PROPOSITION 2.1. *Let E be a vector bundle of rank r over $\mathbb{P}^1_k$. Then E is isomorphic to a direct sum of line bundles, or, in other words,*

$$E \cong \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1_k}(n_i),$$

where $n_i \in \mathbb{Z}$. Furthermore, the integers $\{n_i\}_{i=1}^{r} \in \mathbb{Z}^{\oplus r}$ are determined by E uniquely up to a permutation of the index set $\{1, \ldots, r\}$.

DEFINITION 2.2. Let $X \subset \mathbb{P}^2_k$ be a closed subscheme of dimension one and degree two. If $X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k})$ is reduced and irreducible, where $\overline{k}$ is an algebraically closed field containing k, then X is called a *nondegenerate conic*. A nondegenerate conic that has no k-rational points is called a *nondegenerate anisotropic conic*.

Let X be a nondegenerate conic. If X has a k-rational point x_0, then the incomplete linear system

$$V := \{s \in H^0(X, \mathcal{O}_{\mathbb{P}^2_k}(1)|_X) \mid s(x_0) = 0\}$$

gives an isomorphism of X with the projective line $\mathbb{P}(V) \cong \mathbb{P}^1_k$.

3. A vector bundle on a nondegenerate anisotropic conic

In this section we will show that there is a unique indecomposable vector bundle of rank two and degree two over a nondegenerate anisotropic conic. By 'degree of a vector bundle' we mean the degree of any divisor corresponding to the top exterior product of the vector bundle.

REMARK 3.1. Let X be a nondegenerate conic over a field k. If the field k is algebraically closed or if k is a finite field, then it can be shown that X has a k-rational point. Indeed, if k is a finite field, this is a consequence of the Chevalley–Warning theorem (see [7]). If k is algebraically closed, then this fact is a consequence of the Hilbert Nullstellensatz (see [1]).

LEMMA 3.2. *Let X be a nondegenerate anisotropic conic over a field k. Then there is a degree-two Galois extension k' of k such that $X_{k'} = X \times_k k'$ admits a k'-rational point. In other words, $X_{k'}$ is isomorphic to $\mathbb{P}^1_{k'}$.*

PROOF. The nondegenerate anisotropic conic X is a subscheme of $\mathbb{P}^2_k$ that is defined by some homogeneous polynomial $F(Y_1, Y_2, Y_3)$ of degree two in three variables Y_1, Y_2, Y_3.

First assume that the characteristic of the field k is different from two. By specializing two of the variables suitably, we get an irreducible polynomial in one variable. Set k' to be the splitting field of this irreducible polynomial in one variable. Then it is easy to see that the field k' has the required property.

Now assume that the characteristic of the field k is two. If

$$F(Y_1, Y_2, Y_3) = a_1Y_1^2 + a_2Y_2^2 + a_3Y_3^2 + aY_1Y_2 + bY_1Y_3 + cY_2Y_3,$$

then from our assumption that X is nondegenerate it follows that at least one of a, b and c is not zero. Say $a \neq 0$. Since X has no k-rational point, the polynomial

$$F(Y_1, 1, 0) = a_1Y_1^2 + a_2 + aY_1$$

is an irreducible separable polynomial of degree two. The splitting field k' of this polynomial $F(Y_1, 1, 0)$ has the required property. This completes the proof of the lemma. □

LEMMA 3.3. *Let k be an infinite field and L be a field extension of k. Let V be a finite-dimensional vector space over k. The subset*

$$V = V \otimes 1 \subset V \otimes_k L =: V_L$$

is dense in the Zariski topology.

PROOF. Using induction on n, we will show that any nonempty open subset of L^n contains points of k^n. The field k being infinite, any nonempty Zariski open subset of L contains points of k, hence the statement is true for $n = 1$. Assume that, for all $j \in [1, n-1]$, any nonempty open subset of L^j contains points of k^j.

Let $U \subset L^n$ be a nonempty Zariski open subset. Take any point $(c_1, \ldots, c_n) \in U$. Consider the nonempty Zariski open subset

$$U_{c_n} := \{\lambda \in L \mid (c_1, \ldots, c_{n-1}, \lambda) \in U\} \subset L.$$

Fix any $x \in U_{c_n} \cap k$, and consider the nonempty Zariski open subset

$$U'_x := \{(\lambda_1, \ldots, \lambda_{n-1}) \in L^{n-1} \mid (\lambda_1, \ldots, \lambda_{n-1}, x) \in U\} \subset L^{n-1}.$$

By the induction hypothesis, $U'_x \cap k^{n-1} \neq \emptyset$. For any $(x_1, \ldots, x_{n-1}) \in U'_x \cap k^{n-1}$, we have $(x_1, \ldots, x_{n-1}, x) \in U \cap k^n$.

If V is a finite-dimensional vector space over k, then by choosing a basis V we can identify V with k^n and V_L with L^n. This identifies the inclusion of V in V_L with the natural inclusion of k^n in L^n. Therefore, the earlier observation completes the proof of the lemma. □

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LEMMA 3.4. *Let Y be a variety defined over an infinite field k such that Y does not admit any nonconstant regular functions. Let E and E' be two vector bundles over Y, and let L be a field extension of k. If E and E' are isomorphic after base change to L, then they are already isomorphic over k.*

PROOF. Let Y_L, E_L and E'_L be the base changes to L of Y, E and E' respectively. If E_L and E'_L are isomorphic, then they remain isomorphic over any extension field of L. Therefore, we can assume without any loss of generality that Y_L has an L-rational point. We will assume so.

Set

$$V := H^0(Y, \underline{\mathrm{Hom}}(E, E')), \tag{3.1}$$

where $\underline{\mathrm{Hom}}(E, E')$ is the sheaf of $\mathcal{O}_Y$-module homomorphisms from E to E'. By our assumption on Y that it does not admit any nonconstant regular functions, the k-vector space V is finite-dimensional. Consider the L-vector space

$$V_L := H^0(Y_L, \underline{\mathrm{Hom}}(E_L, E'_L)) \cong V \otimes_k L. \tag{3.2}$$

Since the two vector bundles E_L and E'_L are isomorphic, it can be shown that there is a nonempty Zariski open subset of the affine variety defined by V_L (see (3.2)) that parametrizes all the global isomorphisms of E_L with E'_L. To explain this we fix an L-rational point $x_0 \in Y_L$. By sending any $\alpha \in V_L$ to the homomorphism

$$\bigwedge^r \alpha(x_0) : \bigwedge^r (E_L)_{x_0} \longrightarrow \bigwedge^r (E'_L)_{x_0},$$

where $r = \mathrm{rank}(E) = \mathrm{rank}(E')$, we obtain a section of the trivial line bundle over V_L with fibre $\mathrm{Hom}(\bigwedge^r (E_L)_{x_0}, \bigwedge^r (E'_L)_{x_0})$. This section constructed using x_0 will be denoted by s_L. It is easy to see that $s_L(\alpha) = 0$ if and only if the homomorphism $\alpha : E_L \longrightarrow E'_L$ fails to be an isomorphism. Note that, since E_L and E'_L are isomorphic, the section s_L is nonzero somewhere.

Let

$$U_L \subset V_L$$

be the nonempty Zariski open subset parametrizing isomorphisms of E_L with E'_L. Now from Lemma 3.3 it follows that $V \cap U_L$ is nonempty, where V is defined in (3.1). Hence there is a homomorphism $\alpha \in V$ that is an isomorphism of the vector bundle E with E'. This completes the proof of the lemma. □

PROPOSITION 3.5. *Let X be a nondegenerate anisotropic conic defined over a field k. Then there is an indecomposable vector bundle S of rank two and degree two over X. Two such vector bundles over X are isomorphic.*

PROOF. Let T_X denote the tangent bundle of X. Using Serre duality

$$H^1(X, T_X^\vee) = H^0(X, \mathcal{O}_X)^\vee = k^\vee = k.$$

Let

$$0 \longrightarrow \mathcal{O}_X \longrightarrow S \longrightarrow T_X \longrightarrow 0 \tag{3.3}$$

be the extension corresponding to $1 \in k$. For any $\lambda \in k \setminus \{0\}$, the extension bundle corresponding to λ is isomorphic to S.

We will show that the vector bundle S in (3.3) is indecomposable.

To prove this, fix a Galois extension field k' of k of degree two such that $X_{k'} = X \times_k k'$ has a k'-rational point; such a field exists by Lemma 3.2. Therefore, $X_{k'}$ is isomorphic to $\mathbb{P}^1_{k'}$.

Consider the exact natural exact sequence of vector bundles

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1_{k'}}(-1) \longrightarrow H^0(\mathbb{P}^1_{k'}, \mathcal{O}_{\mathbb{P}^1_{k'}}(1)) \otimes_{k'} \mathcal{O}_{\mathbb{P}^1_{k'}} \longrightarrow \mathcal{O}_{\mathbb{P}^1_{k'}}(1) \longrightarrow 0,$$

defined by the evaluation morphism. Tensoring this with the line bundle $\mathcal{O}_{\mathbb{P}^1_{k'}}(1)$ we get

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1_{k'}} \longrightarrow H^0(\mathbb{P}^1_{k'}, \mathcal{O}_{\mathbb{P}^1_{k'}}(1)) \otimes_{k'} \mathcal{O}_{\mathbb{P}^1_{k'}}(1) \longrightarrow \mathcal{O}_{\mathbb{P}^1_{k'}}(2) \cong T_{\mathbb{P}^1_{k'}} \longrightarrow 0, \tag{3.4}$$

which is a nonsplit extension of $T_{\mathbb{P}^1_{k'}}$ by $\mathcal{O}_{\mathbb{P}^1_{k'}}$; see [5, p. 182, Example 8.20.1].

Since $\dim H^1(\mathbb{P}^1_{k'}, T^\vee_{\mathbb{P}^1_{k'}}) = 1$ and $X_{k'} \cong \mathbb{P}^1_{k'}$, the vector bundle

$$H^0(\mathbb{P}^1_{k'}, \mathcal{O}_{\mathbb{P}^1_{k'}}(1)) \otimes_{k'} \mathcal{O}_{\mathbb{P}^1_{k'}}(1) \cong \mathcal{O}_{\mathbb{P}^1_{k'}}(1) \oplus \mathcal{O}_{\mathbb{P}^1_{k'}}(1)$$

in (3.4) is isomorphic to the vector bundle $S \otimes_k k'$ over $X_{k'}$, where S is defined in (3.3).

Let ξ denote the unique line bundle of degree one over $X_{k'}$. So ξ corresponds to $\mathcal{O}_{\mathbb{P}^1_{k'}}(1)$ by any isomorphism of $X_{k'}$ with $\mathbb{P}^1_{k'}$.

The vector bundle $S \otimes_k k'$ decomposes by Proposition 2.1. Let

$$\xi^{\otimes d_1} \oplus \xi^{\otimes d_2} = S \otimes_k k'$$

be a decomposition of $S \otimes_k k'$. Note that $d_1 + d_2 = \text{degree}(S) = 2$. On the other hand, as we noted above,

$$S \otimes_k k' = \xi \oplus \xi.$$

If $d_1 > 1$, then

$$H^0(X_{k'}, \underline{\text{Hom}}(\xi^{\otimes d_1}, S \otimes_k k')) = H^0(X_{k'}, \underline{\text{Hom}}(\xi^{\otimes d_1}, \xi^{\oplus 2})) = 0.$$

But

$$H^0(X_{k'}, \underline{\text{Hom}}(\xi^{\otimes d_1}, S \otimes_k k')) \neq 0,$$

because $\xi^{\otimes d_1}$ is a subbundle of $S \otimes_k k'$. Hence $d_1 \leq 1$. Similarly, $d_2 \leq 1$. Consequently,

$$d_1 = 1 = d_2.$$

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The complete linear system for any line bundle of degree one over X gives an isomorphism of X with $\mathbb{P}^1_k$. In view of the earlier observation, we therefore conclude that the vector bundle S is indecomposable.

Let S' be another indecomposable vector bundle over X of rank two and degree two. Using the Riemann–Roch theorem,

$$\dim H^0(X, S') \geq 4.$$

In particular, S' admits nonzero sections. Take any nonzero section θ of S'. We have a short exact sequence of coherent sheaves

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\theta\cdot} S' \xrightarrow{\varphi} Q := S'/\mathrm{image}(\theta\cdot) \longrightarrow 0$$

over X. Set

$$\eta := \varphi^{-1}(\mathrm{Torsion}(Q)),$$

where $\mathrm{Torsion}(Q)$ is the torsion part of the above quotient Q. We note that η is a line subbundle of S'. Also $\mathrm{degree}(\eta) \geq 0$, because θ is a section of η.

Consider the exact sequence of coherent sheaves

$$0 \longrightarrow \eta \longrightarrow S' \longrightarrow S'/\eta \longrightarrow 0 \tag{3.5}$$

over X. We note that, since η is a line subbundle of S', the quotient S'/η is a line bundle. If $\mathrm{degree}(\eta) = 0$, then $\eta = \mathcal{O}_X$ and $S'/\eta \cong T_X$. In other words, the exact sequence (3.5) makes S' a nontrivial extension of T_X by $\mathcal{O}_X$. Therefore, if $\mathrm{degree}(\eta) = 0$, then the vector bundle S' is isomorphic to S defined in (3.3).

Assume that $\mathrm{degree}(\eta) > 0$. Then $\mathrm{degree}(\underline{\mathrm{Hom}}(S'/\eta, \eta)) \geq 0$. Therefore,

$$H^1(X, \underline{\mathrm{Hom}}(S'/\eta, \eta)) = 0.$$

Consequently, the exact sequence (3.5) splits, which contradicts the assumption that S' is indecomposable. Hence $\mathrm{degree}(\eta) = 0$ and S' is isomorphic to S. This completes the proof of the proposition. □

REMARK 3.6. Let S be the indecomposable vector bundle in Proposition 3.5. The vector bundle $S_{k'} := S \otimes_k k'$ on $X_{k'} \cong \mathbb{P}^1_{k'}$ is semistable but not stable. Since $S_{k'} \cong \xi^{\oplus 2}$, where ξ is as in the proof of Proposition 3.5, and X does not admit any line bundle of degree one, we conclude that the vector bundle S is stable. Therefore, any nonzero global endomorphism of S is an isomorphism.

REMARK 3.7. Let $\mathrm{End}_k(S)$ denote the k algebra of global endomorphisms of the vector bundle S. From Remark 3.6 it follows that $\mathrm{End}_k(S)$ is a division algebra. Since

$$\mathrm{End}_k(S) \otimes_k k' \cong \mathrm{End}_{k'}(S_{k'}) \cong M_2(k'),$$

where $M_2(k')$ is the algebra of 2×2 matrices over k', we conclude that $\mathrm{End}_k(S)$ is a quaternion division algebra with k' as one of its splitting fields. Thus the stable vector bundle S is not simple. (A vector bundle is said to be *simple* if all its global endomorphisms are scalars.)

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REMARK 3.8. It is easy to see that the indecomposable vector bundle $S \otimes_{\mathcal{O}_X} T_X$ on a nondegenerate anisotropic conic X is isomorphic to $T_{\mathbb{P}^2_k}|_X$, the restriction to X of the tangent bundle of $\mathbb{P}^2_k$. If the characteristic of k is different from two, then the vector bundle S is isomorphic to the first jet bundle $J^1(T_X)$ of the tangent bundle T_X.

4. Vector bundles over anisotropic conic

The following theorem classifies the isomorphism classes of vector bundles over a nondegenerate anisotropic conic.

THEOREM 4.1. *Let X be a nondegenerate anisotropic conic over a field k. Let T_X be the tangent bundle of X and S the unique indecomposable vector bundle over X of rank two and degree two. Any vector bundle E over X is isomorphic to a vector bundle of the following form:*

$$\left(\bigoplus_{i=1}^{m} T_X^{\otimes a_i}\right) \oplus \left(S \otimes \left(\bigoplus_{j=1}^{n} T_X^{\otimes b_j}\right)\right), \tag{4.1}$$

where m and n are nonnegative integers, and $\{a_i\}_{i=1}^m \in \mathbb{Z}^{\oplus m}$ and $\{b_j\}_{j=1}^n \in \mathbb{Z}^{\oplus n}$. Furthermore, if

$$\left(\bigoplus_{i=1}^{m} T_X^{\otimes a_i}\right) \oplus \left(S \otimes \left(\bigoplus_{j=1}^{n} T_X^{\otimes b_j}\right)\right) \cong \left(\bigoplus_{i=1}^{m'} T_X^{\otimes a'_i}\right) \oplus \left(S \otimes \left(\bigoplus_{j=1}^{n'} T_X^{\otimes b'_j}\right)\right),$$

then $m = m'$, $n = n'$ and $\{a'_i\}_{i=1}^{m'} \in \mathbb{Z}^{\oplus m'}$ (respectively, $\{b'_j\}_{j=1}^{n'} \in \mathbb{Z}^{\oplus n'}$) is a permutation of $\{a_i\}_{i=1}^m$ (respectively, $\{b_j\}_{j=1}^n$).

PROOF. First assume that

$$\left(\bigoplus_{i=1}^{m} T_X^{\otimes a_i}\right) \oplus \left(S \otimes \left(\bigoplus_{j=1}^{n} T_X^{\otimes b_j}\right)\right) \cong \left(\bigoplus_{i=1}^{m'} T_X^{\otimes a'_i}\right) \oplus \left(S \otimes \left(\bigoplus_{j=1}^{n'} T_X^{\otimes b'_j}\right)\right). \tag{4.2}$$

Let k' be a degree-two Galois field extension of k such that $X_{k'} := X \times_k k' \cong \mathbb{P}^1_{k'}$ (see Lemma 3.2). So the vector bundle $S_{k'} := S \otimes_k k'$ over $X_{k'} \cong \mathbb{P}^1_{k'}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1_{k'}}(1)^{\oplus 2}$ (see the proof of Proposition 3.5).

From (4.2) we have

$$\left(\bigoplus_{i=1}^{m} T_{X_{k'}}^{\otimes a_i}\right) \oplus \left(S_{k'} \otimes \left(\bigoplus_{j=1}^{n} T_{X_{k'}}^{\otimes b_j}\right)\right)$$
$$\cong \left(\bigoplus_{i=1}^{m'} T_{X_{k'}}^{\otimes a'_i}\right) \oplus \left(S_{k'} \otimes \left(\bigoplus_{j=1}^{n'} T_{X_{k'}}^{\otimes b'_j}\right)\right).$$

The degree of $T_{X_{k'}}^{\otimes a}$ is even and the degree of $T_{\mathbb{P}^1_{k'}}^{\otimes b} \otimes \mathcal{O}_{\mathbb{P}^1_{k'}}(1)$ is odd, and we have $S_{k'} \cong \mathcal{O}_{\mathbb{P}^1_{k'}}(1)^{\oplus 2}$. Therefore, from Proposition 2.1 it follows that $m = m'$, $n = n'$ and $\{a'_i\}_{i=1}^{m'} \in \mathbb{Z}^{\oplus m'}$ (respectively, $\{b'_j\}_{j=1}^{n'} \in \mathbb{Z}^{\oplus n'}$) is a permutation of $\{a_i\}_{i=1}^{m}$ (respectively, $\{b_j\}_{j=1}^{n}$).

Now we will prove the first part of the theorem.

Take any vector bundle E over X. Let

$$0 = F_0 \subset F_1 \subset \cdots \subset F_{\ell-1} \subset F_\ell = E \tag{4.3}$$

be the Harder–Narasimhan filtration of E (see [4, p. 220, Lemma 1.3.7]).

Let $F'_i := F_i \otimes_k k'$ be the vector bundle over $X_{k'} \cong \mathbb{P}^1_{k'}$, where k' and $X_{k'}$ are as above. From the uniqueness of the Harder–Narasimhan filtration of a vector bundle and the fact that k' is a Galois extension of k, it follows immediately that the filtration

$$0 = F'_0 \subset F'_1 \subset \cdots \subset F'_{\ell-1} \subset F'_\ell = E_{k'} \tag{4.4}$$

obtained from (4.3) coincides with the Harder–Narasimhan filtration of $E_{k'}$. Therefore, each successive quotient F'_i/F'_{i-1}, $i \in [1, \ell]$, is isomorphic to a vector bundle of the form $\mathcal{O}_{\mathbb{P}^1_{k'}}(n_i)^{\oplus m_i}$, and $n_i > n_j$ if $i < j$.

As $H^1(\mathbb{P}^1_{k'}, \mathcal{O}_{\mathbb{P}^1_{k'}}(n)) = 0$ for all $n \geq 0$, from the above properties of the successive quotients F'_i/F'_{i-1} it follows immediately that

$$H^1(X_{k'}, \mathrm{Hom}(F'_j/F'_{j-1}, F'_{j-1})) = 0, \tag{4.5}$$

for all $j \in [1, \ell]$.

The filtration in (4.3) gives a sequence of short exact sequences

$$0 \longrightarrow F_{j-1} \longrightarrow F_j \longrightarrow F_j/F_{j-1} \longrightarrow 0,$$

$j \in [1, \ell]$. Since the obstruction to the splitting of the above short exact sequence is an element of $H^1(X, \mathrm{Hom}(F_j/F_{j-1}, F_{j-1}))$, and

$$H^1(X, \mathrm{Hom}(F_j/F_{j-1}, F_{j-1})) \otimes_k k' = H^1(X_{k'}, \mathrm{Hom}(F'_j/F'_{j-1}, F'_{j-1}))$$

(the cohomology base changes), using (4.5) we conclude that the filtration in (4.3) splits completely. Therefore,

$$E \cong \bigoplus_{i=1}^{\ell} (F_i/F_{i-1}). \tag{4.6}$$

As each successive quotient F_i/F_{i-1}, $i \in [1, \ell]$, in (4.3) is semistable, from (4.6) we conclude the following. To prove the first part of the theorem, it is enough to prove it under the assumption that the vector bundle E is semistable (note that the collection of vector bundles of the form (4.1) is closed under the direct sum operation). Henceforth, we will assume that the vector bundle E is semistable.

Consequently, the vector bundle $E_{k'} = E \otimes_k k'$ over $X_{k'}$ is semistable. Therefore,

$$E_{k'} \cong \zeta^{\oplus r}, \tag{4.7}$$

where ζ denotes a line bundle over $X_{k'}$ and $r = \text{rank}(E)$.

First assume that degree(ζ) is even. In that case,

$$\zeta = (T_{X_{k'}})^{\otimes d},$$

where $d \in \mathbb{Z}$. Hence from (4.7) it follows that the base change to k' of the vector bundle $((T_X)^{\otimes d})^{\oplus r}$ over X is isomorphic to $E_{k'}$. Now using Lemma 3.4, we have

$$E \cong ((T_X)^{\otimes d})^{\oplus r}.$$

(Note that, since X is a nondegenerate anisotropic conic defined over k, the field k must be infinite; see Remark 3.1.) Therefore, the theorem is proved when E is semistable and degree(ζ) is even.

Next we assume that degree(ζ) is odd, say degree$(\zeta) = 2d + 1$. So

$$\zeta = (T_{X_{k'}})^{\otimes d} \otimes_{\mathcal{O}_{X_{k'}}} \xi, \tag{4.8}$$

where ξ denotes the unique line bundle of degree one on $X_{k'}$ (as in the proof of Proposition 3.5).

We note that X does not admit any line bundle of odd degree. Indeed, the conic X being anisotropic, there is no line bundle over X of degree one. On the other hand, degree$(T_X) = 2$. Hence X does not admit any line bundle of odd degree.

For the vector bundle $E_{k'}$ in (4.7), from (4.8) it follows that the degree of the top exterior product $\bigwedge^r E_{k'}$ is $r(2d + 1)$. Since degree$(\bigwedge^r E_{k'}) = $ degree$(\bigwedge^r E)$, and X does not admit any line bundle of odd degree, we conclude that $r = 2r_0$, where $r_0 \in \mathbb{N}$.

Therefore, the base change to k' of the vector bundle $(T_X)^{\otimes d} \otimes S^{\oplus r_0}$ over X is isomorphic to $E_{k'}$, where S is the vector bundle in Proposition 3.5. Hence from Lemma 3.4 it follows that E is isomorphic to $(T_X)^{\otimes d} \otimes S^{\oplus r_0}$. This completes the proof of the theorem. □

https://doi.org/10.1017/S1446788708000529 Published online by Cambridge University Press

Acknowledgements

We thank the referee for providing very detailed comments to improve the manuscript. We thank the Université des Sciences et Technologies de Lille, where the work was carried out, for its hospitality.

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