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Elements in a Numerical Semigroup with Factorizations of the Same Length

S. T. Chapman, P. A. García-Sánchez, D. Llena, and J. Marshall

Abstract. Questions concerning the lengths of factorizations into irreducible elements in numerical monoids have gained much attention in the recent literature. In this note, we show that a numerical monoid has an element with two different irreducible factorizations of the same length if and only if its embedding dimension is greater than two. We find formulas in embedding dimension three for the smallest element with two different irreducible factorizations of the same length and the largest element whose different irreducible factorizations all have distinct lengths. We show that these formulas do not naturally extend to higher embedding dimensions.

Let $\mathbb{N}$ denote the set of nonnegative integers. For a subset A of $\mathbb{N}$, set

$$\langle A\rangle = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid n \in \mathbb{N}, \lambda_1, \dots, \lambda_n \in \mathbb{N}, a_1, \dots, a_n \in A\}.$$

This set is a submonoid of $\mathbb{N}$, since it contains the zero element and it is closed under addition. Every submonoid of $\mathbb{N}$ is of this form. Moreover, if S is a submonoid of $\mathbb{N}$ and $S^* = S \setminus \{0\}$, then $A = S^* \setminus (S^* + S^*)$ has finitely many elements and it is the smallest subset of S such that $\langle A\rangle = S$. We will refer to A as the *minimal system of generators* of S. Its cardinality is the *embedding dimension* of S.

Let S be a submonoid of $\mathbb{N}$ generated by $\{n_1, \dots, n_p\}$. By definition, if s belongs to S, then there exists a p-tuple of nonnegative integers $(a_1, \dots, a_p)$ such that $s = a_1 n_1 + \cdots + a_p n_p$. We say that $(a_1, \dots, a_p)$ is a *factorization* of s in S. If $(a_1, \dots, a_p)$ and $(b_1, \dots, b_p)$ are two factorizations of s in S, then we say that they are different if $a_i \neq b_i$ for some $1 \leq i \leq p$. The *length* of a factorization $(a_1, \dots, a_p)$ is defined as $a_1 + \cdots + a_p$. The set of factorizations of s can be seen as the set of nonnegative integer solutions of the equation $x_1 n_1 + \cdots + x_p n_p = s$, which is clearly finite.

A *numerical semigroup* is a submonoid of $\mathbb{N}$ with finite complement in $\mathbb{N}$. In this paper we show that a numerical semigroup has an element with two different factorizations of the same length if and only if its embedding dimension is greater than two. Moreover, if such an element exists, then there is a positive integer such that any other element in the semigroup greater than this integer has two different factorizations with the same length. Hence, we can consider the least element having two different factorizations with the same length, and the largest element in the semigroup all of whose factorizations have different lengths. For embedding dimension three we are able to determine these elements explicitly. We find these results of interest due to the number of recent papers that have explored the factorization properties

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of numerical monoids (see [1–6]). For an overview of the theory of nonunique factorizations in integral domains and monoids, the interested reader is referred to the recent monograph [8].

1 Increasing the Dimension

Let $\mathbb{Z}$ denote the set of integers. Given $z \in \mathbb{Z}^k$ for some positive integer k, we can write $z = z^+ - z^-$, with $z^+, z^- \in \mathbb{N}^k$ and $z^+ \cdot z^- = 0$ (where $\cdot$ stands for the usual dot product).

Theorem 1.1 *Let S be a numerical semigroup minimally generated by the set $\{n_1, \dots, n_p\}$. Then there exists an element in S with at least two different factorizations with the same length if and only if $p \geq 3$.*

Proof For p equal to one, S is $\mathbb{N}$, and thus it is a unique factorization monoid. Assume that $s \in S$ is such that two of its factorizations, $x = (x_1, \dots, x_p)$ and $y = (y_1, \dots, y_p)$, have the same length. Then $x - y$ induces a solution in $\mathbb{Z}^p$ to

$$z_1 n_1 + \cdots + z_p n_p = 0, \quad z_1 + \cdots + z_p = 0. \tag{1.1}$$

The converse is also true. Any nontrivial solution z with integer coefficients of this system of equations yields two different factorizations of an element in S with the same length. Just take z^+ and z^-.

For $p = 2$, the only integer solution to (1.1) is the trivial solution, since $(n_1, 1)$ and $(n_2, 1)$ are linearly independent. Thus in this setting, all the factorizations of all the elements in S have different lengths.

For $p \geq 3$, the above system always has nontrivial integer solutions, since the columns of the equations are linearly dependent over the rationals. ■

Note that if $s \in S$ admits two different factorizations with the same length, then so do all the elements in $s+S$. Denote by $\mathrm{F}(S)$ the largest integer not in S, which is known as *the Frobenius number* of S. Then every integer greater than $s + \mathrm{F}(S)$ admits at least two different factorizations in S with the same length. This proves the following.

Proposition 1.2 *Let S be a numerical semigroup, and let w be the least element in S with at least two different factorizations with the same length in S. Then all the elements in the set $w + S$ have at least two different factorizations with the same length. In particular, every integer x greater than $w + \mathrm{F}(S)$ has two different factorizations of equal length in S.*

For embedding dimension three, the above result can be sharpened. In fact, we prove that $w + S$ is precisely the set of all elements having at least two different factorizations with the same length in S.

Let S be the numerical semigroup minimally generated by n_1, n_2, n_3, with $n_1 < n_2 < n_3$. Let M be the subgroup of $\mathbb{Z}^3$ whose defining equations are

$$z_1 n_1 + z_2 n_2 + z_3 n_3 = 0,$$

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Set $d = \gcd\{n_2 - n_1, n_3 - n_1\}$. Observe that all integer solutions of the above system are integral multiples of $(\frac{n_3-n_2}{d}, \frac{n_1-n_3}{d}, \frac{n_2-n_1}{d})$. Let $s \in S$ be such that there exist two factorizations (x_1, x_2, x_3) and (y_1, y_2, y_3) of s with $x_1+x_2+x_3 = y_1+y_2+y_3$. Then $(x_1 - y_1, x_2 - y_2, x_3 - y_3)$ belongs to M. Assume without loss of generality that $x_1 \geq y_1$. Then there exists a positive integer λ such that

$$(x_1 - y_1, x_2 - y_2, x_3 - y_3) = \lambda\Big(\frac{n_3 - n_2}{d}, \frac{n_1 - n_3}{d}, \frac{n_2 - n_1}{d}\Big).$$

This implies that $x_2 = y_2 - \lambda\frac{n_3-n_1}{d} \geq 0$ and

$$\begin{aligned} s &= x_1n_1 + x_2n_2 + x_3n_3 = y_1n_1 + y_2n_2 + y_3n_3 \\ &= y_1n_1 + \Big(y_2 - \lambda\frac{n_3 - n_1}{d}\Big)n_2 + y_3n_3 + \lambda\frac{n_3 - n_1}{d}n_2. \end{aligned}$$

If we write

$$s' = y_1n_1 + \Big(y_2 - \lambda\frac{n_3 - n_1}{d}\Big)n_2 + y_3n_3,$$

then $s = s' + \lambda\frac{n_3-n_1}{d}n_2$. In particular, $w = \frac{n_3-n_1}{d}n_2$ is the least element in S having two different factorizations $((\frac{n_3-n_2}{d}, 0, \frac{n_2-n_1}{d})$ and $(0, \frac{n_3-n_1}{d}, 0))$ with the same length, and any other element fulfilling this property belongs to $w + S$. Thus, we have proved the following.

Theorem 1.3 *Let $S = \langle n_1 < n_2 < n_3\rangle$ be an embedding dimension three numerical semigroup. Then the set of elements with at least two different factorizations with the same length is*

$$\frac{n_3 - n_1}{d}n_2 + S.$$

In particular, $\frac{n_3-n_1}{d}n_2$ is the least element in S having at least two different factorizations with the same length, and $\frac{n_3-n_1}{d}n_2 + \mathrm{F}(S)$ is the largest element in S all of whose factorizations have different lengths.

This idea will not work for higher embedding dimensions, as the following example shows.

Example 1.4 Let $S = \langle 7, 8, 9, 11, 15\rangle$. Then $\mathrm{F}(S) = 19$ and $w = 18$. The factorizations of 18 are $\{(0, 2, 0, 0), (1, 0, 1, 0)\}$, and those of 18 + 19 are $\{(0, 0, 2, 1), (1, 0, 0, 2), (4, 1, 0, 0)\}$. Hence 37 has two different factorizations of the same length and $37 \notin w + S$. These computations can be easily performed by using the GAP package numericalsgps ([7]).

2 Decreasing the Embedding Dimension

There is an alternative way to find the first element in a numerical semigroup having at least two different factorizations with the same length. Let S be a numerical semigroup minimally generated by $\{n_1 < n_2 < \cdots < n_p\}$. In view of Theorem 1.1, we assume that $p \geq 3$. Set

$$d_i = n_{i+1} - n_i, i \in \{1, \dots, p - 1\} \quad \text{and} \quad D = \langle d_1, \dots, d_{p-1}\rangle.$$

Let M be the subgroup of $\mathbb{Z}^p$ with defining equations (1.1). We know that if an element $s \in S$ has two different factorizations $x = (x_1, \dots, x_p)$ and $y = (y_1, \dots, y_p)$ with the same length, then $z = x - y$ is an element of M. It follows that

$$(z_2 + \cdots + z_p, z_3 + \cdots + z_p, \dots, z_{p-1} + z_p, z_p)$$

belongs to the subgroup N of $\mathbb{Z}^{p-1}$ with defining equation

$$x_1 d_1 + \cdots + x_{p-1} d_{p-1} = 0.$$

Analogously, if $(s_1, \dots, s_{p-1})$ is an element of N, then $(-s_1, s_1 - s_2, \dots, s_{p-2} - s_{p-1}, s_{p-1})$ belongs to M. This provides a one-to-one correspondence between elements in M and elements in N. This correspondence is a group isomorphism. Let us denote it by $f\colon M \to N$. Using this notation, it is easy to prove the following.

Proposition 2.1 *Let S be a numerical semigroup minimally generated by $\{n_1 < n_2 < \cdots < n_p\}$ with $p \geq 3$. Let s be an element in S with at least two different factorizations x and y with the same length. Then $f(x-y)^+$ and $f(x-y)^-$ are distinct factorizations of an element in D. Conversely, for every element in D with two different factorizations a and b, $f^{-1}(a-b)^+$ and $f^{-1}(a-b)^-$ are factorizations of an element in S with the same length.*

Example 2.2 Let S be minimally generated by $\{n_1 < n_2 < n_3\}$. Then $D = \langle d_1 = n_2 - n_1, d_2 = n_3 - n_2 \rangle$. The first element having two factorizations in D is $\frac{d_1 d_2}{d}$ with $d = \gcd\{d_1, d_2\}$. Observe that $a = (\frac{d_2}{d}, 0)$ and $b = (0, \frac{d_1}{d})$ are its factorizations. Then

$$f^{-1}(a-b) = \Big(\frac{d_2}{d}, -\frac{d_2}{d} - \frac{d_1}{d}, \frac{d_1}{d}\Big) = \Big(\frac{n_3 - n_2}{d}, \frac{n_1 - n_3}{d}, \frac{n_2 - n_1}{d}\Big).$$

This idea can be used for other families of numerical semigroups such as those generated by arithmetic progressions, quasi-arithmetic progressions, or those whose three smallest minimal generators are in arithmetic progression. For the general case, nothing more relevant can be said, since we cannot control which are the minimal relations defining a presentation for D (and thus control the factorizations in D).

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Sam Houston State University, Department of Mathematics and Statistics, Huntsville, TX, U.S.A.
e-mail: scott.chapman@shsu.edu

Departamento de Álgebra, Universidad de Granada, Granada, España
e-mail: pedro@ugr.es

Departamento de Geometría, Topología y Química Orgánica, Universidad de Almería, Almería, España
e-mail: dllena@ual.es

Sandia National Laboratories, P.O. Box 5800, Albuquerque, NM, U.S.A.
e-mail: jormars@sandia.gov

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