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A Space of Harmonic Maps from a Sphere into the Complex Projective Space

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Abstract. Guest–Ohnita and Crawford have shown the path-connectedness of the space of harmonic maps from S^2 to $\mathbf{C}P^n$ of a fixed degree and energy. It is well known that the ∂ transform is defined on this space. In this paper, we will show that the space is decomposed into mutually disjoint connected subspaces on which ∂ is homeomorphic.

1 Introduction

Let $\mathcal{H}arm(S^2, \mathbf{C}P^n)$ be the space of harmonic maps from the Riemann sphere S^2 into the complex projective space $\mathbf{C}P^n$ contained in a Banach manifold $W^{1,4}(S^2, \mathbf{C}P^n)$. Then $\mathcal{H}arm(S^2, \mathbf{C}P^n)$ is the disjoint union of $\mathcal{H}arm_{k,E}(S^2, \mathbf{C}P^n)$ consisting of maps of degree k and energy E. By [GO, C], it is known that $\mathcal{H}arm_{k,E}(S^2, \mathbf{C}P^n)$ is path-connected. Let $\mathcal{H}ol_k(S^2, \mathbf{C}P^n)$ be the space of either holomorphic maps of degree $k > 0$ or anti-holomorphic maps of degree $k < 0$. For an integer $r \geq 0$, we denote by $\mathcal{H}ol_{k,r}(S^2, \mathbf{C}P^2)$ the subspace of $\mathcal{H}ol_k(S^2, \mathbf{C}P^2)$ consisting of maps with ramification index r. By [C], for $0 \leq r \leq k-2$, the map

$$\partial\colon \mathcal{H}ol_{k,r}(S^2, \mathbf{C}P^2) \to \mathcal{H}arm_{k-2-r,3k-2-r}(S^2, \mathbf{C}P^2)$$

is a homeomorphism. Inspired by those works, Lemaire and Wood [LW1, LW2] showed the smoothness of the map ∂, the injectivity of its differential, and that any Jacobi field of a harmonic map of S^2 into $\mathbf{C}P^2$ is always integrable. For general $n \geq 3$, $\partial\colon \mathcal{H}arm_{k,E}(S^2, \mathbf{C}P^n) \to \mathcal{H}arm_{k',E'}(S^2, \mathbf{C}P^n)$ is not necessarily continuous. See Example 6.3. In this paper, we consider the subspaces of $\mathcal{H}arm_{k,E}(S^2, \mathbf{C}P^n)$ on which ∂ is continuous. Any new terminology or notation is explained in the following sections.

Theorem 1.1 *Take an integer $n \geq 2$ and an $(n-1)$-tuple $R_J = (R_0, R_1, \dots, R_{n-2})$.*

(i) *If $\mathcal{H}ol^*_{k,R_J}(S^2, \mathbf{C}P^n)$ is not empty, it is a path-connected complex submanifold of the complex manifold $\mathcal{H}ol^*_k(S^2, \mathbf{C}P^n)$ of dimension*

$$(k+1)(n+1) - 1 - \sum_{s=0}^{n-2}(n-s-1)R_s.$$

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(ii) *For* $1 \leq s \leq n$, *the map* $\partial^s\colon \mathcal{H}ol^*_{k,R_J}(S^2, \mathbf{C}P^n) \to \mathcal{H}arm^*_{k,s,R_J}(S^2, \mathbf{C}P^n)$ *is a homeomorphism. So* $\mathcal{H}arm^*_{k,s,R_J}(S^2, \mathbf{C}P^n)$ *can be given the structure of a complex manifold.*

As a direct result, we get the following corollary.

Corollary 1.2 *Put* $\mathcal{H}arm^*_{k,0,R_J}(S^2, \mathbf{C}P^n) = \mathcal{H}ol^*_{k,R_J}(S^2, \mathbf{C}P^n)$. *Then, for* $0 \leq s \leq n-1$, *the map*

$$\partial\colon \mathcal{H}arm^*_{k,s,R_J}(S^2, \mathbf{C}P^n) \to \mathcal{H}arm^*_{k,s+1,R_J}(S^2, \mathbf{C}P^n)$$

is a homeomorphism, where $\mathcal{H}arm^*_{k,n,R_J}(S^2, \mathbf{C}P^n)$ *consists of anti-holomorphic maps.*

The contents of this paper are as follows. In Section 2, we recall required facts of harmonic maps into the complex projective space and their harmonic sequences. In Section 3, we also recall facts about bubble tree maps and their convergence theorem. Then we show a gluing theorem of a holomorphic bubble tree map. In Section 4, we prove Theorem 1.1. In Section 5, we apply Theorem 1.1 to consider gluings of harmonic bubble tree maps. Finally, in Section 6, we give some examples.

2 Harmonic Maps and their Sequences

Let $\mathbf{C}P^n$ be the complex projective space with the Fubini–Study metric g and the complex structure J on $\mathbf{C}P^n$ induced by the multiplication $\sqrt{-1}$. We identify the sphere S^2 with $\mathbf{C}P^1$ through a stereographic projection sending the north-pole to the origin, the south-pole to the infinity and the equator to the unit circle:

$$S^2 - \{\text{the south pole}\} \simeq \mathbf{C} \simeq \mathbf{C}P^1 - \{[0\,;1]\}.$$

The Riemann sphere S^2, g_0 has the induced metric g_0 from the Fubini–Study metric that is represented by $ds_0^2 = \varphi\overline{\varphi}$. Here φ is a one-form defined up to a factor of absolute value 1.

For $p > 2$ and $r \geq 0$, let $W^{r,p}(S^2, \mathbf{C}P^n)$ be a Banach manifold consisting of maps $f\colon S^2 \to \mathbf{C}P^n$ whose derivatives of order $\leq r$ are L_p integrable. A harmonic map f is a critical point of the energy functional $E\colon W^{1,4}(S^2, \mathbf{C}P^n) \to \mathbf{R}$ defined by

$$E(f) = \int_{S^2} |df|^2 \frac{\sqrt{-1}}{2} \varphi \wedge \overline{\varphi},$$

where $|df|^2$ is the Hilbert–Schmidt's norm $\langle g_0, f^*g \rangle_{HS}$. We normalize E so that $E(f) = k$ for $f \in \mathcal{H}ol_k(S^2, \mathbf{C}P^n)$. Consider $\mathcal{H}arm(S^2, \mathbf{C}P^n)$ as the subspace of $W^{1,4}(S^2, \mathbf{C}P^n)$. Denote by $\mathcal{H}arm_k(S^2, \mathbf{C}P^n)$ the subspace of $\mathcal{H}arm(S^2, \mathbf{C}P^n)$ consisting of maps of degree k. Let $C^j(S^2, \mathbf{C}P^n)$ be the space of C^j maps from S^2 into $\mathbf{C}P^n$. By the Sobolev embedding theorem $W^{1,4} \subset C^0$ and the regularity theorem, $\mathcal{H}arm(S^2, \mathbf{C}P^n)$ is contained in $C^j(S^2, \mathbf{C}P^n)$ for any $j \geq 0$. Since $(\mathbf{C}P^n, g)$ is a Kähler manifold, the space $\mathcal{H}ol_k(S^2, \mathbf{C}P^n)$ is also contained in $\mathcal{H}arm_k(S^2, \mathbf{C}P^n)$. Note that $\mathcal{H}arm_k(S^2, \mathbf{C}P^1) = \mathcal{H}ol_k(S^2, \mathbf{C}P^1)$.

We say that $f \in \mathcal{H}arm(S^2, \mathbf{C}P^n)$ is full if its image lies in no proper projective subspace of $\mathbf{C}P^n$. Let $\mathcal{H}arm_k^*(S^2, \mathbf{C}P^n)$ be the subspace of $\mathcal{H}arm_k(S^2, \mathbf{C}P^n)$ consisting of full maps. Put $\mathcal{H}ol_k^*(S^2, \mathbf{C}P^n) = \mathcal{H}arm_k^*(S^2, \mathbf{C}P^n) \cap \mathcal{H}ol_k(S^2, \mathbf{C}P^n)$.

Now we introduce a ∂ transform and a $\overline{\partial}$ transform in [CW], which is the same correspondence given in [EW, §3]. Denote by $G(t, s)$ the complex Grassmann manifold consisting of s-dimensional subspaces in $\mathbf{C}^t$. We equip the standard Riemann metric and the complex structure on it. See [KN, IX, Example 6.4].

For $f \in \mathcal{H}ol_k^*(S^2, \mathbf{C}P^n)$ and $1 \leq s \leq n-1$, consider the map $w_s \colon U \to \wedge^{s+1}\mathbf{C}^{n+1}$ defined by

$$\begin{aligned} w_s(z) &= f_U(z) \wedge f_U'(z) \wedge \cdots \wedge f_U^{(s)}(z) \\ &= \sum_{0 \leq i_0 < \cdots < i_s \leq n} D(p_{i_0}, p_{i_1}, \ldots, p_{i_s})\, e_{i_0} \wedge \cdots \wedge e_{i_s}, \end{aligned}$$

where $f_U \colon U \to \mathbf{C}^{n+1} - \{0\}$ is a lift of f, $\{e_j\}_j$ are the standard one forms of $\mathbf{C}^{n+1}$, and we identify

$$f_U = (p_0, \ldots, p_n) = \sum_j p_j\, e_j, \quad f_U^{(s)} = \frac{d^s}{dz^s} f_U = \sum_j \Big(\frac{d^s}{dz^s} p_j \Big)\, e_j.$$

Let S_f be the set of $z \in S^2$ so that the dimension of the space defined by $w_s(z)$ is less than $s+1$. When $z \notin S_f$, $w_s(z)$ defines an $(s+1)$-dimensional subspace of $\mathbf{C}^{n+1}$. When $z_0 \in S_f$, $w_s(z) = (z - z_0)^\rho W_s(z)$ with $W_s(z_0) \neq 0$ on a neighbourhood of z_0. Since W_s is holomorphic and defines an $(s+1)$-dimensional subspace of $\mathbf{C}^{n+1}$ that is equal to the space defined by w_s except at z_0, we get a holomorphic map $f_s \colon S^2 \to G(n+1, s+1)$ defined by $f_s(z)$ the space $w_s(z)$ or $W_s(z)$. For details, see [EW, Lemma and Definition 3.3]. The subspace $\mathcal{H}ol_k(S^2, G(n+1, s+1))$ of $\mathcal{H}ol(S^2, G(n+1, s+1))$ consisting of maps of degree k is connected. See [GO, Example 6.3].

Denote by $R_s(f)$ the ramification index of f_s. By [GH] or [EW], we get

$$\deg f_s = 2 \cdot \deg f_{s-1} - \deg f_{s-2} - 2 - R_{s-1}(f),$$

where $\deg f_{-1} = 0$ and $f_0 = f$. Hence

$$\deg f_s = (k-s)(s+1) - \sum_{u=0}^{s-1} \sum_{\alpha=0}^{u} R_\alpha(f) \geq 1$$

for $1 \leq s \leq n-1$ if $\deg f = k$. For a $\mathbf{C}$-subspace X in $\mathbf{C}^{n+1}$, denote by $X^\perp$ the orthogonal complement of X in $\mathbf{C}^{n+1}$. Put

$$\begin{aligned} &\mathcal{H}ol_\pm(S^2, \mathbf{C}P^n) := \cup_{\pm k > 0} \mathcal{H}ol_k(S^2, \mathbf{C}P^n), \\ &\mathcal{H}ol_\pm^*(S^2, \mathbf{C}P^n) = \mathcal{H}ol_\pm(S^2, \mathbf{C}P^n) \cap \mathcal{H}ol^*(S^2, \mathbf{C}P^n). \end{aligned}$$

Theorem 2.1 ([EW, Theorem 6.9]) *There is a bijective correspondence between $\widetilde{f} \in \mathcal{H}arm^*(S^2, \mathbf{C}P^n)$ and a pair (f, s), where $f \in \mathcal{H}ol^*_+(S^2, \mathbf{C}P^n)$ and s is an integer with $0 \le s \le n$ satisfying*

$$\widetilde{f} = f_s \cap f_{s-1}^{\perp}.$$

Here $f_{-1} = 0$ and $f_n = \mathbf{C}^{n+1}$.

When $\widetilde{f}$ is harmonic, by Theorem 2.1, we can get a holomorphic map $f\colon S^2 \to \mathbf{C}P^n$ and an integer $s \ge 0$ with $\widetilde{f} = f_s \cap f_{s-1}^{\perp}$. If $\widetilde{f}$ is not anti-holomorphic, we define the ∂-transform of $\widetilde{f}$ by

$$\partial \widetilde{f} = f_{s+1} \cap f_s^{\perp}.$$

Similarly, if $\widetilde{f}$ is not holomorphic, we define its $\overline{\partial}$-transform by

$$\overline{\partial} \widetilde{f} = f_{s-1} \cap f_{s-2}^{\perp}.$$

By Theorem 2.1, both $\partial \widetilde{f}$ and $\overline{\partial} \widetilde{f}$ are harmonic. When $\widetilde{f}$ is anti-holomorphic or holomorphic, we define $\partial \widetilde{f}$ or $\overline{\partial} \widetilde{f}$ as a zero map respectively. Since these ∂ or $\overline{\partial}$ transforms are the same ones to those defined in [CW], we get the following theorem.

Theorem 2.2 ([CW, Theorem 2.2]) *For $f \in \mathcal{H}arm(S^2, \mathbf{C}P^n)$, if ∂f is non-trivial, $\overline{\partial}\partial f = f$. When $\overline{\partial} f$ is non-trivial, $\partial\overline{\partial} f = f$.*

Denote by $R_\partial(\widetilde{f})$ the ramification index of $\widetilde{f}$. By [CW], if $\widetilde{f} = f_s \cap f_{s-1}^{\perp}$ for $1 \le s \le n-1$, we get

$$\deg \widetilde{f} = \deg f_s - \deg f_{s-1} = k - 2s - \sum_{\alpha=0}^{s-1} R_\alpha(f),$$

where $f_0 = f$. For the following lemma, we refer the reader to [GH] and [W, §3].

Lemma 2.3 *For $f \in \mathcal{H}arm(S^2, \mathbf{C}P^n)$, if ∂f is non-trivial, we get*

$$\deg \partial f = \deg f - 2 - R_\partial(f).$$

By Lemma 2.3, $f \in \mathcal{H}ol^*_k(S^2, \mathbf{C}P^n)$ satisfies

$$\deg \partial^s f = \deg \partial^{s-1} f - 2 - R_{s-1}(f) = \deg \partial^{s-1} f - 2 - R_\partial(\partial^{s-1} f),$$

and so $R_\partial(\partial^{s-1} f) = R_{s-1}(f)$ for $1 \le s \le n-1$. Any $f \in \mathcal{H}ol(S^2, \mathbf{C}P^n)$ defines a sequence of harmonic maps

$$\mathrm{seq}(f,0)\colon 0 \stackrel{\overline{\partial}}{\leftarrow} f \stackrel{\partial_0}{\rightarrow} \partial f \stackrel{\partial_1}{\rightarrow} \cdots \stackrel{\partial_{r-1}}{\rightarrow} \partial^r f \stackrel{\partial_r}{\rightarrow} \cdots \stackrel{\partial_{n_0-1}}{\rightarrow} \partial^{n_0} f \stackrel{\partial_{n_0}}{\rightarrow} 0,$$

which is called a harmonic sequence of f of length n_0. Since any non-trivial $\widetilde{f} \in \mathcal{H}arm(S^2, \mathbf{C}P^n)$ has $f \in \mathcal{H}ol_+(S^2, \mathbf{C}P^n)$ and an integer s with $\widetilde{f} = f_s \cap f_{s-1}^{\perp}$ by Theorem 2.1, $\mathrm{seq}(f,0)$ contains $\widetilde{f}$ with $\partial^s f = \widetilde{f}$, and so we also call $\mathrm{seq}(f,0)$ a harmonic sequence of $\widetilde{f}$ and denote by $\mathrm{seq}(\widetilde{f}, s)$. Obviously $\widetilde{f} \in \mathcal{H}arm(S^2, \mathbf{C}P^n)$ is full exactly when the length of $\mathrm{seq}(\widetilde{f}, s)$ is n. Put

$$\mathcal{H}ol^*_{k,R_J}(S^2, \mathbf{C}P^n) := \{f \mid R_s(f) = R_s \text{ for } 0 \leq s \leq n-2\} \subset \mathcal{H}ol^*_k(S^2, \mathbf{C}P^n)$$

where $R_J = \{R_0, \dots, R_{n-2}\}$. We denote by $\mathcal{H}arm^*_{k,s,R_J}(S^2, \mathbf{C}P^n)$ the image of

$$\partial^s \colon \mathcal{H}ol^*_{k,R_J}(S^2, \mathbf{C}P^n) \to \mathcal{H}arm^*(S^2, \mathbf{C}P^n).$$

Here $\mathcal{H}arm^*_{k,n,R_J}(S^2, \mathbf{C}P^n) \subset \mathcal{H}ol^*_-(S^2, \mathbf{C}P^n)$. The frame $\{Z_s\}_{0 \leq s \leq n-1}$ is called the Frenet frame of f if $\{Z_u\}_{0 \leq u \leq s}$ expands the space $f_U \wedge \cdots \wedge f_U^{(s)}$ for a lift $f_U \colon S^2 \supset U \to \mathbf{C}^{n+1}$. By [W], we get the following lemma.

Lemma 2.4 ([W, §2 and §3]) *For $f \in \mathcal{H}ol^*_+(S^2, \mathbf{C}P^n)$, choose the Frenet frame $\{Z_s\}_s$ of f and put*

$$dZ_s = -\overline{a}_{s-1}\overline{\varphi} Z_{s-1} + \omega_s Z_s + a_s \varphi Z_{s+1}$$

for $0 \leq s \leq n$, where $a_{-1} = a_n = 0$. Then each $\partial^s f$ defined by Z_s holds

$$E(\partial^s f) = \int \left(|a_{s-1}|^2 + |a_s|^2\right) \frac{\sqrt{-1}}{2} \varphi \wedge \overline{\varphi},$$

$$\deg \partial^s f = \int \left(|a_s|^2 - |a_{s-1}|^2\right) \frac{\sqrt{-1}}{2} \varphi \wedge \overline{\varphi}.$$

By Lemma 2.4, $\sum_{s=0}^{n} \deg \partial^s f = 0$, and so we get

$$\deg \partial^n f = -(k-n+1)n + \sum_{u=0}^{n-2} \sum_{\alpha=0}^{u} R_\alpha.$$

By Lemmas 2.3 and 2.4, we also get the following inequalities.

Lemma 2.5 ([W, Theorem 3.1]) *For $f \in \mathcal{H}ol^*_+(S^2, \mathbf{C}P^n)$, choose the Frenet frame $\{Z_s\}_s$ of f. Then we get the following for any s.*

(i) $\sum_{s+1 \leq q \leq n} \sum_{s \leq u \leq q-1} R_\partial(\partial^u f) < E(\partial^s f) + (n+1) \cdot |\deg \partial^s f|$.

(ii) $\sum_{0 \leq q \leq s-1} \sum_{q \leq u \leq s-1} R_\partial(\partial^u f) < E(\partial^s f) + (n+1) \cdot |\deg \partial^s f|$.

For an integer $k \geq 0$, denote by V_k the set of polynomials of the degree no greater than k and by V_k^+ the subset of V_k consisting of monics of the degree k. For $p_j \in V_k$, put

$$D(p_0, p_1, \cdots, p_s) = \begin{vmatrix} p_0 & p_1 & \cdots & p_s \\ p_0' & p_1' & \cdots & p_s' \\ \vdots & \vdots & \ddots & \vdots \\ p_0^{(s)} & p_1^{(s)} & \cdots & p_s^{(s)} \end{vmatrix}.$$

Lemma 2.6 *For $0 \leq s \leq n \leq k$ and $p_j \in V_k$, the degree of $D(p_0, p_1, \cdots, p_s)$ is no greater than $(k-s)(s+1)$.*

Proof Put $p_j(z) = \sum_{0\le u\le k} a_u^j z^u$. Then $D(p_0, p_1, \cdots, p_s)$ is the sum of monomials

$$\sum_{\rho} \epsilon(\rho) \cdot \frac{k_0!}{(k_0 - s_0)!} \cdots \frac{k_s!}{(k_s - s_s)!} a_{k_0}^0 \cdots a_{k_s}^s \, z^{k_0+\cdots+k_s-(s_0+\cdots+s_s)},$$

where $k_j \le k$, $\rho = (s_0, \ldots, s_s)$ is a permutation of $(0, 1, 2, \ldots, s)$ and $\epsilon(\rho)$ is denoted for the sign of ρ. When

$$k_0 + \cdots + k_s \ge k(s+1) - \frac{s(s+1)}{2} + 1,$$

$k_i = k_j$ for some $i \ne j$, and so the corresponding monomials vanish. Therefore

$$k_0 + \cdots + k_s - (s_0 + \cdots + s_s) \le k(s+1) - \frac{s(s+1)}{2} - \frac{s(s+1)}{2} = (k-s)(s+1). \quad \blacksquare$$

A map $f \in \mathcal{H}ol_+(S^2, \mathbf{C}P^n)$ is represented by using homogeneous coordinates on $\mathbf{C}P^n$:

$$f(z) = \left[\, p_0(z) \,;\, \cdots \,;\, p_n(z) \right].$$

We define $h_f = [q_0 \,;\, q_1 \,;\, \cdots \,;\, q_n]$ by

$$q_s = (-1)^s \, D(p_0, \ldots, p_{s-1}, \check{p}_s, p_{s+1}, \ldots, p_n).$$

This is uniquely defined by f. By definition, the complex conjugate of h_f defines the anti-holomorphic map $\partial^n f$, which we call the polar of f. See [EW, §3].

Lemma 2.7 *For any $f \in \mathcal{H}ol_n^*(S^2, \mathbf{C}P^n)$, we get*

$$R_\partial(\partial^s f) = 0 \quad \textit{and} \quad \partial^s f \in \mathcal{H}arm_{n-2s}^*(S^2, \mathbf{C}P^n)$$

with $E(\partial^s f) = n(2s+1) - 2s^2$ for any $0 \le s \le n-1$. We also get $\partial^n f \in \mathcal{H}ol_{-n}^*(S^2, \mathbf{C}P^n)$.

Proof By Lemma 2.6, $\deg h_f \le n$. Moreover, as f is full and h_f defines $\partial^n f$, $\deg h_f = n$ and $\partial^n f \in \mathcal{H}ol_{-n}^*(S^2, \mathbf{C}P^n)$. By Lemma 2.3, ramification indexes $R_\partial(\partial^s f)$ hold

$$\deg \partial^s f = n - 2s - \sum_{\alpha=0}^{s-1} R_\partial(\partial^\alpha f)$$

for $1 \le s \le n-1$, and so

$$\deg \partial^n f = -\deg f - \sum_{s=1}^{n-1} \deg \partial^s f = -n + \sum_{s=1}^{n-1} \sum_{\alpha=0}^{s-1} R_\partial(\partial^\alpha f) = -n.$$

Hence $R_\partial(\partial^s f) = 0$ for any $0 \le s \le n-2$. Since

$$E(\partial^{s+1} f) - E(\partial^s f) = \deg \partial^{s+1} f + \deg \partial^s f$$

by Lemma 2.4, we get the required equality for $E(\partial^s f)$. $\blacksquare$

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3 Bubble Tree Map

As for bubbling phenomena, we refer the reader to Parker and Wolfson [PW] and Parker [P]. Let $TS^2 \to S^2$ be the complex tangent bundle over the complex manifold S^2, g_0. Compactifying each vertical fiber, we get a bundle $\Sigma(S^2) \to S^2$ with fibers $S_z = S^2$, where we identify z of S_z with the south pole ∞ of S^2 and equip the complex structure on $\Sigma(S^2)$. By the induction on $k \geq 1$, we define a bundle

$$\Sigma^k(S^2) := \Sigma(\Sigma^{k-1}(S^2)) \longrightarrow \Sigma^{k-1}(S^2).$$

A bubble domain at level k is a fiber $S_z^k = S^2$ of $\Sigma^k(S^2) \to \Sigma^{k-1}(S^2)$, and a bubble domain tower is a union $T^I = \bigvee_{\ell \in I} S^{(\ell)}$ of the base space $S^{(0)}$ of $\Sigma(S^2) \to S^2$ and finite number of bubble domains $S^{(\ell)}$ ($\ell \in I$, $\ell \geq 1$) with

$$\pi_\ell \colon \Sigma S^{(\ell')} \supset S^{(\ell)} = S_{z_\ell}^{k_\ell} = \pi_\ell^{-1}(z_\ell) \longrightarrow z_\ell \in S^{(\ell')}.$$

We denote by ∞_ℓ the south pole of $S^{(\ell)}$. If a map

$$f^I = \bigvee_{\ell \in I} f^{(\ell)} \colon T^I = \bigvee_{\ell \in I} S^{(\ell)} \longrightarrow \mathbf{C}P^n$$

consists of non-trivial maps $f^{(\ell)}$ satisfying $f^{(\ell)}(\infty_\ell) = f^{(\ell')}(z_\ell)$ when $\pi_\ell^{-1}(z_\ell) = S^{(\ell)}$, we call f^I a bubble tree map, $f^{(0)}$ a base map, $f^{(\ell)}$ a bubble map for $\ell \in I - \{0\}$, and $z_\ell \in S^{(\ell')}$ a bubble point of $f^{(\ell')}$. Here, without loss of generality, we can assume that $\ell > \ell'$ if the level of $S^{(\ell)}$ is greater than that of $S^{(\ell')}$. Denote by $B_{f^{(\ell)}}$ the set of bubble points of $f^{(\ell)}$.

We call f^I a harmonic bubble tree map if $f^{(\ell)} \in \mathcal{H}arm(S^{(\ell)}, \mathbf{C}P^n)$ for each $\ell \in I$. Similarly we call f^I a holomorphic (resp. anti-holomorphic) bubble tree map if $f^{(\ell)} \in \mathcal{H}ol_+(S^{(\ell)}, \mathbf{C}P^n)$ (resp. $f^{(\ell)} \in \mathcal{H}ol_-(S^{(\ell)}, \mathbf{C}P^n)$) for any $\ell \in I$.

Two maps f_0 and f_1 in $\mathcal{H}arm(S^2, \mathbf{C}P^n)$ are said to be equivalent if $f_1 = f_0 \circ \sigma$ by a fractional transformation σ fixing the infinity. We say that

$$f_0^{I_0} = \bigvee_{\ell \in I_0} f_0^{(\ell)} \colon T^{I_0} = \bigvee_{\ell \in I_0} S_0^{(\ell)} \longrightarrow \mathbf{C}P^n$$

is equivalent to $f_1^{I_1} = \bigvee_{\ell' \in I_1} f_1^{(\ell')} \colon T^{I_1} = \bigvee_{\ell' \in I_1} S_1^{(\ell')} \to \mathbf{C}P^n$ if $T^{I_0} = T^{I_1}$ and $f_0^{(\ell)}$ is equivalent to $f_1^{(\ell)}$ for any $\ell \in I_0$:

$$f_1^{(\ell)} = f_0^{(\ell)} \circ \sigma^\ell \colon S_1^{(\ell)} \xrightarrow{\sigma^\ell} S_0^{(\ell)} \xrightarrow{f_0^{(\ell)}} \mathbf{C}P^n.$$

We say that a sequence $\{f^\lambda\}_{\lambda \geq 1}$ in $\mathcal{H}arm(S^2, \mathbf{C}P^n)$ converges to a harmonic bubble tree map $f^I \colon T^I \to \mathbf{C}P^n$ if each f^λ defines a bubble tree map $f^{\lambda,I} = \bigvee_{\ell \in I} f^{\lambda,\ell} \colon T^I \to \mathbf{C}P^n$ by the iterated renormalization procedure and if $\{f^{\lambda,I}\}_\lambda$ converges to a harmonic bubble tree map $\widetilde{f}^I$ equivalent to f^I uniformly in $C^0 \cap W^{1,2}$ and uniformly in C^r $(r \geq 1)$ on any compact set of $T^I - \cup_\ell(\{\infty_\ell\} \cup B_{f^{(\ell)}})$. Here $f^{\lambda,\ell} = f^\lambda \circ \sigma_{\lambda,\ell}$

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on a compact set in $S^{(\ell)} - \{\infty_\ell\} \cup B_{f^{(\ell)}}$ by a fractional linear transformation $\sigma_{\lambda,\ell}$ of $S^{(\ell)} = S^2$ fixing the south pole. For details, see [PW, §4].

A harmonic bubble tree map f^I is said to be gluable if a sequence of harmonic maps converges to a harmonic bubble tree map f^I. By [P, Theorem 2.2 and Corollary 2.3], we get the following convergence theorem.

Theorem 3.1 *Let $\{f^\lambda\}_\lambda$ be a sequence in $\mathcal{H}arm(S^2, \mathbf{C}P^n)$ with $\sup_\lambda E(f^\lambda) < \infty$. Then a subsequence (we denote it by the same way) converges to a harmonic bubble tree map $f^I = \bigvee_\ell f^{(\ell)}\colon T^I \to \mathbf{C}P^n$ satisfying*

$$\lim_\lambda E(f^\lambda) = \sum_\ell E(f^{(\ell)}) \quad \text{and} \quad \lim_\lambda \deg f^\lambda = \sum_\ell \deg f^{(\ell)}.$$

We say that a bubble domain tower $T^I = \bigvee_\ell S^{(\ell)}$ is simple if $S^{(\ell)} = \pi_\ell^{-1}(z_\ell)$ by $z_\ell \in S^{(0)} - \{\text{the south pole}\}$ for any $\ell \in I - \{0\}$. We begin to glue a holomorphic bubble tree map defined on a simple bubble domain tower.

Lemma 3.2 (The first step of gluing) *Let $f^I = \bigvee_\ell f^{(\ell)}\colon T^I \to \mathbf{C}P^n$ be a holomorphic bubble tree map with a base map $f^{(0)}$ defined on a simple bubble domain tower $T^I = \bigvee_\ell S^{(\ell)}$. Put $k = \sum_\ell k_\ell$ for $k_\ell = \deg f^{(\ell)}$. Then there is a sequence of holomorphic maps $f_R \in \mathcal{H}ol_k(S^2, \mathbf{C}P^n)$ with $f_R(\infty) = f^{(0)}(\infty)$ converging to f^I when $R \to +\infty$.*

Proof By the assumption, each $f^{(\ell)} \in \mathcal{H}ol_{k_\ell}(S^2, \mathbf{C}P^n)$ holds $f^{(0)}(z_\ell) = f^{(\ell)}(\infty)$ for $z_\ell \in B_{f^{(0)}} \subset S^{(0)}$. Put $f^{(\ell)} = [p_0^{(\ell)} ; \cdots ; p_n^{(\ell)}]$, where $\{p_j^{(\ell)}\}_j$ are coprime and $p_j^{(\ell)}$ satisfies

$$p_j^{(0)}(z_\ell) = \frac{1}{k_\ell!}\frac{d^{k_\ell}}{dz^{k_\ell}} p_j^{(\ell)}(z)$$

for $\ell \geq 1$. For $R > 0$ large enough, we define a gluing map $f^R = [p_0^R ; \cdots ; p_n^R]$ by

$$p_j^R(z) = p_j^{(0)}(z) + \sum_{\ell \geq 1}\Big\{\frac{1}{R^{k_\ell} w_\ell^{k_\ell}}\widetilde{p}_j^{(\ell)}(Rw_\ell) - \frac{1}{k_\ell!}\frac{d^{k_\ell}}{dz^{k_\ell}} p_j^{(\ell)}(z)\Big\}$$

where $\widetilde{p}_j^{(\ell)}(w_\ell) := p_j^{(\ell)}(z)$ for $w_\ell := z - z_\ell$. Though p_j^R is a rational function, f^R is a well-defined map in $\mathcal{H}ol_k(S^2, \mathbf{C}P^n)$.

For a fixed $R_0 > 0$ large enough, if $|z - z_\ell| \geq \frac{1}{\sqrt{R_0}}$ for any $\ell \geq 1$, by $R \to +\infty$, f^R converges to the restriction of $f^{(0)}$. We can extend this convergence by $R_0 \to +\infty$. When $|z - z_\ell| \leq \frac{1}{\sqrt{R}}$, put $w = Rw_\ell$ and define $\widetilde{f}_R^{(\ell)}\colon S^2 \to \mathbf{C}P^n$ by

$$\widetilde{f}_R^{(\ell)}(w) = f^R\Big(z_\ell + \frac{1}{R}w\Big)$$

for $|w| \leq \sqrt{R}$. For a fixed $R_1 > 0$ large enough, passing through a subsequence, the restrictions of $\{\widetilde{f}_R^{(\ell)}\}_R$ on $|w| \leq \sqrt{R_1}$ converge to the restriction of

$$f^{(0)}(z_\ell) + f^{(\ell)}(w) - f^{(\ell)}(\infty) = f^{(\ell)}(w)$$

by $R \to +\infty$. Extending this convergence by $R_1 \to +\infty$, we get a convergence to $f^{(\ell)}$. These facts show that a subsequence of $\{f^R\}_R$ converges to f^I. Moreover, we can calculate to get $f_R(\infty) = f^{(0)}(\infty)$. ■

To continue the procedure, we establish another lemma.

Lemma 3.3 (The second step of gluing) *Let*

$$f^I = f^{(0)} \vee \Big(\bigvee_{\lambda} f^{I_\lambda}\Big) : T^I \longrightarrow \mathbf{C}P^n$$

be a holomorphic bubble tree map with a base map $f^{(0)}$, where

$$f^{I_\lambda} := f^{(\ell_\lambda)} \vee \big(\bigvee_{\mu} f^{(\ell_{\lambda\mu})}\big) \colon T^{I_\lambda} \longrightarrow \mathbf{C}P^n$$

is a bubble tree map defined on a simple domain tower T^{I_λ} with $f^{(\ell_{\lambda\mu})}(\infty) = f^{(\ell_\lambda)}(z_{\ell_{\lambda\mu}})$ by $z_{\ell_{\lambda\mu}} \in S^{(\ell_\lambda)} - \{\infty\}$ and $f^{(\ell_\lambda)}(\infty) = f^{(0)}(z_{\ell_\lambda})$ for some $z_{\ell_\lambda} \in S^{(0)} - \{\infty\}$. Then we can get a sequence of $\{f^k\}_k$ in $\mathcal{H}ol^*(S^2, \mathbf{C}P^n)$ with $f^k(\infty) = f^{(0)}(\infty)$ converging to f^I.

Proof Put $f^{(\ell_\lambda)} = [p_0^\lambda ; \cdots ; p_n^\lambda]$ with $\deg f^{(\ell_\lambda)} = k_\lambda$ and $f^{(\ell_{\lambda\mu})} = [p_0^{\lambda\mu} ; \cdots ; p_n^{\lambda\mu}]$ with $\deg f^{(\ell_{\lambda\mu})} = k_{\lambda\mu}$. By Lemma 3.2, for $S > 0$ large enough, we get a gluing map $f_\lambda^S = [p_{\lambda 0}^S ; \cdots ; p_{\lambda n}^S]$ defined by

$$p_{\lambda j}^S(z) = p_j^\lambda(z) + \sum_{\mu} \Big\{ \frac{1}{S^{k_{\lambda\mu}} w_{\lambda\mu}^{k_{\lambda\mu}}} \widetilde{p}_j^{\lambda\mu}(S w_{\lambda\mu}) - \frac{1}{k_{\lambda\mu}!} \frac{d^{k_{\lambda\mu}}}{dz^{k_{\lambda\mu}}} p_j^{\lambda\mu}(z) \Big\}$$

where $\widetilde{p}_j^{\lambda\mu}(w_{\lambda\mu}) = p_j^{\lambda\mu}(z)$ for $w_{\lambda\mu} := z - z_{\ell_{\lambda\mu}}$ and $p_j^\lambda(\infty) = p_{\lambda j}^S(\infty)$. In fact,$\{f_\lambda^S\}_S$ converges to f^{I_λ} by $S \to \infty$.

Since $f^{(0)} \vee \big(\bigvee_\lambda f_\lambda^S\big)$ also becomes a bubble tree map defined on a simple domain tower with $f_\lambda^S(\infty) = f^{(\ell_\lambda)}(\infty)$, we define a map $f^{RS} = [p_0^{RS} ; \dots ; p_n^{RS}]$ by

$$p_j^{RS}(z) = p_j^{(0)}(z) + \sum_{\lambda} \Big\{ \frac{1}{R^{k_\lambda} w_\lambda^{k_\lambda}} \widetilde{p}_{\lambda j}^S(R w_\lambda) - \frac{1}{k_\lambda!} \frac{d^{k_\lambda}}{dz^{k_\lambda}} p_{\lambda j}(z) \Big\},$$

where $\widetilde{p}_{\lambda j}^S(w_\lambda) = p_{\lambda j}^S(z)$ for $w_\lambda = z - z_\lambda$. Then a subsequence of $\{f^{RS}\}_{RS}$ converges to f^I. This completes the proof. ∎

Now we glue a holomorphic bubble tree map by an elementary way without using the implicit function theorem and without changing the Fubini–Study metric on $\mathbf{C}P^n$.

Proposition 3.4 *Any holomorphic bubble tree map $f^I = \bigvee_\ell f^{(\ell)} \colon T^I = \bigvee_\ell S^{(\ell)} \to \mathbf{C}P^n$ is gluable.*

Proof When $I \neq \{0\}$, we can choose a bubble map $f^{(\ell)}$ defined on $S^{(\ell)}$ whose level is the maximum among any level of $S^{(\ell')}$. If $z_\ell \in S^{(\ell_0)}$, we can get a bubble tree map defined on a simple bubble domain tower

$$f^{(\ell_0)} \vee \Big(\bigvee_{\ell_j} f^{(\ell_j)}\Big) : S^{(\ell_0)} \vee \Big(\bigvee_{\ell_j} S^{(\ell_j)}\Big) \longrightarrow \mathbf{C}P^n,$$

where $f^{(\ell_0)}$ is a base map,$f^{(\ell_j)}(\infty) = f^{(\ell_0)}(z_{\ell_j})$ by bubble points $z_{\ell_j} \in S^{(\ell_0)}$. Especially $z_\ell = z_{\ell_j}$ for some j. By Lemma 3.2, we can glue to get a new holomorphic map $f_R^{(\ell_0)}$. Next we repeat the procedure shown in the proofs of Lemmas 3.2 or 3.3 to get a required holomorphic map . This completes the proof. ∎

4 A Space of Harmonic Maps

From now on, we assume that $n \geq 2$. Let $V_s(\mathbf{C}^m)$ be the Stiefel manifolds of s-frames in $\mathbf{C}^m$. For the proof of the following lemma, we follow the proof of [C, §3].

Lemma 4.1 *For $1 \leq s \leq n-1, 0 \leq i_0 < i_1 < \cdots < i_s \leq n$, $k \geq n$ and $\phi \in V_r^+$, take any $(p_0, \dots, p_n) \in V_{n+1}(\mathbf{C}^{k+1})$ so that ϕ, p_0 are coprime monics. Then ϕ divides $D(p_{i_0}, p_{i_1}, \dots, p_{i_s})$ for any $(i_0, i_1, \dots, i_s)$ if ϕ divides $D(p_0, p_{i_1}, \dots, p_{i_s})$ for any $(i_1, \dots, i_s)$.*

Proof Since

$$p_0 \cdot D(p_{i_0}, p_{i_1}, \dots, p_{i_s}) = \sum_{0 \leq u \leq s} (-1)^u p_{i_u} \cdot D(p_0, p_{i_0}, \dots, \check{p_{i_u}}, \dots, p_{i_s}),$$

we get a required result. ∎

For an $(n-1)$-tuple $R_J = \{R_0, R_1, \dots, R_{n-2}\}$ of non-negative integers with

$$(k-s)(s+1) - 1 \geq \sum_{u=0}^{s-1} \sum_{\alpha=0}^{u} R_\alpha$$

for any $1 \leq s \leq n-1$, consider a subspace

$$F_{k,R_0} = \left\{ f \in \mathcal{H}ol_k^*(S^2, \mathbf{C}P^n) \mid R_0(f) \geq R_0 \right\} \subset \mathcal{H}ol_k^*(S^2, \mathbf{C}P^n)$$

and put $F^*_{k,R_0} = F_{k,R_0} - F_{k,R_0+1}$. By the induction on s, we continue this procedure. Define

$$F_{k,R_0,\dots,R_{s-1},R_s} = \left\{ f \in F^*_{k,R_0,\dots,R_{s-1}} \mid R_s(f) \geq R_s \right\}$$

and put

$$F^*_{k,R_0,\dots,R_s} = F_{k,R_0,\dots,R_s} - F_{k,R_0,\dots,R_s+1}.$$

By definition, $\mathcal{H}ol^*_{k,R_J}(S^2, \mathbf{C}P^n) = F^*_{k,R_0,\dots,R_{n-2}}$. We also consider another sequence of spaces. For $s \geq 0$, consider a subspace

$$\widetilde{F}_{k,R_0,\dots,R_s} \subset V^+_{R_0} \times \cdots \times V^+_{R_s} \times F_{k,R_0,\dots,R_s}$$

consisting of $(\phi_0, \dots, \phi_s, f = [p_0 ; \cdots ; p_n])$ such that ϕ_u divides the ramification divisor of f_u for $0 \leq u \leq s$. Denote by $\pi_s \colon \widetilde{F}_{k,R_0,\dots,R_s} \to F_{k,R_0,\dots,R_s}$ the projection. Let Z_ϕ be the set of zeros of ϕ. If $s \geq 1$, consider the map

$$\Psi_s \colon V^+_{R_0} \times \cdots \times V^+_{R_s} \longrightarrow \mathcal{P}(\mathbf{C}^{s+1})$$

defined by $\Psi_s(\phi_0, \dots, \phi_s) = Z_{\phi_0} \times \cdots \times Z_{\phi_s}$. Here $\mathcal{P}(\mathbf{C}^{s+1})$ is the set of $(s+1)$-tuples of finite sets in $\mathbf{C}$. Let $\mathcal{P}_0(\mathbf{C}^{s+1})$ be the subset consisting of $Z_0 \times \cdots \times Z_s$ with $Z_u \cap Z_{u'} = \phi$ for $u \neq u'$. Then $\mathcal{P}_0(\mathbf{C}^{s+1})$ is a subspace deleting a proper algebraic subset from $\mathcal{P}(\mathbf{C}^{s+1})$ and so is path-connected. Put $V^s_{R_J} := \Psi_s^{-1}(\mathcal{P}_0(\mathbf{C}^{s+1}))$ and

$$\widetilde{F}^0_{k,R_0,\dots,R_s} = \widetilde{F}_{k,R_0,\dots,R_s} \cap (V^s_{R_J} \times F_{k,R_0,\dots,R_s}).$$

It is obvious that $\widetilde{F}^0_{k,R_0} = \widetilde{F}_{k,R_0}$. Let $\widetilde{F}'_{k,R_0,\dots,R_s}$ be the subspace of $\widetilde{F}^0_{k,R_0,\dots,R_s}$ consisting of

$$\big(\phi_0, \dots, \phi_s, f = [p_0 ; \dots ; p_n]\big)$$

with $Z_{\phi_s} \cap Z_{D(p_{i_0},\dots,p_{i_s})} = \phi$ for any $0 \le i_0 < \cdots < i_s \le n$. The group $PGL(2,\mathbf{C}) \times PGL(n,\mathbf{C})$ acts on the last component of $V^+_{R_0} \times \cdots \times V^+_{R_s} \times \mathcal{H}ol^*_k(S^2, \mathbf{C}P^n)$ in a natural way. Let PV^m_k be the projective space defined as the quotient space $V^m_k/\sim$ where $(q_1, \dots, q_m) \sim (\lambda q_1, \dots, \lambda q_m)$ for $\lambda \in \mathbf{C} - \{0\}$. We also denote by $PV_m(\mathbf{C}^{k+1})$ the projective space of $V_m(\mathbf{C}^{k+1})$. Let $\pi \colon V_m(\mathbf{C}^{k+1}) \to PV_m(\mathbf{C}^{k+1})$ be the projection. For the proof of the next lemma, see [C, Lemma 3.2].

Lemma 4.2 *Any point of $\widetilde{F}^0_{k,R_0,\dots,R_s}$ is contained in a neighbourhood biholomorphically equivalent to $\widetilde{F}'_{k,R_0,\dots,R_s}$.*

Proof Take $(\phi_0, \dots, \phi_s, f = [p_0 ; \dots ; p_n]) \in \widetilde{F}_{k,R_0,\dots,R_s}$ by coprime polynomials $\{p_j\}_{0\le j\le n}$. Because of the condition of $V^s_{R_J}$, each $\alpha \in \bigcup_{u=0}^s Z_{\phi_u}$ satisfies $D(p_{i_0}, \dots, p_{i_s})(\alpha) \neq 0$ for some $0 \le i_0 < \cdots i_s \le n$. Therefore, by an action of $PGL(n,\mathbf{C})$, we can choose $q_s = \sum_{j=0}^s c_{sj} p_j$ so that $g_f = [q_0 ; \cdots ; q_n]$ defines ramification divisors $\{\phi_u\}_u$ with $Z_{\phi_s} \cap Z_{D(q_{i_0},\dots,q_{i_s})} = \phi$ for any $0 \le i_0 < \cdots < i_s \le n$. ∎

We denote by $M(s,m)$ the space of (s,m)-matrices. For the proof of the following lemma, we follow the proof of [C, Lemma 3.3].

Lemma 4.3 *If $k - n \ge R_0$, both $\widetilde{F}_{k,R_0}$ and F^*_{k,R_0} are path-connected complex manifold of dimension*

$$(k+1)(n+1) - 1 - (n-1)R_0.$$

Proof We begin to show the assertion for $\widetilde{F}'_{k,R_0}$. Let Z_0 be the set of $(\phi, p_0) \in V^+_{R_0} \times V_k$ so that they are coprime. For $(\phi, p_0) \in Z_0$, define a linear map $T_{\phi,p_0} \colon V_k \to V_{R_0-1}$ by $T_{\phi,p_0}(p)$ being the remainder of $D(p_0, p)$ divided by ϕ which we consider as a $(R_0, k+1)$-matrix. Let $\Phi_0 \colon Z_0 \to M(R_0, k+1) \times V_k$ be the map defined by $\Phi_0(\phi, p_0) = (T_{\phi,p_0}, p_0)$. Consider a subspace

$$E_0 = \big\{ (M, p_0, p_1, \dots, p_n) \mid M p_j = 0 \text{ for } 1 \le j \le n \big\}$$

of $M(R_0, k+1) \times V_{n+1}(\mathbf{C}^{k+1})$. As in the proof of [C, Lemma 3.3], the projection $E_0 \to V_{n+1}(\mathbf{C}^{k+1})$ is a vector bundle, because $k - n \ge R_0$, and so E_0 is a complex manifold. Therefore the projection $\Pi_0 \colon E_0 \to M(R_0, k+1) \times V_k$ defined by $\Pi_0(M, p_0, p_1, \dots, p_n) = (M, p_0)$ is also a complex vector bundle. So the pull-back $\Phi_0^* E_0$ becomes a connected complex manifold with

$$\begin{aligned} \dim Z_0 + \dim E_0 - \dim M(R_0, k+1) \times V_k &= k + 1 + R_0 + (k + 1 - R_0)n \\ &= (k+1)(n+1) - (n-1)\,R_0. \end{aligned}$$

Let $\pi \colon V^+_{R_0} \times V_{n+1}(\mathbf{C}^{k+1}) \to V^+_{R_0} \times PV_{n+1}(\mathbf{C}^{k+1})$ be the projection. By Lemma 4.1, $\pi(\Phi_0^* E_0) = \widetilde{F}'_{k,R_0}$, and so, by Lemma 4.2, we show the assertion for $\widetilde{F}^0_{k,R_0} = \widetilde{F}_{k,R_0}$. Since F_{k,R_0+1} is a proper algebraic subset and cannot disconnect F_{k,R_0},

$$F^*_{k,R_0} = F_{k,R_0} - F_{k,R_0+1} \simeq \widetilde{F}_{k,R_0} - \pi_0^{-1}(F_{k,R_0+1})$$

is a required one. ∎

If $f \in \mathcal{H}ol^*_{k,R_J}(S^2, \mathbf{C}P^n)$,

$$\deg h_f = n(k-n+1) - \sum_{s=0}^{n-2} \sum_{\alpha=0}^{s} R_\alpha(f) \geq n.$$

Except at finite points, $\partial^s f$ is locally defined by the Frenet frame $\{Z_s\}_s$ of f, which is the Gram–Schmidt orthogonalization of $\{\frac{d^s}{dz^s} f_U\}_s$ for a lift $f_U \colon S^2 \supset U \to \mathbf{C}^{n+1}$.

Lemma 4.4 *For $f \in \mathcal{H}ol^*_k(S^2, \mathbf{C}P^n)$, $h_{h_f} = f$.*

Proof Take $f = [p_0 ; \cdots ; p_n] \in \mathcal{H}ol^*_k(S^2, \mathbf{C}P^n)$, where $\{p_j\}_j$ are coprime. For a lift $f_U = (p_0, \dots, p_n)$, put $g_U = (q_0, \dots, q_n)$ and $h_U = (r_0, \dots, r_n)$ where

$$q_j = (-1)^j D(p_0, \dots, \check{p}_j, \dots, p_n) \quad \text{and} \quad r_j = (-1)^j D(q_0, \dots, \check{q}_j, \dots, q_n).$$

Since h_f and h_{h_f} are determined independently on a lift, it is enough to show the assertion for f_U, g_U, and h_U. Both the inner product $f_U^{(s)} \cdot \bar{g}_U$ and $g_U^{(s)} \cdot \bar{h}_U$ vanish for $0 \leq s \leq n-1$. By the induction on s, we can calculate to get $f_U^{(u)} \cdot \bar{g}_U^{(s-u)} = 0$ for any s and $0 \leq u \leq s$. Especially $f_U \cdot \bar{g}_U^{(s)} = 0$ for any s, and so both f_U and h_U are transverse to the space $g_U \wedge \cdots \wedge g_U^{(n-1)}$ in $\mathbf{C}^{n+1}$. This shows that we have ϕ with $r_j = \phi \cdot p_j$ for any j. Since

$$\begin{pmatrix} q_0 & \cdots & q_n \\ \vdots & \ddots & \vdots \\ q_0^{(n)} & \cdots & q_n^{(n)} \end{pmatrix} \begin{pmatrix} p_0 & \cdots & p_0^{(n)} \\ \vdots & \ddots & \vdots \\ p_n & \cdots & p_n^{(n)} \end{pmatrix} := (c_{ij})_{0 \leq i,j \leq n}$$

with $c_{ij} = 0$ if $0 \leq i + j \leq n-1$ and

$$c_{s,n-s} = \sum_{j=0}^{n} q_j^{(s)} p_j^{(n-s)} = (-1)^s \sum_j q_j p_j^{(n)} = (-1)^{n+s} D(p_0, \dots, p_n),$$

we get $D(q_0, \dots, q_n) D(p_0, \dots, p_n) = D(p_0, \dots, p_n)^{n+1}$, and so $D(q_0, \dots, q_n) = D(p_0, \dots, p_n)^n$, because $D(p_0, \dots, p_n)$ is not zero except at finite points. Since $r_j = \phi \cdot p_j$,

$$\begin{aligned} D(q_0, \dots, q_n) &= (-1)^n (q_0^{(n)} r_0 + \cdots + q_n^{(n)} r_n) = (-1)^n \, \phi \cdot (q_0^{(n)} p_0 + \cdots + q_n^{(n)} p_n) \\ &= \phi \cdot (q_0 p_0^{(n)} + \cdots + q_n p_n^{(n)}) = (-1)^n \cdot \phi \cdot D(p_0 \cdots p_n) \end{aligned}$$

and so we get

$$r_j = (-1)^n \cdot D(p_0, \dots, p_n)^{n-1} \cdot p_j$$

for any $0 \leq j \leq n$. This completes the proof. ∎

Lemma 4.5 *For any $f \in \mathcal{H}ol^*_k(S^2, \mathbf{C}P^n)$, $\deg f_s \geq n$ for any $0 \leq s \leq n-1$.*

Proof Put $d_s = \deg f_s$. Since

$$d_s - d_{s-1} = k - 2s - \sum_{0 \le u \le s-1} R_u(f),$$

there is s_0 such that

$$n \le k = d_0 \le \cdots \le d_{s_0} \ge \cdots \ge d_{n-1}.$$

As $\partial^n f$ is also a full map of degree $-d_{n-1}$, $d_{n-1} \ge n$. ■

Lemma 4.6 *Take* $(\phi_0, \dots, \phi_{s-1}, [p_0 ; \cdots ; p_n]) \in \widetilde{F}^0_{k,R_0,\dots,R_{s-1}}$ *represented by coprime* $\{p_j\}_j$ *with* $\deg \phi_u = R_u(f)$ *for* $0 \le u \le s-1$. *In this case,* $D(p_0, p_{i_1}, \dots, p_{i_{s+1}})$ *can be divided by* $\widetilde{\phi}^*_s := \phi_0^{s+1} \cdot \phi_1^s \cdots \phi_{s-1}^2$ *for any* $0 < i_1 < \cdots < i_{s+1} \le n$.

Proof For $R_u \ge 1$ with $0 \le u \le s-1$, take $\alpha \in Z_{\phi_u}$ and an integer k_α so that

$$\phi_u(z) = (z - \alpha)^{k_\alpha} \cdot \psi_u(z)$$

with $\psi_u(\alpha) \ne 0$. Without loss of generality, we can assume that $\alpha = 0$. Moreover, we get

$$\frac{d^N}{dz^N} D(p_{i_0}, \dots, p_{i_u})\Big|_{z=0} = 0$$

for $0 \le i_0 < \cdots < i_u \le n$ and $0 \le N \le k_0 - 1$. Put

$$\mathbf{a}_u = \left(p_0^{(u)}(0), \dots, p_n^{(u)}(0) \right).$$

We start with $N = 0$. Under the above situation, if $D(p_{i_0}, \dots, p_{i_u})|_{z=0} = 0$ for any $0 \le i_0 < \cdots < i_u$, by making standard calculations, we can show that a family $\{\mathbf{a}_0, \dots, \mathbf{a}_u\}$ is linearly dependent. Moreover, by the assumption on $\widetilde{F}^0_{k,R_0,\dots,R_{s-1}}$, we can put

$$\mathbf{a}_u = \sum_{j=0}^{u-1} \lambda_j^0 \, \mathbf{a}_j.$$

Take $0 \le t \le k_0 - 1$. By the induction on $t \ge 0$, we show that

$$\mathbf{a}_{u+t} = \sum_{j=0}^{u-1} \lambda_j^t \, \mathbf{a}_j.$$

Suppose that we have shown this for $t-1 \ge 0$. Differentiating $D(p_{i_0}, \dots, p_{i_u})$ t-times, we get a linearly dependent family

$$\sum_{t_u \le u+t-1} \{\mathbf{a}_{t_0}, \dots, \mathbf{a}_{t_u}\} + \{\mathbf{a}_0, \dots, \mathbf{a}_{u-1}, \mathbf{a}_{u+t}\}.$$

Here, by the assumption of the induction, we can show the linear dependence of any $\{\mathbf{a}_{t_0}, \ldots, \mathbf{a}_{t_u}\}$ with $t_u \le u + t - 1$, and so, by the assumption on $\widetilde{F}^0_{k,R_0,\ldots,R_{s-1}}$, we get

$$\mathbf{a}_{u+t} = \sum_{j=0}^{u-1} \lambda^t_j \,\mathbf{a}_j.$$

This means that any $\{\mathbf{a}_0, \ldots, \mathbf{a}_{u-1}, \mathbf{a}_{t_u}, \ldots, \mathbf{a}_{t_s}\}$ is linearly dependent as far as $u \le t_u < \cdots < t_s$ with $0 \le t_{u'} - u' \le k_0 - 1$ for some $u \le u' \le s$.

On the other hand, as

$$\frac{d^{k_0}}{dz^{k_0}} D(p_{i_0}, \ldots, p_{i_u})|_{z=0} \ne 0$$

for some $0 \le i_0 < \cdots < i_u$, we can get a linearly independent family

$$\{\mathbf{a}_0, \ldots, \mathbf{a}_{u-1}, \mathbf{a}_{u+k_0}, \ldots, \mathbf{a}_{s+k_0}\}$$

with

$$0 + 1 + \cdots + (u-1) + (u+k_0) + \cdots + (s+k_0) - (0 + 1 + \cdots + s) = k_0(s-u+1).$$

These imply that $\widetilde{\phi}^*_s(z) = z^{k_0(s+1-u)}\psi_s(z)$ with $\psi_s(0) \ne 0$ for $s \ge u$, which completes the proof. ∎

Consider the projection

$$\widetilde{\pi}_s \colon \widetilde{F}_{k,R_0,\ldots,R_{s-1}} \longrightarrow \left(\Pi_{u=0}^{s-1} V^+_{R_u}\right) \times PV_{s+1}(\mathbf{C}^{k+1})$$

defined by $\widetilde{\pi}_s\big(\phi_0, \ldots, \phi_{s-1}, [p_0\,;\ldots;\,p_n]\big) = \big(\phi_0, \ldots, \phi_{s-1}, [p_0\,;\ldots;\,p_s]\big)$, where $\{p_j\}_j$ are coprime polynomials. We denote the image of $\widetilde{F}_{k,R_0,\ldots,R_{s-1}}$ by $\widetilde{B}_{k,R_0,\ldots,R_{s-1}}$ and that of $\widetilde{F}^0_{k,R_0,\ldots,R_{s-1}}$ by $\widetilde{B}^0_{k,R_0,\ldots,R_{s-1}}$.

Proposition 4.7 *Suppose that $k-n \ge R_0$. Then $F^*_{k,R_0,\ldots,R_s}$ is a path-connected complex submanifold of $F^*_{k,R_0,\ldots,R_{s-1}}$ of dimension*

$$(k+1)(n+1) - 1 - \sum_{\alpha=0}^{s} (n-\alpha-1)R_\alpha$$

for any $1 \le s \le n-2$.

Proof By Lemma 4.3, we have shown that $\widetilde{F}_{k,R_0}$ is a path-connected complex manifold of dimension $(k+1)(n+1) - 1 - (n-1)R_0$.

By the induction on $s \ge 1$, we will show that $\widetilde{F}_{k,R_0,\ldots,R_s}$ is a path-connected complex submanifold of $\widetilde{F}_{k,R_0,\ldots,R_{s-1}}$ of the required dimension. Suppose that we have shown for $s-1 \ge 0$. Then $\widetilde{F}_{k,R_0,\ldots,R_{s-1}}$ is a path-connected complex manifold. Since

$\widetilde{F}^0_{k,R_0,\dots,R_{s-1}}$ is obtained by deleting a proper subvariety from $\widetilde{F}_{k,R_0,\dots,R_{s-1}}$, $\widetilde{F}^0_{k,R_0,\dots,R_{s-1}}$ is also a path-connected complex submanifold and so is $\widetilde{F}'_{k,R_0,\dots,R_{s-1}}$ by Lemma 4.2.

By the definition of $\widetilde{F}^0_{k,R_0,\dots,R_{s-1}}$, we can extend the result of Lemma 4.6 for $\widetilde{F}_{k,R_0,\dots,R_{s-1}}$. For $(\phi_0,\dots,\phi_{s-1},[p_0\,;\cdots;\,p_n])\in\widetilde{F}_{k,R_0,\dots,R_{s-1}}$,

$$\{D(p_0,p_{i_1},\dots,p_{i_{s+1}})\}_{0<i_1<\cdots<i_{s+1}\leq n}$$

already has a common divisor $\widetilde{\phi}^*_s$. Put

$$\widetilde{x}=\big(\phi_0,\dots,\phi_{s-1},[p_0\,;\cdots;\,p_n]\big),\quad x=\widetilde{\pi}_s\big(\widetilde{x}\big)=\big(\phi_0,\dots,\phi_{s-1},[p_0\,;\cdots;\,p_s]\big).$$

Let $q_xD(p_0,p_1,\dots,p_s,p)$ denote the quotient of $D(p_0,p_1,\dots,p_s,p)$ divided by $\widetilde{\phi}^*_s$. It's degree k_s is given by

$$k_s=(k-s-1)(s+2)-\sum_{u=0}^{s}\sum_{\alpha=0}^{u}R_\alpha+R_s=\deg f_{s+1}+R_s.$$

For $x\in\widetilde{B}_{k,R_0,\dots,R_{s-1}}$, define a map $S_x\colon V_k\to V_{k_s}$ by

$$S_x(p)=q_xD(p_0,p_1,\dots,p_s,p).$$

The kernel of S_x is the space expanded by $\{p_j\}_{0\leq j\leq s}$ whose orthgonal complement is isomorphic to the image $\operatorname{Im}S_x$. As the dimension of $\operatorname{Im}S_x$ is $k-s$ for any x,

$$\widetilde{F}^1_{k,R_0,\dots,R_{s-1}}=\cup_{x\in\widetilde{B}_{k,R_0,\dots,R_{s-1}}}\big(\{x\}\times V_{n-s}(\operatorname{Im}S_x)\big)$$

is a complex bundle over $\widetilde{B}_{k,R_0,\dots,R_{s-1}}$. Consider the map

$$F\colon M(R_s,k_s+1)\times\widetilde{F}_{k,R_0,\dots,R_{s-1}}\longrightarrow M(R_s,k_s+1)\times\widetilde{F}^1_{k,R_0,\dots,R_{s-1}}$$

defined by $F(T,\widetilde{x})=(T,x,\{S_xp_j\}_{s+1\leq j\leq n})$ with $\widetilde{\pi}_s(\widetilde{x})=x$. By the assumption $k\geq n$, F is a bundle map over $M(R_s,k_s+1)\times\widetilde{B}_{k,R_0,\dots,R_{s-1}}$. Since $k_s\geq R_s+n$ by Lemma 4.5, the subspace

$$\widetilde{E}_s=\big\{\,(T,x,\{q_j\}_{s+1\leq j\leq n})\mid Tq_j=0\text{ for any }j\big\}$$

of $M(R_s,k_s+1)\times\widetilde{F}^1_{k,R_0,\dots,R_{s-1}}$ is also a bundle over $\widetilde{F}^1_{k,R_0,\dots,R_{s-1}}$ and so is a path-connected complex manifold. As $\widetilde{E}_s$ is also a bundle over $M(R_s,k_s+1)\times\widetilde{B}_{k,R_0,\dots,R_{s-1}}$, we get the following commutative diagram:

$$\begin{array}{ccc}E_s=F^*\widetilde{E}_s & \subset & M(R_s,k_s+1)\times\widetilde{F}_{k,R_0,\dots,R_{s-1}}\\ \downarrow & & \downarrow\\ M(R_s,k_s+1)\times\widetilde{B}_{k,R_0,\dots,R_{s-1}} & = & M(R_s,k_s+1)\times\widetilde{B}_{k,R_0,\dots,R_{s-1}}\end{array}$$

and so E_s is a path-connected complex manifold.

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For $\phi \in V^+_{R_s}$, we also consider a linear map $T_\phi \colon V_{k_s} \to V_{R_s-1}$ defined by $T_\phi q$ being the remainder of q divided by ϕ. Consider the diagram

$$\begin{array}{ccc} & & E_s \\ & & \downarrow \\ V^+_{R_s} \times \widetilde{B}_{k,R_0,\dots,R_{s-1}} & \xrightarrow{\Phi_s} & M(R_s, k_s+1) \times \widetilde{B}_{k,R_0,\dots,R_{s-1}}, \end{array}$$

where $\Phi_s(\phi, x) := (T_\phi,\ x)$. Then the induced bundle $\Phi_s^* E_s$ is a path-connected manifold of dimension

$$R_s + \dim \widetilde{F}_{k,R_0,\dots,R_{s-1}} - (n-s)R_s = (k+1)(n+1) - 1 - \sum_{\alpha=0}^{s} (n-\alpha-1)R_\alpha.$$

Put $E^0_s = E_s \cap (M(R_s, k_s+1) \times \widetilde{F}^0_{k,R_0,\dots,R_{s-1}})$. The restriction Φ^0_s of Φ_s on $V^+_{R_s} \times \widetilde{B}^0_{k,R_0,\dots,R_{s-1}}$ also induces the bundle $\Phi_s^* E^0_s$ over $V^+_{R_s} \times \widetilde{B}^0_{k,R_0,\dots,R_{s-1}}$ of the same dimension to that of $\Phi_s^* E_s$.

Let $\widetilde{E}^0_s$ be a subspace of $\Phi_s^* E^0_s$ consisting of $(\phi, \phi_0, \dots, \phi_{s-1}, [p_0 \,;\, p_1 \,;\, \cdots \,;\, p_n])$ with $(\phi_0, \dots, \phi_{s-1}, \phi) \in V^s_{R_J}$. Since we get $\widetilde{E}^0_s$ by deleting a proper subvariety from $\Phi_s^* E^0_s$, $\widetilde{E}^0_s$ is also a path-connected complex submanifold of $\Phi_s^* E^0_s$. We also consider a subspace $\widetilde{E}'_s$ of $\widetilde{E}^0_s$ consisting of $(\phi, \phi_0, \dots, \phi_{s-1}, [p_0 \,;\, p_1 \,;\, \cdots \,;\, p_n])$ with

$$Z_\phi \cap Z_{D(p_{i_0}, \dots, p_{i_s})} = \phi$$

for any $0 \le i_0 < \cdots < i_s \le n$. In a similar way to the proof of Lemma 4.2, any point of $\widetilde{E}^0_s$ is locally equivalent to $\widetilde{E}'_s$ and so $\widetilde{E}'_s$ is path-connected.

Take $(\phi, \phi_0, \dots, \phi_{s-1}, [p_0 \,;\, p_1 \,;\, \cdots \,;\, p_n]) \in \widetilde{E}'_s$, where

$$\phi = \Pi_{1 \le j \le R_s} (z - \alpha_j)^{\lambda_j} := \Pi_{1 \le j \le R_s} \varphi_j.$$

For $1 \le s \le n-2$, $s < i < j \le n$ and $0 \le u \le s$, consider the equality

$$\begin{aligned} 0 &= \begin{vmatrix} p_0^{(u)} & \cdots & p_s^{(u)} & p_i^{(u)} & p_j^{(u)} \\ p_0 & \cdots & p_s & p_i & p_j \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ p_0^{(s+1)} & \cdots & p_s^{(s+1)} & p_i^{(s+1)} & p_j^{(s+1)} \end{vmatrix} \\ &= \sum_{\nu=0}^{s} (-1)^\nu p_\nu^{(u)} \cdot D(p_0, \dots, \check{p}_\nu, \dots, p_s, p_i, p_j) \\ &\quad + (-1)^{s+1} p_i^{(u)} \cdot D(p_0, \dots, p_s, p_j) + (-1)^s p_j^{(u)} \cdot D(p_0, \dots, p_s, p_i) \end{aligned}$$

which are linear simultaneous equations of $\{D(p_0, \dots, \check{p}_\nu, \dots, p_s, p_i, p_j)\}_{0 \le \nu \le s}$. Put

$$\begin{aligned} w_\nu &:= D(p_0, \dots, \check{p}_\nu, \dots, p_s, p_i, p_j), \\ \alpha_\nu &:= (-1)^s p_i^{(\nu)} \cdot D(p_0, \dots, p_s, p_j) + (-1)^{s+1} p_j^{(\nu)} \cdot D(p_0, \dots, p_s, p_i). \end{aligned}$$

We have

$$\begin{pmatrix} p_0 & \cdots & (-1)^s p_s \\ \vdots & \ddots & \vdots \\ p_0^{(s)} & \cdots & (-1)^s p_s^{(s)} \end{pmatrix} \begin{pmatrix} w_0 \\ \vdots \\ w_s \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_s \end{pmatrix}.$$

Since the coefficient matrix is regular except at finite points, by Cramer's formula, we get

$$\begin{aligned} &D(p_0, \dots, p_s) w_\nu \\ &= (-1)^{\frac{s(s+1)}{2}} \begin{vmatrix} p_0 & \cdots & \alpha_0 & \cdots & (-1)^s p_s \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ p_0^{(s)} & \cdots & \alpha_s & \cdots & (-1)^s p_s^{(s)} \end{vmatrix} \\ &= (-1)^{\nu+s} \Big\{ D(p_0, \dots, p_{\nu-1}, p_i, p_{\nu+1}, \dots, p_s) \cdot D(p_0, \dots, p_s, p_j) \\ &\qquad - D(p_0, \dots, p_{\nu-1}, p_j, p_{\nu+1}, \cdots, p_s) \cdot D(p_0, \dots, p_s, p_i) \Big\}. \end{aligned}$$

Note that $D(p_0, \dots, p_s)$ is divisible by $\widetilde{\phi}_{s-1}$ and w_ν is divisible by $\widetilde{\phi}_s^*$. By the assumption of the induction, the right-hand side is also divisible by $\widetilde{\phi}_{s-1} \cdot \widetilde{\phi}_s^* \cdot \varphi_1$. Moreover, by the assumption of $\widetilde{E}'_s$, φ_1 is not a factor of $q_s D(p_0, \dots, p_s)$. Hence w_ν can be divided by $\widetilde{\phi}_s^* \cdot \varphi_1$. Thus φ_1 is a factor of $q_s D(p_0, \dots, \check{p}_\nu, \dots, p_s, p_i, p_j)$ for any $j > i \geq s+1$. We repeat the procedure and show that any $D(p_0, p_{i_1}, \dots, p_{i_s}, p_{i_{s+1}})$ can be divided by $\widetilde{\phi}_s^* \cdot \varphi_1$. We also repeat this procedure for φ_j with $j \geq 2$ to show that any $D(p_0, p_{i_1}, \dots, p_{i_s}, p_{i_{s+1}})$ can be divided by $\widetilde{\phi}_s = \widetilde{\phi}_s^* \cdot \phi_s$ for $\phi_s := \phi$. Therefore $\widetilde{E}'_s = \widetilde{F}'_{k,R_0,\dots,R_s}$ and so $\widetilde{E}^0_s = \widetilde{F}^0_{k,R_0,\dots,R_s}$ is a required submanifold of $\widetilde{F}_{k,R_0,\dots,R_{s-1}}$, as the closure of $\widetilde{E}^0_s$ is $\Phi_s^* E_s$, $\Phi_s^* E_s = \widetilde{F}_{k,R_0,\dots,R_s}$. Since the projection $\pi_s \colon \widetilde{F}_{k,R_0,\dots,R_s} \to F_{k,R_0,\dots,R_s}$ induces an isomorphism

$$F^*_{k,R_0,\dots,R_s} = F_{k,R_0,\dots,R_s} - F_{k,R_0,\dots,R_s+1} \simeq \widetilde{F}_{k,R_0,\dots,R_s} - \pi_s^{-1}(F_{k,R_0,\dots,R_s+1})$$

and $F_{k,R_0,\dots,R_s+1}$ is a proper subvariety, we get the assertion. ■

Finally we prove Theorem 1.1. Since the degree of h_f for $f \in F^*_{k,R_0,\dots,R_s}$ is determined only by $F^*_{k,R_0,\dots,R_s}$, without loss of generality, we can assume that $\deg h_f \geq \deg f$. That means that

$$(k-n+1)n - \sum_{u=0}^{n-2} \sum_{\alpha=0}^{u} R_\alpha \geq k$$

by which we get $k - n \geq R_0$. So we can apply Lemma 4.3 and Proposition 4.7. By definition, $F^*_{k,R_0,\dots,R_{n-2}} = \mathcal{H}ol^*_{k,R_J}(S^2, \mathbf{C}P^n)$, and so we get the first assertion of Theorem 1.1.

For $1 \leq s \leq n-1$, the correspondence

$$\mathcal{F}_s \colon F^*_{k,R_0,\dots,R_{n-2}} \longrightarrow \mathcal{H}ol_{d_s}\big(S^2, G(n+1, s+1)\big)$$

defined by $\mathcal{F}_s(f) = f_s$ is continuous, where $d_s = (k-s)(s+1) - \sum_{u=0}^{s-1}\sum_{\alpha=0}^{u} R_\alpha$. Hence

$$\partial^s\colon \mathcal{H}ol^*_{k,R_J}(S^2, \mathbf{C}P^n) \to \mathcal{H}arm^*_{k,s,R_J}(S^2, \mathbf{C}P^n)$$

given by $\partial^s f = f_s \cap f_{s-1}^{\perp}$ is continuous. Here $f_0 = f$. When $s = n$,

$$\partial^n\colon \mathcal{H}ol^*_{k,R_J}(S^2, \mathbf{C}P^n) \to \mathcal{H}ol^*_{-d_{n-1}}(S^2, \mathbf{C}P^n)$$

defined by $\partial^n f = f_{n-1}^{\perp}$ is also continuous.

Next we will show that $\partial^s\colon \mathcal{H}ol^*_{k,R_J}(S^2, \mathbf{C}P^n) \to \mathcal{H}arm^*_{k,s,R_J}(S^2, \mathbf{C}P^n)$ is proper. Let S_r be the subspace of PV_k^m consisting of $[q_1 ; \cdots ; q_m]$ so that $\{q_j\}_j$ has the greatest common divisor ϕ with $\deg\phi \ge r$. Put

$$S_{R_J} := \Pi_{s=0}^{n-2} S_{\widetilde{R}_s} \subset \Pi_{s=1}^{n-1} PV_{d_s+1}^{n_s}$$

for $\widetilde{R}_s = \sum_{u=0}^{s}\sum_{\alpha=0}^{u} R_\alpha$, $n_s = {}_{n+1}C_{s+1}$ and consider the map

$$\mathcal{H}ol^*_{k,R_J}(S^2, \mathbf{C}P^n) \xrightarrow{\Phi} S_{R_J} \quad \text{and} \quad \operatorname{Im}\Phi \xrightarrow{\Psi} \Pi_{s=1}^{n-1} \mathcal{H}arm^*_{k,s,R_J}(S^2, \mathbf{C}P^n)$$

defined by $\Phi(f) = \{f_s\}_s$ and $\Psi(\{f_s\}_s) = \{f_s \cap f_{s-1}^{\perp}\}_s$. Take a sequence $\{f^j\}_j$ in $\mathcal{H}ol^*_{k,R_J}(S^2, \mathbf{C}P^n)$ so that $\{\Psi\circ\Phi(f^j)\}_j$ converges to $\{h_u\}_u$. This is equivalent to say that $\{\Phi(f^j)\}_j$ converges to $\{g_u\}_u$ where $g_u \cap g_{u-1}^{\perp} = h_u$. As mentioned in [C], $\mathcal{H}ol^*_k(S^2, \mathbf{C}P^n)$ is equal to S_0 in a compact space PV_{k+1}^{n+1} and so we can assume that $\{f^j\}_j$ converges to $f \in PV_{k+1}^{n+1}$. Suppose that $f \in S_m$. Then we get

$$\deg\partial f = \big(2k-2-R_0(f)\big) - (k-m) = k-2-R_0 = \deg h_1,$$
$$E(\partial f) = 2(k-m) + \big(2k-2-R_0(f)\big) - (k-m) = 3k-2-R_0 = E(h_1),$$

and so $m = 0$, $R_0(f) = R_0$. We also get

$$\deg\partial^s f = k-2s-\sum_{u=0}^{s-1} R_u(f) = k-2s-\sum_{u=0}^{s-1} R_u = \deg h_s$$

for $2 \le s \le n-1$. Hence $R_u(f) = R_u$ for any $0 \le u \le n-2$ by which $f \in \mathcal{H}ol^*_{k,R_J}(S^2, \mathbf{C}P^{n+1})$. This shows the second assertion of Theorem 1.1.

5 Gluing of Harmonic Maps

We will consider the gluing of harmonic bubble tree maps. We begin with some lemmas.

Lemma 5.1 *Put $p_s(z) = z^s + (Rz)^{-s}$ for any integer $s \ge 0$ and any real $R > 0$. When $0 \le s_0 < s_1 < \cdots < s_\ell$, we get*

$$D(p_{s_0}, p_{s_1}, \ldots, p_{s_\ell})(z) = q_{s_0 s_1 \cdots s_\ell} \cdot \Big(1 - \frac{1}{Rz^2}\Big)^{\frac{\ell(\ell+1)}{2}},$$

where $z^K q_{s_0 s_1 \cdots s_\ell}(z)$ is a polynomial in z for $K = s_0 + \cdots + s_\ell - (0 + 1 + \cdots + \ell)$. In particular,

$$D(p_0, p_1, \ldots, p_\ell)(z) = 2 \cdot 1! \cdots \ell! \cdot \Big(1 - \frac{1}{Rz^2}\Big)^{\frac{\ell(\ell+1)}{2}}.$$

Proof When $k \le s$, we get

$$p_s^{(k)}(z) = s(s-1)\cdots(s-k+1)z^{s-k} + (-1)^k s(s+1)\cdots(s+k-1)R^{-s}z^{-(s+k)}.$$

If $s < k$, the first term vanishes. So we can use the same expression for any $k \ge 0$. Put $q_k(s) = \sqrt{R}^{s-k} p_s^{(k)}(\sqrt{R}^{-1})$. We have $q_0(s) \equiv 2$, $q_1(s) \equiv 0$ and

$$\begin{aligned} q_{2k}(s) &= s(s-1)\cdots(s-2k+1) + s(s+1)\cdots(s+2k-1) \\ &= 2\big(s^{2k} + c_{2k-2}s^{2k-2} + \cdots + c_2 s^2\big), \\ q_{2k+1}(s) &= s(s-1)\cdots(s-2k) - s(s+1)\cdots(s+2k) \\ &= 2\big(d_{2k}s^{2k} + d_{2k-2}s^{2k-2} + \cdots + d_2 s^2\big) \end{aligned}$$

for $k \ge 1$, where $d_{2k} = -2k(2k+1)$ and constants c_j, d_j are determined by k. For $0 \le s_0 < s_1 < \cdots < s_\ell$ and $0 \le k_0 \le k_1 \le \cdots \le k_\ell$, put

$$\mathbf{a}_k(\mathbf{s}) := \mathbf{a}_k(s_0, s_1, \ldots, s_\ell) = \big(q_k(s_0), q_k(s_1), \cdots, q_k(s_\ell)\big)$$

and

$$\Phi(k_0, \ldots, k_\ell\,;\mathbf{s}) = \begin{vmatrix} \mathbf{a}_{k_0}(\mathbf{s}) \\ \mathbf{a}_{k_1}(\mathbf{s}) \\ \vdots \\ \mathbf{a}_{k_\ell}(\mathbf{s}) \end{vmatrix} = \sqrt{R}^{K_0} \cdot \begin{vmatrix} p_{s_0}^{(k_0)}(\sqrt{R}^{-1}) & \cdots & p_{s_0}^{(k_\ell)}(\sqrt{R}^{-1}) \\ \vdots & \ddots & \vdots \\ p_{s_\ell}^{(k_0)}(\sqrt{R}^{-1}) & \cdots & p_{s_\ell}^{(k_\ell)}(\sqrt{R}^{-1}) \end{vmatrix},$$

where $K_0 = \sum_{j=0}^{\ell}(s_j - k_j)$. We get

$$\frac{d^N}{dz^N} D(p_{s_0}, p_{s_1}, \ldots, p_{s_\ell})(\sqrt{R}^{-1}) = \sum \sqrt{R}^{-K_0} \cdot \Phi(k_0, \ldots, k_i, \ldots, k_\ell\,;\mathbf{s}),$$

where $k_i \ge i$ and the summations are taken over $0 \le k_0 \le k_1 \le \cdots \le k_\ell$ with

$$\sum_{i=0}^{\ell}(k_i - i) = k_0 + \cdots + k_\ell - \frac{\ell(\ell+1)}{2} = N.$$

Starting from $\Phi(0, 1, \ldots, \ell\,;\mathbf{s})$, we consider $\Phi(k_0, \ldots, k_\ell\,;\mathbf{s})$ determined by the differentiation of $D(p_{s_0}, \ldots, p_{s_\ell})$ at $z = \sqrt{R}^{-1}$.

As $q_1(s) \equiv 0$, $\Phi(0, 1, \ldots, k_\ell\,;\mathbf{s}) = 0$ for any $\mathbf{s}$. Since $q_2(s) = 2s^2$ and $q_3(s) = -6s^2$, $\{\mathbf{a}_2(\mathbf{s}), \mathbf{a}_3(\mathbf{s})\}$ is linearly dependent and so $\Phi(0, 2, 3, \ldots, k_\ell\,;\mathbf{s}) = 0$. Now suppose

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that we have shown

$$\Phi\big(0,2,\dots,2(k'-1),2k'-1,k_{k'+1},\dots,k_\ell\;;\mathbf{s}\big)=0$$

for any $k' \geq 2$. Then consider $\Phi(0,\dots,2(k'-1),2k',2k'+1,k_{k'+2},\dots,k_\ell\;;\mathbf{s})$. Since $\mathbf{a}_{2k'}(\mathbf{s})$ and $\mathbf{a}_{2k'+1}(\mathbf{s})$ are polynomials of degree $2k'$, we get a non-trivial $\mathbf{b}(\mathbf{s})$ consisting of polynomials of degree $2k'-2$:

$$\mathbf{b}(\mathbf{s})=2k'(2k'+1)\mathbf{a}_{2k'}(\mathbf{s})+\mathbf{a}_{2k'+1}(\mathbf{s}).$$

Repeating fundamental row operations finitely many times, we get non-trivial $\mathbf{c}(\mathbf{s})$ of degree 2 by making use of $\mathbf{a}_{2k'+1}(\mathbf{s}),\mathbf{a}_{2k'}(\mathbf{s}),\dots,\mathbf{a}_4(\mathbf{s})$. As $\{\mathbf{c}(\mathbf{s}),\mathbf{a}_2(\mathbf{s})\}$ are linearly dependent,

$$\Phi(0,2,\dots,2(k'-1),2k',2k'+1,k_{k'+2},\dots,k_\ell\;;\mathbf{s})=0.$$

Repeating this procedure, we get the Vandermonde determinant

$$\Phi(0,2,\dots,2\ell\;;\mathbf{s})=2^{\ell+1}\cdot\begin{vmatrix}1&1&\cdots&1\\ s_0^2&s_1^2&\cdots&s_\ell^2\\ \vdots&\vdots&\ddots&\vdots\\ s_0^{2\ell}&s_1^{2\ell}&\cdots&s_\ell^{2\ell}\end{vmatrix}=2^{\ell+1}\cdot\prod_{0\le i<j\le\ell}(s_j^2-s_i^2),$$

which does not vanish. So the first non-zero value appears after differentiating $D(p_{s_0},p_{s_1},\cdots,p_{s_\ell})$ $\frac{\ell(\ell+1)}{2}$ times at $\sqrt{R}^{-1}$. As the expression consists of polynomials of even degree, this means that

$$D(p_{s_0},p_{s_1},\dots,p_{s_\ell})(z)=q_{s_0s_1\cdots s_\ell}\cdot\Big(1-\frac{1}{Rz^2}\Big)^{\frac{\ell(\ell+1)}{2}},$$

where $z^{s_0+\cdots+s_\ell+0+1+\cdots+\ell}D(p_{s_0},p_{s_1},\dots,p_{s_\ell})(z)$ is a polynomial. Hence $q_{s_0s_1\cdots s_\ell}$ is a required one. Since the degree of $z^{\ell(\ell+1)}D(p_0,p_1,\dots,p_\ell)$ is $\ell(\ell+1)$, $q_{01\cdots\ell}$ is a constant K.

Now put $N=\frac{\ell(\ell+1)}{2}$. By direct calculation, we get $p_0(z)=2$ and

$$R^sz^{s+k}p_s^{(k)}(z)\big|_{z=0}=(-1)^k\frac{(s+k-1)!}{(s-1)!}$$

for $1\le s,k\le\ell$. Since

$$D(p_0,p_1,\dots,p_\ell)=2\begin{vmatrix}p_1'&\cdots&p_\ell'\\ \vdots&\ddots&\vdots\\ p_1^{(\ell)}&\cdots&p_\ell^{(\ell)}\end{vmatrix}=K\,\Big(1-\frac{1}{Rz^2}\Big)^N,$$

we get

$$\begin{aligned}
K &= (-1)^N\, 2\, R^N z^{2N} \begin{vmatrix} p_1' & \cdots & p_\ell' \\ \vdots & \ddots & \vdots \\ p_1^{(\ell)} & \cdots & p_\ell^{(\ell)} \end{vmatrix}\Bigg|_{z=0} \\
&= 2 \begin{vmatrix} \frac{1!}{0!} & \frac{2!}{1!} & \cdots & \frac{\ell!}{(\ell-1)!} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\ell!}{0!} & \frac{(\ell+1)!}{1!} & \cdots & \frac{(2\ell-1)!}{(\ell-1)!} \end{vmatrix} \\
&= 2\,\ell! \begin{vmatrix} 1 & \frac{2!}{1!} & \cdots & \frac{s!}{1!} & \cdots & \frac{\ell!}{1!} \\ 1 & \frac{3!}{2!} & \cdots & \frac{(s+1)!}{2!} & \cdots & \frac{(\ell+1)!}{2!} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & \frac{(\ell+1)!}{\ell!} & \cdots & \frac{(s+\ell-1)!}{\ell!} & \cdots & \frac{(2\ell-1)!}{\ell!} \end{vmatrix}.
\end{aligned}$$

Following the above equality, for $1 \le j \le \ell$, we define x_j as the determinant of a square matrix of order $\ell - j + 1$:

$$x_j = \begin{vmatrix} 1 & \frac{(j+1)!}{j!} & \cdots & \frac{s!}{j!} & \cdots & \frac{\ell!}{j!} \\ 1 & \frac{(j+2)!}{(j+1)!} & \cdots & \frac{(s+1)!}{(j+1)!} & \cdots & \frac{(\ell+1)!}{(j+1)!} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & \frac{(\ell+1)!}{\ell!} & \cdots & \frac{(s+\ell-j)!}{\ell!} & \cdots & \frac{(2\ell-j)!}{\ell!} \end{vmatrix}.$$

Then we get

$$\begin{aligned}
x_j &= \begin{vmatrix} 1 & * & * & \cdots & * \\ 0 & 1 & 2\frac{(j+2)!}{(j+1)!} & \cdots & (\ell-j)\frac{\ell!}{(j+1)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 2\frac{(\ell+1)!}{\ell!} & \cdots & (\ell-j)\frac{(2\ell-j-1)!}{\ell!} \end{vmatrix} \\
&= (\ell-j)! \begin{vmatrix} 1 & \frac{(j+2)!}{(j+1)!} & \cdots & \frac{\ell!}{(j+1)!} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{(\ell+1)!}{\ell!} & \cdots & \frac{(2\ell-j-1)!}{\ell!} \end{vmatrix} = (\ell-j)!\, x_{j+1}
\end{aligned}$$

and $x_{\ell-1} = \begin{vmatrix} 1 & \frac{\ell!}{(\ell-1)!} \\ 1 & \frac{(\ell+1)!}{\ell!} \end{vmatrix} = 1$ by which $K = 2\,\ell!\,(\ell-1)!\cdots 3!\,2!\,1!$. ■

Lemma 5.2 *For $n \ge 2$, if $R_J = \{\underbrace{2, 2, \ldots, 2}_{n-1}\}$, then $\mathcal{H}ol^*_{2n,R_J}(S^2, \mathbf{C}P^n)$ is non-empty.*

Proof For any $R > 0$, define $f \in \mathcal{H}ol_{2n}(S^2, \mathbf{C}P^n)$ by

$$f(z) = [1\,;\cdots;\, z^s + \frac{1}{R^s z^s}\,;\cdots;\, z^n + \frac{1}{R^n z^n}].$$

Then, by Lemma 5.1, we get

$$\deg f_s = (2n-s)(s+1) - s(s+1) = 2(n-s)(s+1)$$

for $0 \le s \le n-1$, and so $\deg \partial^s f = 2(n-2s)$ for $0 \le s \le n-1$. Hence

$$R_s(f) = \partial^s f - 2 - \partial^{s+1} f = 2$$

for any $0 \le s \le n-2$. ■

Let $\mathcal{H}ol_n^*(S^2, \mathbf{C}P^n) * \mathcal{H}ol_n^*(S^2, \mathbf{C}P^n)$ be the subspace of the product manifold $\mathcal{H}ol_n^*(S^2, \mathbf{C}P^n) \times \mathcal{H}ol_n^*(S^2, \mathbf{C}P^n)$ consisting of pairs $(f^{(0)}, f^{(1)})$ with $\partial^s f^{(0)}(0) = \partial^s f^{(1)}(\infty)$ for any $0 \le s \le n$. In Lemma 5.3, for the case $n = 2$, see [K, 5. Example].

Lemma 5.3 *For $n \ge 2$, $\mathcal{H}ol_n^*(S^2, \mathbf{C}P^n) * \mathcal{H}ol_n^*(S^2, \mathbf{C}P^n)$ is a smooth submanifold of $\mathcal{H}ol_n^*(S^2, \mathbf{C}P^n) \times \mathcal{H}ol_n^*(S^2, \mathbf{C}P^n)$ of $\mathbf{R}$-dimension $n(3n+7)$.*

Proof Let $F(\mathbf{C}^{n+1})$ be the flag manifold consisting of sequences of vector spaces $\{V_s\}_{0 \le s \le n}$ in $\mathbf{C}^{n+1}$ satisfying $V_0 \subset \cdots \subset V_{n-1} \subset \mathbf{C}^{n+1}$ with $\dim V_s = s+1$. As is well known, $F(\mathbf{C}^{n+1})$ is isomorphic to the quotient space

$$U(n+1)/\underbrace{U(1) \times \cdots \times U(1)}_{(n+1)-times}$$

as complex manifolds. Hence $F(\mathbf{C}^{n+1})$ is a complex manifold of dimension $\frac{n(n+1)}{2}$. Consider the map

$$\Phi\colon \mathcal{H}ol_n^*(S^2, \mathbf{C}P^n) \times \mathcal{H}ol_n^*(S^2, \mathbf{C}P^n) \to F(\mathbf{C}^{n+1}) \times F(\mathbf{C}^{n+1})$$

defined by $\Phi(f,g) = (\{f_s(0)\}_{0 \le s \le n-1}, \{g_s(\infty)\}_{0 \le s \le n-1})$. Since Φ is smooth and transverse to the diagonal Δ, $\Phi^{-1}(\Delta) = \mathcal{H}ol_n^*(S^2, \mathbf{C}P^n) * \mathcal{H}ol_n^*(S^2, \mathbf{C}P^n)$ is a smooth manifold of dimension

$$2 \times 2\big\{(n+1)^2 - 1\big\} - n(n+1) = n(3n+7). \qquad ■$$

Let $T^I = S^{(0)} \vee S^{(1)}$ be a bubble domain tower with the base space $S^{(0)} = S^2$ and a bubble domain $S^{(1)} = \pi_1^{-1}(0)$ for $0 \in S^{(0)}$. Any $f^I \in \mathcal{H}ol_n^*(S^2, \mathbf{C}P^n) * \mathcal{H}ol_n^*(S^2, \mathbf{C}P^n)$ can be considered as a holomorphic bubble tree map $f^I = f^{(0)} \vee f^{(1)}\colon T^I \to \mathbf{C}P^n$ with $f^{(\ell)} \in \mathcal{H}ol_n^*(S^2, \mathbf{C}P^n)$ for $\ell = 0, 1$.

Any $f^I \in \mathcal{H}ol_n^*(S^2, \mathbf{C}P^n) * \mathcal{H}ol_n^*(S^2, \mathbf{C}P^n)$ is said to be completely gluable if there is a sequence $\{f_k\}_k$ so that $\{\partial^s f_k\}_k$ converges to

$$\partial^s f^I = \partial^s f^{(0)} \vee \partial^s f^{(1)}\colon T^I = S^{(0)} \vee S^{(1)} \to \mathbf{C}P^n$$

for any $0 \le s \le n$. We show the following proposition.

Proposition 5.4 *Any $f^I \in \mathcal{H}ol_n^*(S^2, \mathbf{C}P^n) * \mathcal{H}ol_n^*(S^2, \mathbf{C}P^n)$ is completely gluable exactly when a sequence $\{f_k\}_{k\geq 1}$ in $\mathcal{H}ol_{2n,R_J}(S^2, \mathbf{C}P^n)$ converges to f^I. Here $R_J = \{2, 2, \dots, 2\}$.*

Proof Put $R_J' = \{0, 0, \dots, 0\}$. By Lemma 2.7, $\partial^s f^{(\ell)} \in \mathcal{H}arm^*_{n,s,R_J'}(S^2, \mathbf{C}P^n)$. Hence

$$\deg \partial^s f^{(\ell)} = n - 2s \quad \text{and} \quad E(\partial^s f^{(\ell)}) = n(2s+1) - 2s^2.$$

By Lemma 5.2, $\mathcal{H}ol^*_{2n,R_J}(S^2, \mathbf{C}P^n)$ is not empty and any $f_k \in \mathcal{H}ol^*_{2n,R_J}(S^2, \mathbf{C}P^n)$ holds $\deg \partial^s f_k = 2n - 4s$ and $E(\partial^s f_k) = 2n(2s+1) - 4s^2$ for $1 \leq s \leq n$. ∎

Proposition 5.5 *When $n \geq 2$, there is a holomorphic bubble tree map f^I in $\mathcal{H}ol_n^*(S^2, \mathbf{C}P^n) * \mathcal{H}ol_n^*(S^2, \mathbf{C}P^n)$ so that $\partial^s f^I$ is well defined for any s but non-gluable for some s.*

Proof Put $R_J = \{2, \dots, 2\}$. By Lemma 5.2 and Theorem 1.1, the dimension of $\mathcal{H}ol^*_{2n,R_J}(S^2, \mathbf{C}P^n)$ is equal to

$$(2n+1)(n+1) - 1 - 2\sum_{s=0}^{n-2}(n-s-1) = n^2 + 4n.$$

By Lemma 5.3, we get $n(3n+7) - 2(n^2+4n) = n(n-1) > 0$. ∎

6 Examples

Example 6.1 We will show an example of gluing a holomorphic bubble tree map by doing the procedure given in Lemma 3.2. Let $T^I = S^{(0)} \vee S^{(1)} \vee S^{(2)}$ be a simple bubble tower domain consisting the base space $S^{(0)}$ and bubble domains $S^{(1)} = \pi_1^{-1}(0)$ by $0 \in S^{(0)}$, $S^{(2)} = \pi_2^{-1}(1)$ by $1 \in S^{(0)}$. Consider the holomorphic bubble tree map $f^I = f^{(0)} \vee f^{(1)} \vee f^{(2)} \colon T^I \to \mathbf{C}P^2$ defined by

$$f^{(0)}(z) = [z^2\,;\, z^2 + z\,;\, z+1] = [p_0(z)\,;\, p_1(z)\,;\, p_2(z)],$$
$$f^{(1)}(z) = [1\,;\, z+1\,;\, z^2] = [q_0(z)\,;\, q_1(z)\,;\, q_2(z)],$$
$$f^{(2)}(z) = [(z-1)^2 + (z-1)\,;\, 2(z-1)^2 + 1\,;\, 2(z-1)^2] = [r_0(z)\,;\, r_1(z)\,;\, r_2(z)].$$

As defined in the proof of Lemma 3.2, we define $f^R = [p_0^R\,;\, p_1^R\,;\, p_0^R]$ by

$$p_j^R(z) = p_j(z) + \Big\{ \frac{1}{R^2 z^2} q_j(Rz) - \frac{1}{2!}\frac{d^2}{dz^2} q_j(z) \Big\} + \Big\{ \frac{1}{R^2 w^2} \widetilde{r}_j(Rw) - \frac{1}{2!}\frac{d^2}{dw^2} \widetilde{r}_j(w) \Big\}$$

where $\widetilde{r}_j(w) = r_j(z)$ for $w = z - 1$. Then we get

$$f^R(z) = \Big[z^2 + \frac{1}{R^2 z^2} + \frac{1}{R(z-1)}\,;\, z^2 + z + \frac{1}{Rz} + \frac{1}{R^2 z^2} + \frac{1}{R^2 (z-1)^2}\,;\, z+1 \Big].$$

Choose $R_0 > 0$ large enough. When $|z| \geq 1/\sqrt{R_0}$ and $|z-1| \geq 1/\sqrt{R_0}$, the restriction of f^R converges to the restriction of $f^{(0)}$. By $R_0 \to +\infty$, we can extend the

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convergence of f^R to $f^{(0)}$. When $|z| \leq 1/\sqrt{R}$, put $w = Rz$. Then we can get the convergence of $f^R(w)$ to $f^{(1)}(w)$ for $|w| \leq \sqrt{R_1}$ for $R_1 > 0$ large enough. By $R_1 \to +\infty$, we can show the convergence to $f^{(1)}$. Similarly, if $|z-1| \leq 1/\sqrt{R}$, put $w = R(z-1)$. Then we can get the convergence of f^R to $f^{(2)}$.

Example 6.2 We will show another example of gluing a holomorphic bubble tree map. Let $T^I = S^{(0)} \vee S^{(1)} \vee S^{(2)}$ be a bubble tower domain consisting of the base space $S^{(0)}$ and bubble domains $S^{(1)} = \pi_1^{-1}(0)$ by $0 \in S^{(0)}$ and $S^{(2)} = \pi_2^{-1}(0)$ by $0 \in S^{(1)}$. Consider the holomorphic bubble tree map $f^I = f^{(0)} \vee f^{(1)} \vee f^{(2)} \colon T^I \to \mathbf{C}P^2$ defined by

$$f^{(0)}(z) = [2\,;z\,;z^2+1] = [p_0(z)\,;p_1(z)\,;p_2(z)],$$
$$f^{(1)}(z) = [2z^2+1\,;z\,;1+z^2] = [q_0(z)\,;q_1(z)\,;q_2(z)],$$
$$f^{(2)}(z) = [z^2+1\,;z\,;z^2] = [r_0(z)\,;r_1(z)\,;r_2(z)].$$

As defined in the proofs of Lemmas 3.2 and 3.3, first define a map

$$q_j^S(z) = q_j(z) + \Big\{ \frac{1}{S^2z^2} r_j(Sz) - \frac{1}{2!}\frac{d^2}{dz^2} r_j(z) \Big\}$$

for $S \geq 1$ and then define

$$p_j^{RS}(z) = p_j(z) + \Big\{ \frac{1}{R^2z^2} q_j^S(Rz) - \frac{1}{2!}\frac{d^2}{dz^2} q_j(z) \Big\}$$

for $S \geq R^2 \geq 1$ large enough. Then we get a well-defined map $f_{RS} = [p_0^{RS}\,;p_1^{RS}\,;p_2^{RS}] \in \mathcal{H}ol_6(S^2, \mathbf{C}P^2)$, which is given by

$$\begin{aligned} f_{RS}(z) &= \Big[2 + \frac{1}{S^2R^4z^4} + \frac{1}{R^2z^2}\,;z + \frac{1}{Rz} + \frac{1}{SR^3z^3}\,;z^2 + 1 + \frac{1}{R^2z^2} \Big] \\ &= \Big[2R^2z^2 + \frac{1}{S^2R^2z^2} + 1\,;R^2z^3 + Rz + \frac{1}{SRz}\,;R^2z^4 + R^2z^2 + 1 \Big]. \end{aligned}$$

For a fixed real $R_0 > 0$ large enough, a sequence of $\{f_{RS}\}_R$ converges to $f^{(0)}$ on $|z| \geq 1/\sqrt{R_0}$ when $R \to +\infty$. By $R_0 \to +\infty$, we can extend to get the convergence to $f^{(0)}$. For $1/R\sqrt{R} \leq |z| \leq 1/\sqrt{R}$, put $w = Rz$. Then $\{f_{RS}\}_R$ defined on $1/\sqrt{R} \leq |w| \leq \sqrt{R}$ converges to $f^{(1)}$ if $R \to +\infty$. Finally, for $|w| \leq 1/\sqrt{R}$, put $u = Sw$. Then $\{f_{RS}\}_S$ defined on $|u| \leq S/\sqrt{R}$ converges to $f^{(2)}$. Considering the degrees, we get the convergence of $\{f_{RS}\}_{R,S}$ to f^I when $R \to +\infty$. Thus, choose a sequence $\{R_nS_n\}_n$ appropriately so that a sequence $\{f_{R_nS_n}\}_n$ converges to f^I.

Example 6.3 Let $T^I = S^{(0)} \vee S^{(1)}$ be the bubble tower domain defined in Proposition 5.4. Define $f^I = f^{(0)} \vee f^{(1)} \colon T^I \to \mathbf{C}P^3$ by

$$f^{(0)}(z) = [1\,;z\,;z^2\,;z^3] \quad \text{and} \quad f^{(1)}(z) = [z^3\,;z^2\,;z\,;1].$$

Put $R'_J = \{0, 0\}$. By Lemma 2.7,

$$f^{(\ell)} \in \mathcal{H}ol^*_{3,R'_J}(S^2, \mathbf{C}P^3), \quad \partial^s f^{(\ell)} \in \mathcal{H}arm^*_{3,s,R'_J}(S^2, \mathbf{C}P^3)$$

for $s = 1, 2, 3$ and $\ell = 0, 1$, where $\mathcal{H}arm^*_{3,3,R'_J}(S^2, \mathbf{C}P^3) \subset \mathcal{H}ol^*_{-3}(S^2, \mathbf{C}P^3)$. We also get

$$\partial^\ell f^{(0)}(0) = \partial^\ell f^{(1)}(\infty),$$

and so $\partial^\ell f^I$ is well defined for $0 \le \ell \le 3$. Hence $f^I \in \mathcal{H}ol^*_3(S^2, \mathbf{C}P^3) * \mathcal{H}ol^*_3(S^2, \mathbf{C}P^3)$. Consider

$$\begin{aligned} f^R(z) &= [R^3 z^3 \,;\, R^3 z^4 + R^2 z^2 \,;\, R^3 z^5 + Rz \,;\, R^3 z^6 + 1] \\ &= \Big[1 \,;\, z + \frac{1}{Rz} \,;\, z^2 + \frac{1}{R^2 z^2} \,;\, z^3 + \frac{1}{R^3 z^3}\Big], \end{aligned}$$

which is contained in $\mathcal{H}ol^*_6(S^2, \mathbf{C}P^3)$ and converges to f^I if $R \to +\infty$. Moreover, by Lemma 5.1, $f^R \in \mathcal{H}ol^*_{6,R_J}(S^2, \mathbf{C}P^3)$ for $R_J = \{2, 2\}$. Hence, by Theorem 1.1, $\partial^s f_R$ converges to a bubble tree map

$$\partial^s f^I = \partial^s f^{(0)} \vee \partial^s f^{(1)} \colon T^I \longrightarrow \mathbf{C}P^3$$

for $s = 1, 2, 3$.

Example 6.4 Take $f \in \mathcal{H}ol_4(S^2, \mathbf{C}P^3)$ represented by $f(z) = [1 \,;\, z \,;\, z^3 \,;\, z^4]$. By calculations, we get $f_1 \in \mathcal{H}ol_6(S^2, G(4, 2))$. Hence $R_\partial(f) = R_0(f) = 0$. We also get

$$\partial^3 f(z) = [\bar{z}^4 \,;\, -2\bar{z}^3 \,;\, 2\bar{z} \,;\, -1] \in \mathcal{H}ol_{-4}(S^2, \mathbf{C}P^3)$$

by which $f \in \mathcal{H}ol_{4,\{0,2\}}(S^2, \mathbf{C}P^3)$.

Now consider $f_c \in \mathcal{H}ol_4(S^2, \mathbf{C}P^3)$ represented by $f_c(z) = [1 \,;\, z \,;\, z^3 + cz^2 \,;\, z^4]$, which holds $R_\partial(f_c) = 0$ and $\partial^3 f_c \in \mathcal{H}ol_{-5}(S^2, \mathbf{C}P^3)$ represented by

$$\partial^3 f_c(z) = \big[3\bar{z}^4(\bar{z} + c) \,;\, -2\bar{z}^3(3\bar{z} + 4c) \,;\, 6\bar{z}^2 \,;\, -3\bar{z} - c\big]$$

if $c \neq 0$. In this case, $f_c \in \mathcal{H}ol_{4,\{0,1\}}(S^2, \mathbf{C}P^3)$. Since $\deg \partial^2 f = -2$ and $\deg \partial^2 f_c = -1$ for $c \neq 0$, $\partial^2 \colon \mathcal{H}ol_{4,\{0\}}(S^2, \mathbf{C}P^3) \to \mathcal{H}arm(S^2, \mathbf{C}P^3)$ is not continuous. Here $\mathcal{H}ol_{4,\{0\}}(S^2, \mathbf{C}P^3)$ is the subset of $\mathcal{H}ol_4(S^2, \mathbf{C}P^3)$ consisting of f with $R_0(f) = 0$.

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