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Monotonicity Properties of the Hurwitz Zeta Function

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Abstract. Let

$$\zeta(s,x)=\sum_{n=0}^{\infty}\frac{1}{(n+x)^s}\quad(s>1,\ x>0)$$

be the Hurwitz zeta function and let

$$Q(x)=Q(x;\alpha,\beta;a,b)=\frac{(\zeta(\alpha,x))^a}{(\zeta(\beta,x))^b},$$

where $\alpha,\beta>1$ and $a,b>0$ are real numbers. We prove: (i) The function Q is decreasing on $(0,\infty)$ iff $\alpha a-\beta b\geq\max(a-b,0)$. (ii) Q is increasing on $(0,\infty)$ iff $\alpha a-\beta b\leq\min(a-b,0)$. An application of part (i) reveals that for all $x>0$ the function $s\mapsto[(s-1)\zeta(s,x)]^{1/(s-1)}$ is decreasing on $(1,\infty)$. This settles a conjecture of Bastien and Rogalski.

1 Introduction

The Hurwitz zeta function is defined for real numbers $s>1$ and $x>0$ by

$$\zeta(s,x)=\sum_{n=0}^{\infty}\frac{1}{(n+x)^s}.$$

The special case $x=1$ leads to the classical Riemann zeta function

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}.$$

Both functions have important applications in number theory. The Riemann and the Hurwitz zeta functions play an eminent role in the distribution of prime numbers and in the theory of Dirichlet L-functions, respectively. In [5] an application of $\zeta(s,x)$ to probability theory is given. Estimates of the Laurent coefficients of the Hurwitz zeta function are presented in [7]. A collection of the most interesting properties of $\zeta(s,x)$ can be found, for example, in the monographs [2, §1.3; 3, Ch. 12].

Several authors studied various inequalities for the Hurwitz zeta function; see [1, 4, 6]. Using Euler's summation formula we obtain upper and lower bounds

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for $\zeta(s,x)$:

$$\frac{1}{(s-1)x^{s-1}}+\frac{1}{2x^s}+\sum_{k=1}^{2n}\frac{(s)_{2k-1}B_{2k}}{(2k)!\,x^{2k+s-1}}<\zeta(s,x)$$
$$<\frac{1}{(s-1)x^{s-1}}+\frac{1}{2x^s}+\sum_{k=1}^{2n-1}\frac{(s)_{2k-1}B_{2k}}{(2k)!\,x^{2k+s-1}}.$$

Here, $B_2, B_4, \ldots$ are Bernoulli numbers and $(a)_n$ denotes the Pochhammer symbol, *i.e.*, $(a)_n = a(a+1)\cdots(a+n-1)$. The following elegant inequality was published in 2002 by Bastien and Rogalski [4]:

$$[s\zeta(s+1,x)]^{1/s} \le [(s-1)\zeta(s,x)]^{1/(s-1)} \quad (s>1, x\ge 1).$$

This result led the authors to the

Conjecture *For all $x \ge 1$, the function*

$$s \mapsto [(s-1)\zeta(s,x)]^{1/(s-1)}$$

is decreasing on $(1,\infty)$.

It is the aim of this paper to solve two related monotonicity problems. We determine all parameters $\alpha,\beta>1$ and $a,b>0$ such that the function

$$Q(x)=Q(x;\alpha,\beta;a,b)=\frac{(\zeta(\alpha,x))^a}{(\zeta(\beta,x))^b}$$

is decreasing on $(0,\infty)$. And, we determine all parameters $\alpha,\beta>1$ and $a,b>0$ such that Q is increasing on $(0,\infty)$. It turns out that as a by-product we obtain a proof of the Bastien–Rogalski conjecture.

2 Two Lemmas

In this section we provide two technical lemmas, which are important for the proof of our main result.

Lemma 1 *Let*

$$\phi(y,u)=\frac{1-y^2}{(1-e^{-u(1+y)})(1-e^{-u(1-y)})} \quad (0<y<1, u>0). \tag{2.1}$$

The function $y\mapsto\phi(y,u)$ is strictly decreasing on $(0,1)$, whereas $y\mapsto\phi(y,u)/(1-y^2)$ is strictly increasing on $(0,1)$.

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Proof Let $y \in (0,1)$ and $u > 0$. Differentiation yields

$$\frac{\partial}{\partial y}\phi(y,u) = u\phi(y,u)[\delta(u(1+y)) - \delta(u(1-y))],$$

where $\delta(x) = (1/x) - 1/(e^x - 1)$. Since

$$\delta'(x) = -\frac{2e^{x/2}(xe^{x/2} + e^x - 1)(\sinh(x/2) - x/2)}{x^2(e^x-1)^2} < 0 \quad (x > 0),$$

we obtain $(\partial/\partial y)\phi(y,u) < 0$.

Further, we get

$$\frac{\partial}{\partial y}\frac{\phi(y,u)}{1-y^2} = \frac{2u\sinh(uy)}{e^u(1-e^{-u(1+y)})^2(1-e^{-u(1-y)})^2} > 0.$$

This completes the proof of Lemma 1. ■

Lemma 2 *Let $\alpha, \beta > 1$ and $a, b > 0$ with $\alpha a - \beta b \geq \max(a-b, 0)$. Further, let*

$$\begin{aligned} I(y) &= I(y;\alpha,\beta;a,b) \\ &= (1+y)^{\alpha-2}(1-y)^{\beta-2}[a(1+y) - b(1-y)] \\ &\quad + (1-y)^{\alpha-2}(1+y)^{\beta-2}[a(1-y) - b(1+y)]. \end{aligned} \tag{2.2}$$

If $\beta > \alpha$ and $a > b$, then there exists a number $y_0 \in (0,1)$ such that I is positive on $(0, y_0)$ and negative on $(y_0, 1)$. If $\alpha > \beta$ and $a > b$, then $I(y) > 0$ for $y \in (0,1)$. And, if $\alpha > \beta$ and $b > a$, then there exists a number $y_1 \in (0,1)$ such that I is negative on $(0, y_1)$ and positive on $(y_1, 1)$.

Proof We define for $y \in (0,1)$:

$$P(y) = P(y;\alpha,\beta;a,b) = (1-y)^{2-\alpha}(1+y)^{2-\beta}I(y)$$

and

$$R(y) = R(y;\alpha,\beta;a,b) = (1+y)^{2-\alpha}(1-y)^{2-\beta}I(y).$$

(i) $\beta > \alpha$ and $a > b$: Partial differentiation gives

$$\frac{\partial^2}{\partial y\partial a}P(y;\alpha,\beta;a,b) = 2(\alpha-\beta)\Big(\frac{1-y}{1+y}\Big)^{\beta-\alpha-1}\frac{1}{1+y} + \Big(\frac{1-y}{1+y}\Big)^{\beta-\alpha} - 1 < 0.$$

Since $\alpha a - \beta b \geq a - b$, we get $a \geq b(\beta-1)/(\alpha-1)$. This leads to

$$\begin{aligned} \frac{\partial}{\partial y}P(y;\alpha,\beta;a,b) &\leq \frac{\partial}{\partial y}P(y;\alpha,\beta;a,b)\Big|_{a=b(\beta-1)/(\alpha-1)} \\ &= \frac{b}{\alpha-1}\Big[2(\alpha-\beta)(\beta-\alpha+(\alpha+\beta-2)y)\Big(\frac{1-y}{1+y}\Big)^{\beta-\alpha-1}\frac{1}{(1+y)^2} \\ &\quad + (\alpha+\beta-2)\Big\{\Big(\frac{1-y}{1+y}\Big)^{\beta-\alpha} - 1\Big\}\Big] < 0. \end{aligned}$$

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Hence, P is strictly decreasing on $(0,1)$ with $P(0) = 2(a-b) > 0$ and $P(1) = -2b$. This implies that there exists a number $y_0 \in (0,1)$ such that P is positive on $(0, y_0)$ and negative on $(y_0, 1)$.

(ii) $\alpha > \beta$ and $a > b$: From

$$\frac{\partial^2}{\partial y \partial a} P(y;\alpha,\beta;a,b) > 0,$$

we obtain

$$\frac{\partial}{\partial y} P(y;\alpha,\beta;a,b) \geq \frac{\partial}{\partial y} P(y;\alpha,\beta;a,b)\Big|_{a=b}$$
$$= 2b\Big[2(\alpha-\beta)\Big(\frac{1-y}{1+y}\Big)^{\beta-\alpha-1} \frac{y}{(1+y)^2} + \Big(\frac{1-y}{1+y}\Big)^{\beta-\alpha} - 1\Big] > 0.$$

Thus, we have $P(y) > P(0) = 2(a-b) > 0$ for $y \in (0,1)$.

(iii) $\alpha > \beta$ and $b > a$: We get

$$R'(y) = (a+b)\Big[1 - \Big(\frac{1-y}{1+y}\Big)^{\alpha-\beta}\Big]$$
$$+ 2(\alpha-\beta)(b-a+(a+b)y)\Big(\frac{1-y}{1+y}\Big)^{\alpha-\beta-1} \frac{1}{(1+y)^2} > 0$$

and

$$R(0) = 2(a-b) < 0 < 2a = R(1).$$

This implies that there exists a number $y_1 \in (0,1)$ such that R is negative on $(0, y_1)$ and positive on $(y_1, 1)$. ■

3 Main Result

Now we can prove the following monotonicity properties of the ratio of Hurwitz zeta functions.

Theorem *Let*

$$Q(x) = Q(x;\alpha,\beta;a,b) = \frac{(\zeta(\alpha,x))^a}{(\zeta(\beta,x))^b},$$

where $\alpha, \beta > 1$ and $a, b > 0$ are real numbers.

(i) *The function Q is decreasing on $(0,\infty)$ if and only if $\alpha a - \beta b \geq \max(a-b, 0)$.*

(ii) *Q is increasing on $(0,\infty)$ if and only if $\alpha a - \beta b \leq \min(a-b, 0)$.*

Proof (i) First, we assume that Q is decreasing on $(0,\infty)$. Let

$$g(\alpha,x) = \zeta(\alpha,x) - \frac{1}{x^\alpha}.$$

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Then we have

$$Q(x) = \frac{(1+x^{\alpha}g(\alpha,x))^{a}}{(1+x^{\beta}g(\beta,x))^{b}}x^{\beta b-\alpha a} = \frac{(x^{\alpha-1}\zeta(\alpha,x))^{a}}{(x^{\beta-1}\zeta(\beta,x))^{b}}x^{(\beta-1)b-(\alpha-1)a}. \tag{3.1}$$

Since

$$\lim_{x\to 0} g(\alpha,x) = \zeta(\alpha) \quad \text{and} \quad \lim_{x\to\infty} x^{\alpha-1}\zeta(\alpha,x) = \frac{1}{\alpha-1},$$

we conclude from (3.1) that

$$\alpha a - \beta b \geq 0 \quad \text{and} \quad \alpha a - \beta b \geq a - b.$$

Next, we suppose that $\alpha a - \beta b \geq \max(a-b,0)$. If $\alpha = \beta$, then $a - b \geq 0$, so that $Q(x) = (\zeta(\alpha,x))^{a-b}$ is decreasing on $(0,\infty)$. Let $\alpha \neq \beta$. Differentiation gives for $x > 0$:

$$-\zeta(\alpha,x)\zeta(\beta,x)\frac{Q'(x)}{Q(x)} = \alpha a\zeta(\alpha+1,x)\zeta(\beta,x) - \beta b\zeta(\beta+1,x)\zeta(\alpha,x).$$

Applying the integral representation (see [3, p. 251])

$$\zeta(\alpha,x) = \frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-xt}\frac{t^{\alpha-1}}{1-e^{-t}}\,dt$$

and the convolution theorem for Laplace transforms, we obtain

$$-\zeta(\alpha,x)\zeta(\beta,x)\frac{Q'(x)}{Q(x)} = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^\infty e^{-xt}\Delta(t;\alpha,\beta;a,b)\,dt, \tag{3.2}$$

where

$$\Delta(t;\alpha,\beta;a,b) = \int_0^t \frac{(t-s)^{\alpha-1}s^{\beta-1}(a(t-s)-bs)}{(1-e^{-s})(1-e^{-(t-s)})}\,ds.$$

We set $u = t/2 > 0$. The substitution $s = u(1+y)$ leads to:

$$\begin{aligned} u^{-(\alpha+\beta)}&\Delta(2u;\alpha,\beta;a,b) \\ &= \int_{-1}^1 \frac{(1-y)^{\alpha-1}(1+y)^{\beta-1}(a(1-y)-b(1+y))}{(1-e^{-u(1+y)})(1-e^{-u(1-y)})}\,dy \\ &= \int_0^1 \phi(y,u)I(y;\alpha,\beta;a,b)\,dy, \end{aligned} \tag{3.3}$$

where ϕ and I are defined in (2.1) and (2.2), respectively. If $a = b$, then $\alpha > \beta$. This implies that I is positive on $(0,1)$, so that (3.2) and (3.3) reveal that $Q'(x) \leq 0$ for $x > 0$. Next, let $a \neq b$. We distinguish two cases.

Case 1: $\beta > \alpha$.
Then we obtain $a > b$. Applying Lemmas 1 and 2 we conclude that there exists a number $y_0 \in (0,1)$ such that

$$\phi(y,u)I(y) \geq \phi(y_0,u)I(y) \quad \text{for } y \in (0,1).$$

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This implies

$$\int_0^1 \phi(y,u)I(y)\,dy \ge \phi(y_0,u)\int_0^1 I(y)\,dy. \tag{3.4}$$

We have $\alpha a - \beta b \ge a - b$. This gives $a \ge b(\beta-1)/(\alpha-1)$. Since I is increasing with respect to a, we get

$$\begin{aligned}\int_0^1 I(y;\alpha,\beta;a,b)\,dy &\ge \int_0^1 I(y;\alpha,\beta;b(\beta-1)/(\alpha-1),b)\,dy \\ &= \frac{b}{\alpha-1}\Big[(1-y)^{\alpha-1}(1+y)^{\beta-1} \\ &\qquad - (1+y)^{\alpha-1}(1-y)^{\beta-1}\Big]_{y=0}^{y=1} = 0.\end{aligned} \tag{3.5}$$

We have $\phi(y_0,u) > 0$, so that (3.2)–(3.5) imply $Q'(x) \le 0$ for $x > 0$.

Case 2: $\alpha > \beta$.
We consider two subcases.

Case 2.1: $a > b$.
Lemma 2 yields that I is positive on $(0,1)$. Since ϕ is also positive, we obtain from (3.2) and (3.3) that $Q'(x)$ is negative for $x > 0$.

Case 2.2: $b > a$.
Applying Lemmas 1 and 2 we conclude that there exists a number $y_1 \in (0,1)$ such that

$$\phi(y,u)I(y) \ge \frac{\phi(y_1,u)}{1-y_1^2}(1-y^2)I(y) \quad \text{for } y \in (0,1). \tag{3.6}$$

Since $a \ge b\beta/\alpha$, we get

$$\begin{aligned}\int_0^1 (1-y^2)I(y;\alpha,\beta;a,b)\,dy &\ge \int_0^1 (1-y^2)I(y;\alpha,\beta;b\beta/\alpha,b)\,dy \\ &= \frac{b}{\alpha}\Big[(1-y)^{\alpha}(1+y)^{\beta} - (1+y)^{\alpha}(1-y)^{\beta}\Big]_{y=0}^{y=1} = 0.\end{aligned} \tag{3.7}$$

From (3.2), (3.3), (3.6), and (3.7) we obtain that Q' is negative on $(0,\infty)$. The proof of part (i) is complete.
(ii) Since

$$1/Q(x;\alpha,\beta;a,b) = Q(x;\beta,\alpha;b,a) \quad \text{and} \quad -\max(c,0) = \min(-c,0),$$

we conclude from part (i) that Q is increasing on $(0,\infty)$ if and only if $\alpha a - \beta b \le \min(a-b,0)$. ■

Remark Let $t > s > 1$. If we set $\alpha = s$, $\beta = t$, $a = t - 1$, $b = s - 1$, then part (i) of the Theorem yields that

$$Q^*(x) = \frac{(\zeta(s,x))^{t-1}}{(\zeta(t,x))^{s-1}}$$

is decreasing on $(0, \infty)$. Hence, we get

$$Q^*(x) \geq \lim_{y\to\infty} Q^*(y) = \frac{(t-1)^{s-1}}{(s-1)^{t-1}} \quad (x > 0),$$

which implies that

$$s \mapsto [(s-1)\zeta(s,x)]^{1/(s-1)} \quad (x > 0)$$

is decreasing on $(1, \infty)$. This settles a slightly extended version of the Conjecture posed in Section 1.

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