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RESEARCH ARTICLE

Positivity of direct images with a Poincaré type twist

Philipp Naumann

Universität Bayreuth, Mathematisches Institut, 95440 Bayreuth, Deutschland; E-mail: philipp.naumann@uni-bayreuth.de.

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Abstract

We consider a holomorphic family $f : \mathcal{X} \to S$ of compact complex manifolds and a line bundle $\mathcal{L} \to \mathcal{X}$. Given that $\mathcal{L}^{-1}$ carries a singular hermitian metric that has Poincaré type singularities along a relative snc divisor $\mathcal{D}$, the direct image $f_*(K_{\mathcal{X}/S} \otimes \mathcal{D} \otimes \mathcal{L})$ carries a smooth hermitian metric. If $\mathcal{L}$ is relatively positive, we give an explicit formula for its curvature. The result applies to families of log-canonically polarized pairs. Moreover, we show that it improves the general positivity result of Berndtsson-Păun in a special situation of a big line bundle.

Dedicated to Tristan

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1. Introduction

We consider a proper holomorphic submersion $f : \mathcal{X} \to S$ of complex manifolds and a snc divisor $\mathcal{D}$ on $\mathcal{X}$ whose restriction $D_s := \mathcal{D}|_{X_s}$ to each fibre $X_s = f^{-1}(s)$ is also simple normal crossing. Given a line bundle $\mathscr{L} \to \mathscr{X}$ that carries a hermitian metric that is smooth on $\mathscr{X}' := \mathscr{X} \setminus \mathscr{D}$ and whose inverse has Poincaré growth near the divisor $\mathscr{D}$, we study the spaces of square integrable canonical forms on the open fibres $X'_s := X_s \setminus D_s$ that have values in $L_s := \mathscr{L}|_{X'_s}$. By the work of Zucker [24] and Fujiki [8], we can identify these L^2-Dolbeault cohomology groups $H^0_{(2)}(X'_s, K_{X'_s} \otimes L_s)$ with the spaces $H^0(X_s, K_{X_s} \otimes D_s \otimes L_s)$. More globally, we will see that these spaces are the fibres of the coherent sheaf $f_*(K_{\mathscr{X}/S} \otimes \mathscr{D} \otimes \mathscr{L})$ on the base S. Under the condition that this sheaf is locally free, the natural L^2-metrics on the L^2-Dolbeault spaces induce a smooth hermitian metric on the direct image. We give an explicit curvature formula for it in the case the hermitian metric on $\mathscr{L}$ is positive along the fibres X'_s. If $\mathscr{L}$ is globally positive on $\mathscr{X}' := \mathscr{X} \setminus \mathscr{D}$, the direct image $f_*(K_{\mathscr{X}/S} \otimes \mathscr{D} \otimes \mathscr{L})$ is also positive. The result applies to families of log-canonically polarized pairs, where the Poincaré type Kähler-Einstein metrics induce such a singular hermitian metric on the relative canonical bundle. We give another application to illustrate how our result improves the general positivity theorem from [3] in a special situation.

2. Differential geometric setup and statement of results

Let $f : \mathscr{X} \to S$ be a proper holomorphic submersion of complex manifolds with connected fibres and $\mathscr{L}$ a line bundle on $\mathscr{X}$. We assume that $\mathscr{L}$ has a hermitian metric h that is smooth on the complement of a relative simple normal crossing divisor $\mathscr{D} = \sum_{i=1}^{l} \mathscr{D}_i$ on $\mathscr{X}$ with the following asymptotic behaviour along $\mathscr{D}$

$$h^{-1}|_{\mathscr{X}'} = \exp(u) \cdot \frac{h^{C^\infty}_{\mathscr{L}^{-1}}}{\prod_{i=1}^{l} ||\sigma_i||_i^2 \log^2 ||\sigma_i||_i^2}, \tag{2.1}$$

where the notation is as follows:

- $h^{C^\infty}_{\mathscr{L}^{-1}}$ is a smooth metric on $\mathscr{L}^{-1}$.
- $||\sigma_i||_i$ is the norm of the canonical section cutting out $\mathcal{D}_i$ with regard to a smooth metric s.t. $||\sigma_i||_i < 1$.
- u is a function on $\mathscr{X}'$ s.t. $u|_{X'_s} \in \mathscr{C}^{k,\alpha}$ $(k \geq 2)$ for all s, and the map $s \mapsto u|_{X'_s}$ is Fréchet differentiable.
- $\omega_s := -\mathrm{i}\partial\bar{\partial}\log(h)|_{X'_s}$ is a Poincaré type Kähler metric on each fibre X'_s.

Here we have used the Hölder space of functions $\mathscr{C}^{k,\alpha} = \mathscr{C}^{k,\alpha}(X'_{s_0})$ on an open fibre X'_{s_0} that were introduced in [6, 14, 21] and do not depend on the fibre. We refer to this by saying that the inverse metric h^{-1} has *Poincaré type singularities* along $\mathscr{D}$. We write $\mathscr{D} \overset{i}{\hookrightarrow} \mathscr{X} \overset{f}{\to} S$ for the family of smooth log pairs (X_s, D_s).

The reason for choosing this asymptotic will become clear when we consider the family of Poincaré type Kähler-Einstein metrics for a family of log canonically polarized manifolds. We remark that it includes the case where the function u is smooth on $\mathscr{X}$, and it implies that u and its derivatives in the base and fibre direction we will consider are bounded on $\mathscr{X}'$. In a local description, the norm squared of a local trivialising section $e_{\mathscr{L}}$ of $\mathscr{L}$ near a point $p \in \mathscr{D}$ is given by

$$|e_{\mathscr{L}}|_h^2(z,s) = \left(\prod_{i=1}^{r} |z^i|^2 \log^2(|z^i|^2)\right) \cdot v(z,s), \qquad v \in C^k_{\mathrm{loc}}(\mathscr{X}'),$$

where the divisor is given by $\mathscr{D} = \{z^1 \cdots z^r = 0\}$ with respect to local holomorphic coordinates (z,s) around p with $z = z^1, \ldots, z^n$ and $s = s^1, \ldots, s^m$ such that $f(z,s) = s$. Here, $n := \dim X_s$ and $m := \dim S$.

The curvature form of the hermitian line bundle $(\mathscr{L}, h)$ restricted to $\mathscr{X}' := \mathscr{X} \setminus \mathscr{D}$ is given by

$$\omega_{\mathscr{X}'} := -\sqrt{-1}\partial\bar{\partial}\log h.$$

This means we view h as a singular hermitian metric on $\mathscr{L}$ whose curvature current restricted to $\mathscr{X}'$ is given by the smooth form $\omega_{\mathscr{X}'}$. Our assumption on h guarantees that each restriction $\omega_{\mathscr{X}'}|_{X'_s}$ is

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quasi-isometric to the model metric

$$\sqrt{-1}\left(\sum_{i=1}^{k} \frac{dz^i \wedge d\overline{z}^i}{|z^i|^2 \log^2(1/|z^i|^2)} + \sum_{i=k+1}^{n} dz^i \wedge d\overline{z}^i\right)$$

near the point p.

We consider the case where the hermitian line bundle $(\mathscr{L}, h)|_{\mathcal{X}'}$ is relatively positive, which means

$$\omega_s := \omega_{\mathcal{X}'}|_{X'_s}$$

are Kähler forms on the open fibres $X'_s := X_s \setminus D_s$. This implies that $\mathscr{L} \otimes \mathscr{D}$ is relatively big and nef. Then one has the notion of the *horizontal lift* v_i of a tangent vector ∂_i on the base S (see Section 6.1 for the precise definition), and we get a representative of the Kodaira-Spencer class by

$$A_i := \bar{\partial}(v_i)|_{X'_s},$$

which is a $\mathscr{C}^{k,\alpha}$-tensor by [18, Lemma 3] and thus square integrable. Furthermore, one sets

$$\varphi_{i\overline{j}} := \langle v_i, v_j \rangle_{\omega_{\mathcal{X}'}},$$

which is called the *geodesic curvature*. We note that $(\varphi_{i\overline{j}})_{i\overline{j}}$ is positive (semi-)definite if and only if $\mathscr{L}$ is globally (semi-)positive on $\mathcal{X}'$. Again, our assumption on $u(z, s)$ guarantees that each $\varphi_{i\overline{j}}$ is a $\mathscr{C}^{k,\alpha}$ function and thus in particular bounded on each fibre.

Now we turn to the direct image sheaf we want to study. On a fibre X_s, we denote by $\Omega^n_{X_s}(\log D_s) = K_{X_s} \otimes D_s$ the locally free sheaf of germs of logarithmic n-forms with log-poles along D_s. This sheaf is the restriction of $\Omega^n(\log \mathscr{D})_{\mathcal{X}/S} = K_{\mathcal{X}/S} \otimes \mathscr{D}$, the sheaf of relative logarithmic n-forms with log-poles along $\mathscr{D}$, to the fibres X_s.

We assume that the dimension of the cohomology groups

$$H^0(X_s, \Omega^n_{X_s}(\log D_s)(L_s))$$

is constant on S, which in general only holds outside a proper subvariety. Under this assumption, we get the local freeness of the coherent sheaf

$$f_*(\Omega^n(\log \mathscr{D})_{\mathcal{X}/S}(\mathscr{L}))$$

whose fibres are canonically isomorphic to the cohomology groups $H^0(X_s, \Omega^n_{X_s}(\log D_s)(L_s))$. By the work of Zucker [24] and Fujiki [8], we can identify these groups with the L^2-Dolbeault cohomology groups $H^0_{(2)}(X'_s, K_{X'_s} \otimes L_s)$. Hence, the latter spaces also form a vector bundle on the base, which we denote by

$$f_*(K_{\mathcal{X}'/S} \otimes \mathscr{L}|_{\mathcal{X}'})_{L^2}.$$

It turns out that this is nothing but the bundle $f_*(\Omega^n(\log \mathscr{D})_{\mathcal{X}/S}(\mathscr{L}))$. Now we can represent local sections of this bundle by holomorphic sections of $\Omega^n(\log \mathscr{D})_{\mathcal{X}/S}(\mathscr{L})$ whose restrictions to the open fibres X'_s are thus L^2-integrable and holomorphic $(n, 0)$-forms with values in L_s. Let $\{\psi^1, \ldots, \psi^r\}$ be a local frame of the direct image consisting of such sections around a fixed point $s \in S$. We denote by $\{(\partial/\partial s_i) \mid i = 1, \ldots, m\}$ a basis of the complex tangent space $T_s S$ of S over $\mathbb{C}$, where s_i are local holomorphic coordinates on S. The components of the metric tensor for the L^2-metric on the direct image are then defined by

$$H^{\overline{l}k}(s) := \langle \psi^k, \psi^l \rangle := \langle \psi^k|_{X'_s}, \psi^l|_{X'_s} \rangle(s) := \int_{X'_s} \psi^k|_{X'_s} \cdot \psi^{\overline{l}}|_{X'_s}\, dV = \mathrm{i}^{n^2} \int_{X'_s} (\psi^k|_{X'_s} \wedge \psi^{\overline{l}}|_{X'_s})_h.$$

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Here we use the notation $\psi^{\bar{l}} := \overline{\psi^l}$ for the sections ψ^l and write $dV = \omega^n_{X'_s}/n!$. The pointwise inner product $\psi^k \cdot \psi^{\bar{l}}$ is the one given by ω_s and $h|_{X'_s}$. In the last equality, we used the Hodge-Riemann bilinear relation because the holomorphic $(n, 0)$-forms are primitive. Note that by working on the open fibres X'_s, the metrics involved are smooth so that we have a harmonic theory for square integrable forms lying in the domain of the Laplacian.

Let $A^{\alpha}_{i\bar{\beta}}(z, s)\partial_\alpha dz^{\bar{\beta}} = \bar{\partial}(v_i)|_{X'_s}$ be the $\bar{\partial}$-closed representative of the Kodaira-Spencer class of ∂_i described above. We know that A_i lies in the space of smooth and L^2-integrable $(0, 1)$-forms $A^{0,1}_{(2)}(X'_s, T_{X'_s})$. Hence these, together with contraction, define a map

$$A^{\alpha}_{i\bar{\beta}}\partial_\alpha dz^{\bar{\beta}} \cup \; : H^0(X_s, \Omega^n_{X_s}(\log D_s)(L_s)) \to A^{0,1}_{(2)}(X'_s, \Omega^{n-1}_{X'_s}(L_s)).$$

When applying the Laplace operator to (p, q)-forms with values in $L|_{X'_s}$ on the fibres X'_s, we have

$$\Box_\partial - \Box_{\bar{\partial}} = (n - p - q) \cdot \mathrm{id}$$

due to the definition $\omega_{X'_s} = \omega_{\mathcal{X}'}|_{X'_s}$ and the Bochner-Kodaira-Nakano identity. Thus, we write $\Box = \Box_\partial = \Box_{\bar{\partial}}$ in the case $q = n - p$. The main result is

Theorem 2.1. *Let $\mathcal{D} \xhookrightarrow{i} \mathcal{X} \xrightarrow{f} S$ be a family of smooth log pairs and $(\mathcal{L}, h) \to \mathcal{X}$ a hermitian line bundle as described above. With the objects just described, the L^2-metric on $f_*(\Omega^n(\log \mathcal{D})_{\mathcal{X}/S}(\mathcal{L}))$ is smooth, and its curvature is given by*

$$\begin{aligned} R^{\bar{l}k}_{i\bar{j}}(s) = &\int_{X'_s} \varphi_{i\bar{j}} \cdot (\psi^k \cdot \psi^{\bar{l}})\, dV \\ &+ \int_{X'_s} (\Box + 1)^{-1}(A_i \cup \psi^k) \cdot (A_{\bar{j}} \cup \psi^{\bar{l}})\, dV. \end{aligned}$$

In particular, $f_*(K_{\mathcal{X}/S} \otimes \mathcal{D} \otimes \mathcal{L})$ is Nakano (semi-)positive if $\mathcal{L}$ is (semi-)positive on $\mathcal{X}'$ and positive along the fibres X'_s.

We remark that if $\mathcal{L}$ is (semi-)positive, the direct image $f_*(K_{\mathcal{X}/S} \otimes \mathcal{D} \otimes \mathcal{L})$ is locally free by the Ohsawa-Takegoshi extension theorem. The result applies to families of log-canonically polarized pairs where the relative canonical bundle $K_{\mathcal{X}'/S}$ plays the role of $\mathcal{L}$. Here the hermitian metric is induced from the fibrewise Poincaré type Kähler-Einstein metrics. In this case, we first prove

Theorem 2.2 (= Theorem 4.1). *Let $\mathcal{D} \xhookrightarrow{i} \mathcal{X} \xrightarrow{f} S$ be a family of smooth log-canonically polarized pairs. Then the curvature of the hermitian metric on $K_{\mathcal{X}'/S}$ that is induced by the Poincaré type Kähler-Einstein metrics on the fibres is semipositive. If the family is effectively parametrised, then $K_{\mathcal{X}'/S}$ is strictly positive.*

This answers a question raised in [11, Remark 7.1]:

Corollary 1 (= Corollary 5). *For a family of smooth log-canonically polarized pairs $\mathcal{D} \xhookrightarrow{i} \mathcal{X} \xrightarrow{f} S$, the relative log-canonical bundle $K_{\mathcal{X}/S} \otimes \mathcal{D}$ equipped with the metric induced from the fibrewise Kähler-Einstein metrics is nef. If the family is effectively parametrised, $K_{\mathcal{X}/S} \otimes \mathcal{D}$ is big. Here, S is assumed to be compact.*

By combining both theorems, we get

Corollary 2. *For a family of log-canonically polarized pairs $\mathcal{D} \xhookrightarrow{i} \mathcal{X} \xrightarrow{f} S$, the direct image sheaf $f_*((K_{\mathcal{X}/S} \otimes \mathcal{D}) \otimes K_{\mathcal{X}/S})$ is semipositive in the sense of Nakano. If the family of log pairs is effectively parametrised, this direct image is Nakano positive.*

To implement the method of computation given in [19, 17], we have to pass from the compact fibres X_s to the open part X'_s, where the metrics in consideration are smooth. This requires imposing the L^2

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condition on the spaces of forms on X_s'. To show that all steps in the computation are still justified, we have to check integrability. This is possible due to knowledge of the asymptotic behaviour of our sections. Hölder spaces are another important tool with respect to quasi-coordinates. We give the details below.

A proof of the more general curvature formula, including the higher direct images, was claimed in [22]. However, the arguments in this work seem to be incomplete. One major issue is the smooth dependency of the fibrewise harmonic projections *without* the usual assumption of having constant dimension for the space of fibrewise harmonic forms; compare to [22, Lemma 2]. This is impossible given that linear independency is an open property; compare to the argument in the proof of [15, Lemma 7.7]. Moreover, it is unclear how to derive the square integrability of some Lie-differentiated forms from the arguments given in [22, pp. 2958f], which is another key ingredient in the computation. In the present independent work, we give the detailed arguments for the case of the zeroth direct image. The key is that one has a precise asymptotic for square integrable holomorphic sections due to the Laurent series expansion that is lacking for the case of general harmonic forms.

3. Preparations

3.1. L^2-integrable forms

We start with a fibrewise consideration. Let $(X, D = \sum_{i=1}^{l} D_i)$ be a smooth log pair, and set $X' = X\backslash D$. We consider a holomorphic line bundle L on X together with a metric h that is smooth on X' and whose inverse has the asymptotic behaviour from equation (2.1)

$$h^{-1}|_{X'} = \exp(u) \cdot \frac{h^{C^\infty}_{L^{-1}}}{\prod_{i=1}^{l} ||\sigma_i||_i^2 \log^2 ||\sigma_i||_i^2},$$

where the notation is as follows:

- $h^{C^\infty}_{L^{-1}}$ is a smooth metric on $\mathscr{L}^{-1}$.
- $||\sigma_i||_i$ is the norm of the canonical section cutting out D_i with regard to a smooth metric s.t. $||\sigma_i||_i < 1$.
- u is a function in $\mathscr{C}^{k,\alpha}(X')$.
- $\omega_{X'} := -\mathrm{i}\partial\bar\partial \log(h)|_{X'}$ is a Poincaré type Kähler metric.

Then we can identify the holomorphic and locally L^2-integrable $(n,0)$-forms with values in L:

Proposition 1. *We denote by $\mathscr{O}_{(2)}(\Omega^n_{X'}(L|_{X'}), h|_{X'})$ the sheaf of holomorphic L-valued n-forms on X', which are locally L^2 on X with respect to $h|_{X'}$. Then*

$$\mathscr{O}_{(2)}(\Omega^n_{X'}(L|_{X'}), h|_{X'}) = \mathscr{O}(\Omega^n_X(\log D)(L)).$$

The *proof* follows immediately from a Laurent series argument together with the estimates of Poincaré type metrics: sections that are locally square integrable with respect to a metric with a Poincaré type metric extend holomorphically as forms with logarithmic poles to the given snc divisor D (and vice versa). □

Let $\mathscr{A}^{n,q}_{(2)}(L|_{X'})$ denote the sheaf on X of L-valued (n,q)-forms that are locally L^2 integrable with respect to $\omega_{X'}$ and $h|_{X'}$ and whose $\bar\partial$-exterior derivatives, taken in the current sense, are also locally L^2. We refer to [25] or [5] for more details on the L^2-complex of sheaves.

Proposition 2. *The complex $(\mathscr{A}^{n,\bullet}_{(2)}(L|_{X'}), \bar\partial)$ of sheaves on X is a fine resolution of $\mathscr{O}_{(2)}(\Omega^n_{X'}(L|_{X'}), h|_{X'})$. Thus the L^2-Dolbeault cohomology group $H^0_{(2)}(X', \Omega^n_{X'}(L|_{X'}))$ can be identified with $H^0(X, \Omega^n(\log D)(L))$, which is of finite dimension.*

Proof. We decompose the vector bundle locally as a sum of line bundles and apply [8, Prop. 2.1]; also compare to [8, p.870]. This shows that we have a resolution. The fact that we get a *fine* resolution is not automatic (compare to [25, p.175]) because we need cut-off functions with bounded differential.

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But this holds in the context of Poincaré geometry; see [24], where it was successfully employed in the one-dimensional case. □

3.2. Quasi-coordinates and Hölder spaces

We recall from [6] that a *quasi-coordinate map* is a holomorphic map from an open set $V \subset \mathbb{C}$ into X' if it is of maximal rank everywhere in V. In this case, V together with the Euclidean coordinates of $\mathbb{C}^n$ is called a *local quasi-coordinate* of X'. According to [6, 14, 21], we have the following:

Proposition 3. *There exists a family* $\mathscr{V} = \{(V; v^1, \ldots, v_n)\}$ *of local quasi-coordinates of* X' *with the following properties:*

(i) X' *is covered by the images of the quasi-coordinates in* $\mathscr{V}$.
(ii) *The complement of some open neighbourhood of the divisor* D *in* X *is covered by the images of finitely many of the quasi-coordinates in* $\mathscr{V}$, *which are local coordinates in the usual sense.*
(iii) *For each* $(V; v^1, \ldots, v^n) \in \mathscr{V}$, $v \subset \mathbb{C}^n$ *contains an open ball of radius* $\frac{1}{2}$.
(iv) *There are constants* $c > 0$ *and* $A_k > 0, k + 0, 1, \ldots,$ *such that for every* $(V; v^1, \ldots, v^n) \in \mathscr{V}$, *the following inequalities hold:*
   - *We have*
   $$\frac{1}{c}(\delta_{i\bar{j}}) < (g_{i\bar{j}}) < c(\delta_{i\bar{j}})$$
   as matrices in the sense of positive definiteness.
   - *For any multi-indices* $I = (i_1, \ldots, i_p)$ *and* $J = (j_1, \ldots, j_q)$ *of order* $|I| = i_1 + \ldots + i_p$, *respectively* $|J| = j_1 + \ldots + j_q$, *we have*
   $$\left| \frac{\partial^{|I|+|J|} g_{i\bar{j}}}{\partial v^I \partial \bar{v}^J} \right| < A_{|I|+|J|},$$
   where $\partial v^I = (\partial v^1)^{i_1} \cdots (\partial v^p)^{i_p}$ *and* $\partial \bar{v}^J = (\partial \bar{v}^1)^{j_1} \cdots (\partial \bar{v}^q)^{j_q}$.

A complete Kähler manifold (X', g), which admits a family $\mathscr{V}$ of local quasi-coordinates satisfying the conditions of the proposition, is called being of *bounded geometry* (of order ∞).

Although the coordinate system from Proposition 3 is not a coordinate system in the ordinary sense because of the covering map involved, it makes sense to talk about the components of a tensor field on X' (or a neighbourhood $U(p) = (\Delta^*)^k \times \Delta^{n-k}$) with respect to these 'coordinates' v^i by first lifting it to a tensor field Δ^n. The behaviour of the tensor on $U(p)$ can thus be examined by looking at the lifted function in a neighbourhood of $(1, \ldots, 1, *, \ldots, *)$ in Δ.

We mention here the transition of tensors from a local coordinate function z^i, where a component D_i is given by $\{z^i = 0\}$ to a quasi-coordinate v^i, $|v^i| < R < 1$, which is given by

$$\frac{dz^i}{z^i \log |z^i|^2} = \frac{|v^i - 1|^2}{(v^i - 1)^2} \frac{dv^i}{1 - |v^i|^2} \sim dv^i, \tag{3.1}$$

because v is bounded away from 1. The respective equation for $\partial/\partial z^1$ reads

$$z^i \log |z^i|^2 \frac{\partial}{\partial z^i} \sim \frac{\partial}{\partial v^i}. \tag{3.2}$$

This transformation rule is used to derive estimates from Hölder regular functions and tensors that we will now define. The Hölder spaces $\mathscr{C}^{k,\alpha}(X')$ are defined in terms of quasi-coordinates for sufficiently large values of k and $0 < \alpha < 1$. The Hölder norms are computed in terms of the infinite number of quasi-coordinate systems. Following [6, 14, 21], we define

Definition 1. Let $k \in \mathbb{N}_0$ and $\alpha \in (0, 1)$, and denote by $C^k(X')$ the space of k-times differentiable functions $u : X' \to \mathbb{C}$. For $u \in C^k(X')$, let

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$$\|u\|_{k,\alpha} = \sup_{(V,v^1,\dots,v^n)\in\mathscr{V}} \left(\sup_{z\in V} \sum_{|I|+|J|\le k} |\partial_v^J \bar{\partial}_v^J u(z)| + \sup_{z,z'\in V} \sum_{|I|+|J|=k} \frac{|\partial_v^I \bar{\partial}_v^J u(z) - \partial_v^I \bar{\partial}_v^J u(z')|}{|z-z'|^\alpha} \right)$$

be the $\mathscr{C}^{k,\alpha}$-norm of u, where $\partial_v^I \bar{\partial}_v^J = \frac{\partial^{|I|+|J|}}{\partial v^I \partial \bar{v}^J}$. Then let

$$\mathscr{C}^{k,\alpha} := \mathscr{C}^{k,\alpha}(X') := \{u \in C^k(X') \ : \ \|u\|_{k,\alpha} < \infty\}$$

be the function space of $\mathscr{C}^{k,\alpha}$ functions on X' with respect to $\mathscr{V}$.

In a similar way, we can define $\mathscr{C}^{k,\alpha}$-tensors by pulling back via the quasi-coordinate maps. Exterior derivatives and covariant derivatives of $\mathscr{C}^{k,\alpha}$-tensors are of the same type (with k being replaced by $k-1$). The arguments from [14] using Hölder spaces $\mathscr{C}^{k,\alpha}$ (with respect to quasi-coordinates) are valid for all large fixed numbers $k \ge k_0$, where k_0 denotes some minimal degree. During the computations, we will have to take derivatives, products and contractions of such tensors, arriving at $\mathscr{C}^{k,\alpha}$-tensors for some lower value of k. In each of these steps, we will have to increase the lower bound k_0. Only finitely many such steps will be necessary. We will increase k_0 tacitly.

Let $\omega_{X'}$ be the complete Poincaré type Kähler form from the previous section.

Lemma 3.1. *Any $\mathscr{C}^{k,\alpha}$-tensor on X' is globally square-integrable.*

The *proof* follows immediately from the given uniform bounds of such tensors in terms of the above quasi-coordinate systems and the metric $\omega_{X'}$ with respect to the transition equations of the type in equation (3.1): take the pointwise norm of such a tensor with respect to the given metric. We consider the resulting function a bounded $\mathscr{C}^{k,\alpha}$-tensor (for some value of k), which means it is uniformly bounded in terms of quasi-coordinate systems (or any other coordinate systems). Finally, we use the boundedness of the volume of X'. □

3.3. *Hodge theory on the open fibres*

In this section, we summarise the Hodge theory on the complete, noncompact Kähler manifold $(X', \omega_{X'})$, for which we refer to the book of Marinescu and Ma [16, Chapter 3]. We consider the holomorphic vector bundle $E = K_{X'} \otimes L|_{X'}$ equipped with the hermitian metric $h|_{X'}$.

For the operator $\bar{\partial}$ defined on the smooth, compactly supported forms $A_0^{0,q}(X', E) \subset L^{0,q}_{(2)}(X', E)$, we consider its formal adjoint $\bar{\partial}^*$. For $s_1 \in L^{0,q}_{(2)}(X', E)$, we can calculate $\bar{\partial} s_1$ in the sense of currents: $\bar{\partial} s_1$ is the current defined by

$$\langle \bar{\partial} s_1, s_2 \rangle = \langle s_1, \bar{\partial}^* s_2 \rangle \qquad \text{for } s_2 \in A_0^{(0,q+1)}(X', E).$$

Then we have

Lemma 3.2 [16, Lemma 3.1.1]. *The operator $\bar{\partial}_{\max}$ defined by*

$$\mathrm{Dom}(\bar{\partial}_{\max}) = \{s \in L^{0,q}_{(2)}(X', E) \ : \ \bar{\partial} s \in L^{0,q+1}_{(2)}(X', E)\}$$
$$\bar{\partial}_{\max} s = \bar{\partial} s \qquad \textit{for } s \in \mathrm{Dom}(\bar{\partial}_{\max})$$

is a densely defined, closed extension called the maximal extension of $\bar{\partial}$.

In the sequel, we work with the maximal extension and simply write $\bar{\partial} = \bar{\partial}_{\max}$. The Hilbert space adjoint of $\bar{\partial}_{\max}$ is denoted by $\bar{\partial}^*_H$. We note that $\bar{\partial}^*_H \subset (\bar{\partial}^*)_H$ and $\bar{\partial}_{\max} = (\bar{\partial}^*_H)^*_H$. From now on, we work with $\bar{\partial}^*_H$ and just write $\bar{\partial}^*$.

The Laplacian $\Box = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ is a densely defined, positive operator, so one can consider its Friedrichs extension. But in the context of L^2 cohomology, it is useful to consider another extension:

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Proposition 4 [16, Proposition 3.1.2]. *The operator defined by*

$$\begin{aligned}&\mathrm{Dom}(\Box) = \{s \in \mathrm{Dom}(\bar\partial) \cap \mathrm{Dom}(\bar\partial^*) \ : \ \bar\partial s \in \mathrm{Dom}(\bar\partial^*),\ \bar\partial^* s \in \mathrm{Dom}(\bar\partial)\},\\ &\Box s = \bar\partial^*\bar\partial s + \bar\partial\bar\partial^* s \quad \textit{for } s \in \mathrm{Dom}(\Box)\end{aligned}$$

is a positive self-adjoint extension of the Laplacian, called the Gaffney extension.

We note that this result relies on Gaffney's generalisation of Stokes' theorem that we will use tacitly during our computation:

Proposition 5 [9]. *Let (M, g) be an orientable complete Riemannian manifold of real dimension $2n$ whose Riemann tensor is of class C^2. Let γ be a $(2n-1)$-form on M of class C^1 such that both γ and $d\gamma$ are in L^1. Then*

$$\int_M d\gamma = 0.$$

We define the space of harmonic forms $\mathscr{H}^{0,q}(X', E) \subset L^{0,q}_{(2)}(X', E)$ by

$$\mathscr{H}^{0,q}(X', E) := \mathrm{Ker}(\Box) = \{s \in \mathrm{Dom}(\Box) \ : \ \Box s = 0\}.$$

We see that

$$\mathscr{H}^{0,q}(X', E) = \mathrm{Ker}(\bar\partial) \cap \mathrm{Ker}(\bar\partial^*).$$

The qth L^2 Dolbeault cohomology is defined by

$$H^{0,q}_{(2)}(X', E) := \mathrm{Ker}(\bar\partial) \cap L^{0,q}_{(2)}(X', E) \Big/ \mathrm{Im}(\bar\partial) \cap L^{0,q}_{(2)}(X', E),$$

which is of course the same as the group $H^q_{(2)}(X', K^p_{X'}(L|_{X'}))$ from Proposition 2, because the L^2-cohomology can be also computed by smooth forms. From this, we also get

Proposition 6. *The L^2-Dolbeault cohomology $H^{0,q}_{(2)}(X', E)$ is finite-dimensional, and we have*

$$H^{0,q}_{(2)}(X', E) \cong \mathscr{H}^{0,q}(X', E).$$

Moreover, the images of $\bar\partial$ and $\bar\partial^*$ are closed in $L_{(2)}(X', E)$, and thus there exists a constant $C > 0$ such that

$$||s||^2_{L^2} \le C\left(||\bar\partial s||^2_{L^2} + ||\bar\partial^* s||^2_{L^2}\right) \tag{3.3}$$

for all $s \in \mathrm{Dom}(\bar\partial) \cap \mathrm{Dom}\,\bar\partial^* \cap L^{0,q}_{(2)}(X', E)$, $s \perp \mathscr{H}^{0,q}(X', E)$.

Proof. By Proposition 2, we know that we can identify the L^2-Dolbeault cohomology with the sheaf cohomology of the locally free sheaf $\Omega^p(\log D) \otimes K_X$ on the compactification X; hence it must be finite-dimensional. This immediately implies that the images of $\bar\partial$ (and hence also of $\bar\partial^*$) are closed ([5, Proposition 3.5]) and the isomorphism with the space of harmonic forms ([5, Corollary 3.6]). The stated estimate for the L^2-norm for a section s that is orthogonal to the space of harmonic forms follows from the closedness of $\mathrm{Im}(\bar\partial)$ and $\mathrm{Im}(\bar\partial^*)$ ([16, Prop. 3.1.6]). □

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Corollary 3. *We have the strong Hodge decomposition, which is orthogonal:*

$$L^{0,q}_{(2)}(X',E) = \mathscr{H}^{0,q}(X',E) \oplus \operatorname{Im}(\square) = \mathscr{H}^{0,q}(X',E) \oplus \operatorname{Im}(\bar\partial\bar\partial^*) \oplus \operatorname{Im}(\bar\partial^*\bar\partial),$$
$$\operatorname{Ker}(\bar\partial) \cap L^{0,q}_{(2)}(X',E) = \mathscr{H}^{0,q}(X',E) \oplus \left(\operatorname{Im}(\bar\partial) \cap L^{0,q}_{(2)}(X',E)\right).$$

Moreover, there exists a bounded operator G on $L^{0,q}_{(2)}(X',E)$, called the Green operator, such that

$$\square G = G\square = \operatorname{Id} - H, \quad HG = GH = 0,$$

where H is the orthogonal projection form $L^{0,q}_{(2)}(X',E)$ onto $\mathscr{H}^{0,q}(X',E)$.

Proof. See [16, Theorem 3.1.8]. □

Remark 3.3. By the usual elliptic regularity theory, we get that harmonic sections are in fact smooth. G maps smooth forms to smooth forms so that we can cut down the Hodge decomposition to the space of smooth L^2 sections:

$$A^{0,q}_{(2)}(X',E) = \mathscr{H}^{0,q}(X',E) \oplus \operatorname{Im}(\square).$$

3.4. *Families of logarithmic pairs*

Let $(\mathcal{X},\mathscr{D})$ be a smooth log pair: that is, $\mathcal{X}$ a complex manifold and $\mathscr{D} \subset \mathcal{X}$ a reduced snc divisor. The boundary divisor $\mathscr{D}$ is written as a sum of its irreducible components $\mathscr{D} = \mathcal{D}_1 + \ldots + \mathscr{D}_k$. If $I \subset \{1,\ldots,k\}$ is any non-empty subset, we consider the intersection $\mathscr{D}_I := \cap_{i\in I}\mathscr{D}_i$.

Definition 2 [13, Def. 3.4]. For a smooth log pair $(\mathcal{X},\mathscr{D})$ and a proper holomorphic submersion $f : \mathcal{X} \to S$, we say that $\mathscr{D}$ is relatively snc or that f is a snc morphism if for any set I with $D_I \neq \emptyset$, all the restricted morphisms $f|_{\mathscr{D}_I} \to S$ are also smooth of relative dimension $\dim \mathcal{X} - \dim S - |I|$.

If $s \in S$ is any point, set $X_s = f^{-1}(s)$ and $D_s := \mathscr{D} \cap X_s$. Then X_s is smooth and (X_s, D_s) is a snc pair. Moreover, the number of irreducible components of D_s is also k, which is the number of irreducible components of $\mathscr{D}$. This excludes phenomena like the deformation of the smooth hyperbola $\{wz = s \neq 0\}$ into the snc divisor $\{zw = 0\}$ that has two components. The reason for this will become clear later. Moreover, the definition implies

Lemma 3.4. *For a smooth log pair $(\mathcal{X},\mathscr{D})$ and a smooth snc morphism $f : \mathcal{X} \to S$, we can find (after shrinking S to a contractible neighbourhood) a differentiable trivialisation $\Phi : \mathcal{X} \xrightarrow{\sim} X \times S$ such that the restriction $\Phi|_{\mathscr{D}} : \mathscr{D} \xrightarrow{\sim} D \times S$ is also a smooth trivialisation.*

The lemma says that we can trivialise the pair (X_s, D_s) in a differentiable way. By restriction to $\mathcal{X}' := \mathcal{X} \setminus \mathscr{D}$, we find a smooth trivialisation of $\mathcal{X}' \cong X' \times S$.

Given a smooth log pair (X, D) and a point x, there exists an open neighbourhood $U = U(x)$ and holomorphic coordinates $z_1,\ldots,z_n$ such that $D \cap U = \{z_1 \cdots z_r\}$ for some $0 \leq r \leq n$. The following is the relative analogue of this fact:

Lemma 3.5 [13, Lemma 3.7]. *Let $(\mathcal{X},\mathscr{D})$ be a smooth snc pair and $f : \mathcal{X} \to S$ an snc morphism. For any point $x \in \mathcal{X}$, there exist open neighbourhoods $V = V(f(x)) \subset S$ and $U = U(x) \subset f^{-1}(V) \subset \mathcal{X}$ and holomorphic coordinates $z_1,\ldots,z_n,z_{n+1},\ldots,z_{n+m}$ around x and $s_1,\ldots,s_m$ around $f(x)$ and a number $0 \leq r \leq n$ such that the following hold:*

(i) *We have $z_{n+i} = s_i \circ f$ for all indices $1 \leq i \leq m$.*
(ii) $\mathscr{D} \cap \mathcal{X} = \{z_1 \cdots z_r = 0\}$.

In the following, we will make use of these *adapted* coordinates tacitly. In the applications, we also consider the following families:

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Definition 3. A holomorphic family of log-canonically polarised pairs consists of a smooth snc pair $(\mathscr{X}, \mathscr{D})$ and a snc morphism $f : \mathscr{X} \to S$ such that $K_{X_s} + D_s$ is ample for each $s \in S$.

In particular, our families are *logarithmic deformations* in the sense of [12] and generalise the deformations considered in [18] to the case of singular snc divisors.

Remark 3.6. By the C^∞ trivialisation $\mathscr{X}' \cong X' \times S$ and the fact that nearby fibres (X_s, g_s) and bundles $E_s' = K_{X_s'} \otimes L|_{X_s'}$ are quasi-isometric implies that the L^2-spaces $L^{0,q}_{(2)}(X_s', E_s')$ and the domain of the Laplacians $\mathrm{Dom}(\square_s)$ do not depend on the fibre X_s'. The same holds for the Hölder spaces $\mathscr{C}^{k,\alpha}(X_s')$.

3.5. L^2-integrable Kodaira-Spencer forms

We consider a holomorphic family of smooth log pairs $\mathscr{D} \overset{i}{\hookrightarrow} \mathscr{X} \overset{f}{\to} S$ and a hermitian holomorphic line bundle $(\mathscr{L}, h)$ on $\mathscr{X}$ whose inverse metric has Poincaré type singularities along $\mathscr{D}$, as already described in the introduction. Moreover, we recall that we have defined on $\mathscr{X}'$ the global $(1,1)$-form

$$\omega_{\mathscr{X}'} = -\mathrm{i}\partial\overline{\partial}(\log h) = \mathrm{i}\partial\bar{\partial}\log(h^{C^\infty}_{\mathscr{L}^{-1}}) - \sum_{i=1}^{l} \mathrm{i}\partial\bar{\partial}\log(||\sigma_i||^2 \log^2 ||\sigma_i||^2) + \mathrm{i}\partial\bar{\partial}u,$$

whose restrictions to the open fibres X_s' give a smooth family of Poincaré type Kähler forms $\omega_s = \omega_{\mathscr{X}'}|_{X_s'}$.

By using the local description

$$\omega_{\mathscr{X}'} = \sqrt{-1}\Big(g_{\alpha\overline{\beta}}dz^\alpha \wedge dz^{\overline{\beta}} + g_{i\overline{\beta}}ds^i \wedge dz^{\overline{\beta}} + g_{\alpha\overline{\jmath}}dz^\alpha \wedge ds^{\overline{\jmath}} + g_{i\overline{\jmath}}ds^i \wedge ds^{\overline{\jmath}}\Big)$$

with respect to the local coordinates (z, s), we can say that the restrictions of $\omega_{\mathscr{X}'}$ as well as restrictions of contractions depend in a C^∞ way upon the parameter.

Lemma 3.7. *The restrictions*

$$\omega_{\mathscr{X}'}\llcorner(\partial/\partial s^i)|X_s' = g_{i\overline{\beta}}dz^{\overline{\beta}}|X_s'$$
$$\omega_{\mathscr{X}'}\llcorner(\partial/\partial s^{\overline{\jmath}})|X_s' = g_{\alpha\overline{\jmath}}dz^{\alpha}|X_s'$$
$$\omega_{\mathscr{X}'}\llcorner(\partial/\partial s^i \wedge ds^{\overline{\jmath}})|X_s' = g_{i\overline{\jmath}}|X_s'$$

are $\mathscr{C}^{k,\alpha}$-tensors that depend in a C^∞ way upon the parameter $s \in S$. In particular, they are smooth and L^2 integrable tensors ($\mathscr{A}_{(2)}$-tensors for short). The analogous statement holds if $\omega_{\mathscr{X}'}$ is replaced by $\widetilde{\omega}_{\mathscr{X}'}$, where the first two tensors do not change.

Proof. The only critical term appearing in the expression of $\omega_{\mathscr{X}'}$ is the one coming from u. The purely vertical components are dealt with the fact that $u|_{X_s'}$ lies in $\in \mathscr{C}^{k,\alpha}$. But now, by assumption, the map $s \mapsto u(z,s) \in \mathscr{C}^{k,\alpha}$ is Fréchet differentiable, so that the (locally defined) partial derivatives in the base direction of the function u are again of class $\mathscr{C}^{k,\alpha}$. □

Horizontal lifts of tangent vectors from S to the total space $\mathscr{X}'$ are defined as perpendicular to the fibres of $\mathscr{X}' \to S$ with respect to $\omega_{\mathscr{X}'}$. Here we only need the property that the restrictions of $\omega_{\mathscr{X}'}$ to the fibres are positive definite.

We denote by $(g^{\overline{\beta}\alpha})$ the inverses of the metric tensors $g_{\alpha\overline{\beta}}$ of ω_s on the fibres X_s'. Covariant derivatives are always taken with respect to these metrics. We will use both the semicolon notation and the ∇-notation. We continue using Greek indices for tensors in fibre direction and $\partial_\alpha = \partial/\partial z^\alpha$.

The horizontal lift v_i of a tangent vector $\partial_i = \partial/\partial s^i$ is a differentiable lift of ∂_i to $\mathscr{X}'$, which is orthogonal to the fibres with respect to the sesquilinear form $\omega_{\mathscr{X}'}$:

$$\langle v_i, \partial_\alpha \rangle_{\omega_{\mathscr{X}'}} = 0 \quad \text{for all } \alpha = 1 \dots n.$$

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This is well defined since the form $\omega_{\mathcal{X}'}$ is positive when restricted to the fibres. In terms of the coefficients of $\omega_{\mathcal{X}}$, it is given by

$$v_i = \partial_i + a_i^\alpha \partial_\alpha,$$

where

$$a_i^\alpha = -g^{\overline{\beta}\alpha} g_{i\overline{\beta}}.$$

Lemma 3.8. *Let $A_i = \overline{\partial} v_i | X_s'$: that is,*

$$A_i = A^\alpha_{i\overline{\beta}} \partial_\alpha dz^{\overline{\beta}}.$$

Then A_i is of class $\mathscr{C}^{k,\alpha}$ and satisfies $\overline{\partial} A_i = 0$. In particular, it lies in $A^{0,1}_{(2)}(X_s', T_{X_s'})$.

The proof is the same as in [18, Lemma 3]. □

The Kodaira-Spencer map for a family $\mathscr{D} \overset{i}{\hookrightarrow} \mathcal{X} \overset{f}{\to} S$

$$\rho_s : T_s S \to H^1_{(2)}(X_s', T_{X_s'})$$

was already defined in [18]. Analogous results like those of Section 3.1 hold for the sheaf of holomorphic vector fields:

The sequence $(\mathscr{A}^{0,\bullet}_{(2)}(\mathcal{T}_{X_s'}), \overline{\partial})$ is a fine resolution of the sheaf of L^2 holomorphic vector fields $\mathcal{O}_{(2)}(T_{X_s'})$, which is isomorphic to $T_{X_s}(-\log D_s) := (\Omega^1_{X_s}(\log D_s))^\vee$.

In particular,

$$H^1_{(2)}(X_s', T_{X_s'}) \cong H^1(X_s, T_{X_s}(-\log D_s)),$$

and the above explicit construction yields a L^2-Dolbeault representative $A^\alpha_{i\overline{\beta}} \partial_\alpha dz^{\overline{\beta}}$ of $\rho_s(\partial/\partial s^i)$, where the Kodaira-Spencer map is taken as map

$$\rho_s : T_s S \to H^1_{(2)}(X_s', T_{X_s'}).$$

We will also need the following fact that follows from the definition.

Lemma 3.9. *Let $A_i^{\alpha\sigma} = g^{\overline{\beta}\sigma} A^\alpha_{i\overline{\beta}}$. Then*

$$A_i^{\alpha\sigma} = A_i^{\sigma\alpha}.$$

Finally, the pointwise inner product with respect to $\omega_{\mathcal{X}'}$ of two horizontal lifts

$$\varphi_{i\overline{j}} := \langle v_i, v_j \rangle_{\omega_{\mathcal{X}'}},$$

called the geodesic curvature, has an expression

$$\varphi_{i\overline{j}} = g_{i\overline{j}} - g_{\alpha\overline{j}} g_{i\overline{\beta}} g^{\alpha\overline{\beta}}$$

so that we conclude from Lemma 3.7 that this function lies in $\mathscr{C}^{k,\alpha}$.

3.6. Bundles of L^2-integrable forms

Now we consider the L^2 condition on the total space $\mathcal{X}$. Because the form $\omega_{\mathcal{X}'}$ is fibrewise positive, for any $s_0 \in S$ after replacing S by a neighbourhood (contractible and Stein), there exists a positive hermitian form ω_S such that $\widetilde{\omega}'_{\mathcal{X}} = \omega_{\mathcal{X}'} + f^*\omega_S$ is a positive form on $\mathcal{X}'$.

The previous arguments imply that the statements of Proposition 1 and Proposition 2 hold for the total spaces $(\mathscr{X}', \widetilde{\omega}_{\mathscr{X}'})$. The sheaves of locally L^2 sections $\mathscr{O}_{(2)}(\Omega^n_{\mathscr{X}'}(\mathscr{L}|_{\mathscr{X}'}), \widetilde{\omega}_{\mathscr{X}'}, h|_{\mathscr{X}'})$ on the whole total space $\mathscr{X}$ are equal to $\Omega^n_{\mathscr{X}}(\log \mathscr{D})(\mathscr{L})$, and an analogous fine resolution in terms of square-integrable forms exists. However, the sheaves $\Omega^n_{\mathscr{X}}(\log \mathcal{D})$ of log n-forms on the total space $\mathscr{X}$ are not suitable for our methods.

Instead, we will need the coherent sheaf $\Omega^n(\log \mathcal{D})_{\mathscr{X}/S}(\mathscr{L})$ and the sheaves

$$\mathscr{O}_{(2)}(\Omega^n_{\mathscr{X}'/S}(\mathscr{L}|_{\mathscr{X}'}), \widetilde{\omega}_{\mathscr{X}'}, h|_{\mathscr{X}'})$$

of relative $\mathscr{L}$-valued holomorphic n-forms that are square-integrable with respect to the Kähler form $\widetilde{\omega}_{\mathscr{X}'} = \omega_{\mathscr{X}'} + f^*\omega_S$ and the hermitian metric $h|_{\mathscr{X}'}$, where integrability does not depend upon the choice of a hermitian form ω_S.

Let

$$\mathscr{A}^{0,q}_{(2)}(\Omega^n_{\mathscr{X}'/S}(\mathscr{L}), \widetilde{\omega}_{\mathscr{X}'}, h|_{\mathscr{X}'})$$

denote the sheaf of $(0, q)$*-currents on the total space* $\mathscr{X}'$*with values in the coherent sheaf* $\Omega^n_{\mathscr{X}'/S}(\mathscr{L})$ *that are square-integrable along with their exterior* $\overline{\partial}$*-derivatives.*

Proposition 7.

(i)

$$\mathscr{O}_{(2)}(\Omega^n_{\mathscr{X}'/S}(\mathscr{L}|_{\mathscr{X}'}), \widetilde{\omega}_{\mathscr{X}'}, h|_{\mathscr{X}'}) \simeq \Omega^n(\log \mathcal{D})_{\mathscr{X}/S}(\mathscr{L}).$$

(ii) *The complex*

$$(\mathscr{A}^{0,\bullet}_{(2)}(\Omega^n_{\mathscr{X}'/S}(\mathscr{L}), \widetilde{\omega}_{\mathscr{X}'}, h|_{\mathscr{X}'}), \overline{\partial})$$

is a fine resolution of $\mathscr{O}_{(2)}(\Omega^n_{\mathscr{X}'/S}(\mathscr{L}), \widetilde{\omega}_{\mathscr{X}'}, h|_{\mathscr{X}'})$.

Proof. To simplify notation, we assume that $\dim S = 1$. We note that

$$\widetilde{\omega}^{n+1}_{\mathscr{X}'} = \sqrt{-1}\omega^n_{\mathscr{X}'/S}\, \varphi\, ds \wedge d\overline{s} + \omega^n_{\mathscr{X}'/S} \wedge f^*\omega_S$$

and the function φ is (locally with regard to S uniformly) bounded. For an open set $U \subset \mathscr{X}$, we thus have for the L^2-norm of a section $u \in \Omega^n_{\mathscr{X}'/S}(\mathscr{L})$ that

$$\int_U |u|^2_{\omega_{\mathscr{X}'/S}, h_{\mathscr{L}}}\, \widetilde{\omega}^{n+1}_{\mathscr{X}'} \sim \int_{f(U)} \left(\int_{X_s \cap U} (u \wedge \overline{u})_h \right) \omega_S.$$

The first statement follows from Fubini's theorem, the fact that nearby fibres and bundles are quasi-isometric, and Proposition 1. For the second statement, we decompose the vector bundle $\Omega^n_{\mathscr{X}'/S}(\mathscr{L}|_{\mathscr{X}'})$ locally as a sum of line bundles and apply [8, Prop. 2.1]; also compare to [8, p.870]. This shows that we have a resolution. By Proposition 2, fibrewise we find cut-off functions with bounded differentials. Because, in the differentiable sense, we have a product situation, this also holds on the total space. Hence, the sheaves of locally L^2 integrable smooth sections are again fine. □

Corollary 4. *The (local) holomorphic sections of* $f_*(\Omega^n(\log \mathcal{D})_{\mathscr{X}/S}(\mathscr{L}))$ *are given by holomorphic sections of* $\Omega^n(\log \mathcal{D})_{\mathscr{X}/S}(\mathscr{L})$ *on the total space that are precisely the holomorphic sections of* $\Omega^n_{\mathscr{X}'/S}(\mathscr{L}|_{\mathscr{X}'})$ *that are* L^2*-integrable along the fibres. In particular, their restrictions to the open fibres yield holomorphic and* L^2*-integrable* $(n, 0)$*-forms with values in L.*

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3.7. Fibre integrals and Lie derivatives

Given our family $f : \mathscr{X}' \to S$ of open complex manifolds X_s' of dimension n and a relative (n, n)-form η on $\mathscr{X}'$ that is smooth there and integrable along the fibres, the fibre integral

$$\int_{\mathscr{X}'/S} \eta$$

gives a function on the base S (see [19, Sect. 2.1] and [10, Ch. VII] for the general definition of fibre integrals). In our case, the components $H^{\bar{l}k}$ of the metric tensor on the direct image are defined by such fibre integrals, where η is given by the inner product/wedge product of the sections ψ^k:

$$\eta\,(s) = \psi^k|_{X_s'} \cdot \psi^{\bar{l}}|_{X_s'}\; dV = \mathrm{i}^{n^2}\; (\psi^k|_{X_s'} \wedge \psi^{\bar{l}}|_{X_s'})_h.$$

We want to show that these fibre integrals give smooth functions on the base so that we indeed get a smooth hermitian metric on the direct image we consider. Thus, if $s^1, \ldots, s^m$ are local holomorphic coordinates on the base, we need to compute the derivatives

$$\frac{\partial}{\partial s^k} \int_{X_s} \eta \quad \text{for } 1 \le i \le r \quad \text{ and } \quad \frac{\partial}{\partial s^{\bar{l}}} \int_{X_s} \eta, \quad \text{for } 1 \le l \le r.$$

This can be done by using Lie derivatives:

Lemma 3.10. *For $1 \le k \le m$, let v_k be the horizontal lift of $\partial/\partial s^k$. We write $\partial/\partial s^{\bar{l}}$ for $\partial/\partial \overline{s^l}$ and $v_{\bar{l}}$ for $\overline{v_l}$. Then*

$$\frac{\partial}{\partial s^k} \int_{X_s'} \eta = \int_{X_s'} L_{v_k}(\eta) \quad \textit{and} \quad \frac{\partial}{\partial s^{\bar{l}}} \int_{X_s'} \eta = \int_{X_s'} L_{v_{\bar{l}}}(\eta),$$

where L_{v_k} and $L_{v_{\bar{l}}}$ denotes the Lie derivative in the direction of v_k and $v_{\bar{l}}$, respectively.

Proof. The statement is well-known when the fibres are compact [17, Lemma 1]. We only have to show that $L_{v_k}(\eta)$ and $L_{v_{\bar{l}}}(\eta)$ are square integrable. Then the statement follows from the dominated convergence theorem. We only present it for the first. We are using local holomorphic coordinates $z^1, \ldots, z^n$ near a point $p \in D_s$ such that $D_s = \{z^1 \cdots z^k = 0\}$. Because $\psi^k, \psi^l \in H^0(\mathscr{X}, \Omega^n(\log \mathcal{D})_{\mathscr{X}/S}(\mathscr{L}))$ and due to the assumption on the metric h, we have that

$$\eta_{1\ldots n\bar{1}\ldots\bar{n}} = O\left(\prod_{i=1}^{k} \log^2(|z^i|^2)\right),$$

where

$$\eta = \eta_{1\ldots n\bar{1}\ldots\bar{n}}\, dz^1 \wedge \ldots \wedge dz^n \wedge dz^{\bar{1}} \wedge \cdots \wedge dz^{\bar{n}}.$$

Now $v_k = \partial/\partial_k + a_k^\alpha\, \partial/\partial z^\alpha$ so that

$$L_{v_k}\eta = \left(\eta_{1\ldots n,\bar{1},\ldots\bar{n};k} + \sum_{\alpha=1}^{n} a_k^\alpha \eta_{1\ldots n\bar{1}\ldots\bar{n};\alpha} + \sum_{\alpha=1}^{n} a_{k;\alpha}^\alpha \eta_{1\ldots n\bar{1}\ldots\bar{n}}\right) dz^1 \wedge \ldots \wedge dz^n \wedge dz^{\bar{1}} \wedge \ldots \wedge dz^{\bar{n}}.$$

Here, ; denotes the covariant derivative. Now, because of

$$a_k^\alpha = O(|z^\alpha| \log |z^\alpha|) \quad \text{ for } \quad 1 \le \alpha \le k,$$

we see that $L_{v_k}(\eta)$ is indeed integrable. □

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We see that we can iterate this process so that the fibre integral gives a smooth function on S. But this means the L^2-metric is indeed a smooth metric on $f_*(\Omega^n(\log\mathcal{D})_{\mathcal{X}/S}(\mathcal{L}))$. We note here that the square integrability of $L_v(\eta)$ will also follow from Lemma 6.2.

Before we go to the computation of the curvature, which is the most technical part of the article, let us consider the two applications.

4. Families of log-canonically polarised manifolds: (semi-)positivity of the relative canonical bundle

Let $\mathscr{D} \overset{i}{\hookrightarrow} \mathcal{X} \overset{f}{\to} S$ be a holomorphic *family of log-canonically polarised pairs*: that is, a holomorphic family of smooth log pairs (X_s, D_s) with ample adjoint bundle $K_{X_s} + D_s$. The family is called *effectively parametrised* if the Kodaira-Spencer map

$$\rho_s : T_s S \to H^1_{(2)}(X'_s, T_{X'_s})$$

is injective at all points $s \in S$.

Now let Ω be a smooth (relative) volume form on $\mathcal{X}$. The relative volume form $\psi := \Omega/(\prod_i ||\sigma_i||^2 \log^2 ||\sigma_i||^2)$ has the property that $\tilde{\omega}_s := -\mathrm{Ric}(\psi|X'_s)$ gives a family of complete Kähler metrics of finite volume on the open fibres X'_s such that $C^{-1} < \psi/(-\mathrm{Ric}(\psi))^n < C$ for some constant $C > 1$. Let $\{\omega_s\}_{s\in S}$ be the family of complete Kähler-Einstein metrics on $X'_s = X_s \setminus D_s$ with constant negative curvature -1 given by [14, 21]. When we write $\omega_s = \tilde{\omega}_s + \sqrt{-1}\partial_s\bar{\partial}_s u_s$ on X'_s, they fulfill the following Monge-Ampère equation (compare to [18, Eq.(1)]):

$$\omega_s^n = \exp(u_s)\cdot(\psi|X'_s). \tag{4.1}$$

Here, $\{u_s\}_{s\in S}$ is a family of functions in $\mathscr{C}^{k,\alpha}(X \setminus D)$ with $k \geq 6$. By the implicit function theorem applied to the Hölder spaces $\mathscr{C}^{k,\alpha}(X \setminus D)$ of functions, these functions u_s depend smoothly on the parameter $s \in S$ in the sense that the map $s \mapsto \mathscr{C}^{k,\alpha}$ is indeed Fréchet differentiable (compare to [18, Sect. 2.6]). Hence we can consider the relative volume form $\omega^n_{\mathcal{X}'/S}$ on $\mathcal{X}' = \mathcal{X} \setminus \mathscr{D}$ associated to the family $\{\omega_s\}_{s\in S}$ as a singular hermitian metric on $K_{\mathcal{X}/S}$ whose inverse has Poincaré type singularities along $\mathscr{D}$. Its smooth curvature form on $\mathcal{X}'$ is given by

$$\omega_{\mathcal{X}'} := -\mathrm{Ric}(\omega^n_{\mathcal{X}'/S}).$$

Analogous to the canonically polarised case proved in [19], we have the following result:

Theorem 4.1. *The form $\omega_{\mathcal{X}'} \geq 0$ is semi-positive and strictly positive if the family $\mathscr{D} \overset{i}{\hookrightarrow} \mathcal{X} \overset{f}{\to} S$ is effectively parametrised.*

Proof. The computation from [18] can be adopted. We summarise the main points that we need.

Given a coordinate vector field $\partial/\partial s^i$ on S and the horizontal lift v_i, define

$$\varphi_{i\bar{\imath}} = \langle v_i, v_i\rangle_{\omega_{\mathcal{X}'}}.$$

Then

$$(1 + \Box_s)\varphi_{i\bar{\imath}} = ||A_i||^2(z, s), \tag{4.2}$$

where $\Box_s$ denotes the (semi-positive) Laplacian and $||A_i||(z, s)$ the pointwise norm of the harmonic representative of $\rho_s(\partial/\partial s^i)$.

The results of the previous section imply that the quantities occurring in equation (4.2) are $\mathscr{C}^{k,\alpha}$-tensors on the total space and also define such tensors when restricted to the fibres of f and class C^∞ on

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$\mathscr{X}'$. Yau's maximum principle [23, Theorem 1] applies to restrictions of equation (4.2) to the fibres of f and immediately yields that $\varphi_{i\bar{\imath}} \geq 0$.

The integral of equation (4.2) along a fibre yields the Weil-Petersson norm of $\partial/\partial s^i|_s$:

$$\|\partial/\partial s^i|_s\|^2_{WP} = \int_{X_s} \|A_i\|^2(z,s) g dV.$$

(Again, we are using Gaffney's result.)

Let $\rho_s(\partial/\partial s^i) \neq 0$: that is, $A_i \neq 0$. One can show that $\varphi_{i\bar{\imath}}(z,s)$ has no zeroes. This follows from the lower heat kernel estimate in the complete case, as given in [20, Cor. 4.3]. This shows that the heat kernel is strictly positive on the fibre X_s'. Then the argument is the same as in [19, Prop. 1], except that we do not have a fixed positive lower bound in terms of the diameter of the fibres. □

Corollary 5. *For a family of log-canonically polarised manifolds $\mathscr{D} \xhookrightarrow{i} \mathscr{X} \xrightarrow{f} S$ over a compact base S, the relative adjoint bundle $K_{\mathscr{X}/S} \otimes \mathscr{D}$ is nef. If the family is effectively parametrised, $K_{\mathscr{X}/S} \otimes \mathscr{D}$ is big.*

Proof. We now compute the curvature current of the singular hermitian metric $(\omega^n_{\mathscr{X}'/S})^{-1} = (\omega^n_s)^{-1}_{s\in S}$ on $K_{\mathscr{X}/S}$ on the whole $\mathscr{X}$. From the Monge-Ampère equation (4.1), we see that the only additional term that vanishes by restricting to $\mathscr{X}'$ is $-\mathscr{D}$, which comes from the term $-\sum \partial\bar{\partial} \log ||\sigma_i||^2$; it is compensated by adding $\mathscr{D}$. This shows that $K_{\mathscr{X}/S} \otimes \mathscr{D}$ is pseudoeffective. Here we take the canonical singular metric on $\mathscr{D}$. The nefness follows from the fact that the curvature current of the metric on $K_{\mathscr{X}/S} \otimes \mathscr{D}$ has zero Lelong numbers. The bigness in the effectively parametrised situation then follows from the strict positivity of $K_{\mathscr{X}'/S}$ and Boucksom's bigness criterion [2, Cor. 3.3]. □

5. The case of a big line bundle

Let $E \xhookrightarrow{i} \mathscr{X} \xrightarrow{f} S$ be a holomorphic family of smooth log pairs (X_s, E_s) and F a big line bundle on $\mathscr{X}$. We assume that we have a decomposition

$$F = A + E,$$

where A is ample on $\mathscr{X}$. We choose a smooth positive metric h_A on A and a smooth hermitian metric h_i on each irreducible component E_i of E. Using the canonical section σ_i cutting out the divisor E_i, we define another metric on A by setting

$$h_{A,\varepsilon} := h_A \cdot \left(\prod_i |\sigma_i|^2_{h_i} \log^2(|\sigma_i|^2_{h_i})\right)^{\varepsilon}.$$

We now make the assumption that there exists an $\varepsilon > 0$ such that

$$\mathrm{i}\,\Theta_{h_{A,\varepsilon}}(A) = \mathrm{i}\,\Theta_{h_A}(A) - \varepsilon\sqrt{-1}\sum_i \partial\bar{\partial}\log\Big(|\sigma_i|^2_{h_i}\log^2(|\sigma_i|^2_{h_i})\Big) > 0 \quad \text{on} \quad \mathscr{X}'.$$

In general, such an ε need not exist and it depends on the curvatures of h_A and h_i on $\mathscr{X}$. Then we can equip F with the metric $h_{F,\varepsilon}$ defined by

$$h_{F,\varepsilon} = h_{A,\varepsilon} \cdot h_{E,\mathrm{sing}},$$

where $h_{E,\mathrm{sing}}$ is the canonical singular hermitian metric on E given by the section $\sigma = \prod_i \sigma_i$. We note that $(E, h_{E,\mathrm{sing}})|_{\mathscr{X}'}$ is the trivial line bundle equipped with the trivial metric.

Now the hermitian bundle $(A, h_{A,\varepsilon})$ fulfils the requirements of our main theorem (note here that it works with a power $\varepsilon > 0$ instead of 1 as well), and we get the Nakano positivity of

$$f_*(K_{\mathscr{X}/S} + E + A) = f_*(K_{\mathscr{X}/S} + F).$$

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If we instead apply the general result from [3] directly to the hermitian bundle (F, h_F) with $h_F = h_A \cdot h_{E,\mathrm{sing}}$, we get first that

$$f_*((K_{\mathcal{X}/S} + F) \otimes \mathcal{J}(h_F))$$

is positive in the singular sense of Griffiths. But here we have $\mathcal{J}(h_F) = \mathcal{O}(-E)$, so we can conclude that

$$f_*((K_{\mathcal{X}/S} + F) \otimes \mathcal{J}(h_F)) = f_*(K_{\mathcal{X}/S} + A)$$

is positive in the sense of Nakano using the result from [1]. Of course, we should mention here that we have

$$f_*(K_{\mathcal{X}/S} + A) \subset f_*(K_{\mathcal{X}/S} + E + A)$$

as sheaves but not as hermitian bundles because we changed the metric on A for the larger one. If the decomposition $F = A + E$ is a relative Zariski decomposition, both sheaves coincide. If one applies [3] to $(L, h_{F,\varepsilon})$ with a trivial multiplier ideal sheaf $\mathcal{J}(h_{F,\varepsilon})$, we get Griffiths positivity of $f_*(K_{\mathcal{X}/S} + F)$ only in the weaker singular sense.

6. Computation of the curvature

Computing the curvature of the L^2-metric on $f_*(\Omega^n(\log \mathscr{D}))_{\mathcal{X}/S}(\mathscr{L})$ requires taking derivatives in the base direction of fibre integrals, which can be realised by taking Lie derivatives of the integrands. These Lie derivatives can be split up by introducing Lie derivatives of $(n, 0)$-forms with values in L. They are computed in terms of covariant derivatives with respect to the Chern connection on (X'_s, ω_s) and the hermitian holomorphic bundle (L_s, h_s). We use the symbol ; for covariant derivatives and , for ordinary derivatives. Greek letters indicate the fibre direction, whereas Latin indices stand for directions on the base. Because we are dealing with alternating (p, q)-forms, the coefficients are meant to be skew-symmetric. Thus every such (p, q)-form carries a factor $1/p!q!$, which we suppress in the notation. These factors play a role in the process of skew-symmetrising the coefficients of a (p, q)-form by taking alternating sums of the (not yet skew-symmetric) coefficients. We adopt the Einstein convention of summation.

6.1. Setup

By polarisation, it is sufficient to treat the case where $\dim S = 1$ for the computation of the curvature, which simplifies the notation. Therefore, we set $s = s^1, v_s = v_1$, and so on. We write $s, \overline{s}$ for the indices $1, \overline{1}$ so that

$$v_s = \partial_s + a_s^\alpha \partial_\alpha$$

and

$$A_s = A^\alpha_{s\overline{\beta}} \partial_\alpha dz^{\overline{\beta}}.$$

We assume local freeness of the sheaf $f_*(\Omega^n(\log \mathscr{D}))_{\mathcal{X}/S}(\mathscr{L})$. According to Corollary 4, we can represent local sections of this sheaf by holomorphic sections of $(\Omega^n(\log \mathcal{D}))_{\mathcal{X}'/S}(\mathscr{L}|_{\mathcal{X}'})$, which restrict to holomorphic and square integrable $(n, 0)$-forms on the open fibres X'_s. We denote such a section by ψ. In local coordinates, we have

$$\begin{aligned} \psi|_{X'_s} &= \psi_{\alpha_1 \ldots \alpha_n} dz^{\alpha_1} \wedge \ldots dz^{\alpha_n} \\ &= \psi_{A_n} dz^{A_n}, \end{aligned}$$

where $A_n = (\alpha_1, \ldots, \alpha_n)$. The $\bar{\partial}$-closedness of ψ means

$$\psi_{A_n;s} = 0 \quad \text{and} \quad \psi_{A_n;\overline{\beta}} = 0 \quad \text{for all} \quad A_n, 1 \leq \beta \leq n. \tag{6.1}$$

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6.2. Cup product

Definition 4. Let $s \in S$ and $A = A^{\alpha}_{s\overline{\beta}}(z,s)\partial_\alpha dz^{\overline{\beta}}$ be the Kodaira-Spencer form on the fibre X'_s. The wedge product, together with the contraction, defines a map

$$A^{\alpha}_{i\overline{\beta}}\partial_\alpha dz^{\overline{\beta}}\cup\ : H^0(X_s, \Omega^n_{X_s}(\log D_s)(L|_{X_s})) \to A^{0,1}_{(2)}(X'_s, \Omega^{n-1}_{X'_s}(L|_{X'_s})),$$

which can be described locally by

$$\begin{aligned}&\left(A^{\gamma}_{i\overline{\delta}}\partial_\gamma dz^{\overline{\delta}}\right) \cup \left(\psi_{\alpha_1\ldots\alpha_n}\, dz^{\alpha_1} \wedge \ldots \wedge dz^{\alpha_n}\right)\\ &= A^{\gamma}_{i\overline{\beta}}\psi_{\gamma\alpha_1\ldots\alpha_{n-1}}\, dz^{\overline{\beta}} \wedge dz^{\alpha_1} \wedge \ldots \wedge dz^{\alpha_{n-1}}.\end{aligned}$$

The fact that $A_i \cup \psi$ is indeed square integrable will be proved in Lemma 6.2.

6.3. Lie derivatives

Now we choose a local frame $\{\psi^1, \ldots, \psi^r\}$ according to Corollary 4. The components of the metric tensor $H^{\overline{l}k}$ for $f_*(\Omega^n(\log \mathscr{D}))_{\mathcal{X}/S}(\mathscr{L})$ on the base space S are given by

$$H^{\overline{l}k}(s) := \langle \psi^k, \psi^l\rangle := \langle \psi^k|_{X'_s}, \psi^l|_{X'_s}\rangle = \int_{X'_s} \psi^k_{A_n}\psi^{\overline{l}}_{\overline{B}_n} g^{\overline{B}_n A_n} h|_{X'_s}\, dV.$$

We also write

$$\psi^k \cdot \psi^{\overline{l}} = \psi^k_{A_n}\psi^{\overline{l}}_{\overline{B}_n} g^{\overline{B}_n A_n} h|_{X'_s}$$

for the pointwise inner product of $L|_{X'_s}$-valued $(n,0)$-forms. Here and in the following, we write g for the hermitian metric associated to the complete Kähler from ω_s. When we compute derivatives with respect to the base of these fibre integrals, we apply Lie derivatives with respect to the horizontal lifts of the tangent vectors according to Lemma 3.10. This considerably simplifies the computation. To break up the Lie derivative of the pointwise inner product (which is a relative (n,n)-form), we need to introduce Lie derivatives of relative differential forms with values in a line bundle. This can be done by using the hermitian connection ∇ on $\Lambda^{n,0}T^*_{\mathcal{X}'/S} \otimes \mathscr{L}|_{\mathcal{X}'}$ induced by the Chern connections on $(T_{X'_s}, \omega_{X_s})$ and (L_s, h_s). We define the Lie derivative of ψ with respect to the horizontal lift v by using Cartan's formula

$$L_v\psi := L_v(\psi_{\mathcal{X}'/S}) := (\delta_v \circ \nabla + \nabla \circ \delta_v)\psi \tag{6.2}$$

and similar for the Lie derivative with respect to $\overline{v}$.

Taking Lie derivatives is not type-preserving. We have the type decomposition for $\psi = \psi^k$ or $\psi = \psi^l$ and $v = v_s$

$$L_v\psi = L_v\psi' + L_v\psi'',$$

where $L_v\psi'$ is of type $(n,0)$ and $L_v\psi''$ is of type $(n-1,1)$. In local coordinates, we have

$$L_v\psi' = \left(\psi_{A_n;s} + a^{\alpha}_{s}\psi_{A_n;\alpha} + \sum_{j=1}^{n} a^{\alpha}_{s;\alpha_j}\psi_{\alpha_1\ldots\underset{j}{\alpha}\ldots\alpha_n}\right) dz^{A_n} \tag{6.3}$$

$$L_v\psi'' = \sum_{j=1}^{n} A^{\alpha}_{s\overline{\beta}_n}\psi_{\alpha_1\ldots\underset{j}{\alpha}\ldots\alpha_n}\, dz^{\alpha_1} \wedge \ldots \wedge \underset{j}{dz^{\overline{\beta}_n}} \wedge \ldots \wedge dz^{\alpha_n}. \tag{6.4}$$

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One justification for using Lie derivatives is given by the following lemma, which allows us to express some components of the Lie derivatives as cup products with the Kodaira-Spencer form:

Lemma 6.1. *We have*

$$L_v \psi'' = A_s \cup \psi, \tag{6.5}$$

and it is primitive on the fibres.

Proof. First, we note

$$\begin{aligned} L_v \psi'' &= \sum_{j=1}^{n} A^{\alpha}_{s\overline{\beta}_n} \psi_{\alpha_1 \ldots \underset{j}{\alpha} \ldots \alpha_n} dz^{\alpha_1} \wedge \ldots \wedge \underset{j}{dz^{\overline{\beta}_n}} \wedge \ldots \wedge dz^{\alpha_n} \\ &= \sum_{j=1}^{n} A^{\alpha}_{s\overline{\beta}_n} \psi_{\alpha\, \alpha_1 \ldots \alpha_{n-1}} dz^{\overline{\beta}_n} \wedge dz^{\alpha_1} \wedge \ldots \wedge dz^{\alpha_{n-1}}. \end{aligned}$$

To prove that $A_s \cup \psi$ is primitive, we have to show that $\Lambda_s(A_s \cup \psi) = 0$, where Λ_s is the dual Lefschetz operator with respect to the Kähler form

$$\omega_s = \sqrt{-1} g_{\alpha\overline{\beta}}\, dz^{\alpha} \wedge dz^{\overline{\beta}}.$$

We have

$$(\Lambda_s(A_s \cup \psi))_{\alpha_2 \ldots \alpha_{n-1}} = g^{\overline{\beta}_n \alpha_1} A^{\alpha}_{s\overline{\beta}_n} \psi_{\alpha\, \alpha_1 \ldots \alpha_{n-1}} = A_s^{\alpha\alpha_1} \psi_{\alpha\, \alpha_1 \ldots \alpha_{n-1}}.$$

But now, because $A_s^{\alpha\alpha_1} = A_s^{\alpha_1\alpha}$ by Lemma 3.9 and $\psi_{\alpha\, \alpha_1 \ldots \alpha_{n-1}}$ is skew-symmetric, we get that

$$(\Lambda_s(A_s \cup \psi))_{\alpha_2 \ldots \alpha_{n-1}} dz^{\alpha_2} \wedge \ldots \wedge dz^{\alpha_{n-1}} = 0.$$

□

Similarly, we have a type decomposition for the Lie derivative along $\overline{v} = v_{\overline{s}}$

$$L_{\overline{v}} \psi = L_{\overline{v}} \psi' + L_{\overline{v}} \psi'',$$

where $L_{\overline{v}} \psi'$ is of type $(n, 0)$ and $L_{\overline{v}} \psi''$ is of type $(n+1, n-1)$ and hence vanishes by degree reasons. In local coordinates, this is

$$L_{\overline{v}} \psi' = \left(\psi_{A_p \overline{B}_{n-p};\overline{s}} + a^{\overline{\beta}}_{\overline{s}} \psi_{A_p \overline{B}_{n-p};\overline{\beta}} + \sum_{j=p+1}^{n} a^{\overline{\beta}}_{\overline{s};\overline{\beta}_j} \psi_{A_p \overline{\beta}_{p+1} \ldots \underset{j}{\overline{\beta}} \ldots \overline{\beta}_n} \right) dz^{A_p} \wedge dz^{\overline{B}_{n-p}}. \tag{6.6}$$

From this, we infer that $L_{\overline{v}} \psi = L_{\overline{v}} \psi' = 0$ because ψ is holomorphic. The type decomposition can be verified using the definition given by equation (6.2). We refer the reader to [17] for verification.

Lemma 6.2. *The smooth forms $L_v \psi'$ and $L_v \psi''$ are L^2-integrable.*

Proof. We use local coordinates $z^1, \ldots, z^n$ in a neighbourhood U of a point $p \in D_s \subset X_s$, where $D_s \cap U = \{z^1 \cdots z^k = 0\}$. We now have

$$\psi_{A_n} = O\left(\frac{1}{|z^1 \cdots z^k|} \right)$$

and

$$a_s^{\alpha} = O(|z^{\alpha}| \log |z^{\alpha}|) \quad \text{for} \quad 1 \leq \alpha \leq k \quad \text{else} \quad a_s^{\alpha} = O(1).$$

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From the local expression (6.3), we thus see that

$$(L_v\psi)'_{A_n} = O\left(\frac{1}{|z^1\cdots z^k|}\right),$$

so it is again square integrable.

To prove that $L_v\psi''$ is square integrable is more complicated. We first look at the order of $A^{\alpha}_{s\overline{\beta}}$:

$$\begin{aligned}
A^{\alpha}_{\overline{\beta}} &= O(1) \quad \text{for} \quad 1 \le \alpha = \beta \le k \quad \text{or} \quad k+1 \le \alpha, \beta \le n.\\
A^{\alpha}_{\overline{\beta}} &= O\left(\frac{1}{|z^\beta|\log|z^\beta|}\right) \quad \text{for} \quad \alpha > k \text{ and } \beta \le k.\\
A^{\alpha}_{\overline{\beta}} &= O(|z^\alpha|\log|z^\alpha|) \quad \text{for} \quad \alpha \le k \text{ and } \beta > k.\\
A^{\alpha}_{\overline{\beta}} &= O\left(\frac{|z^\alpha|\log|z^\alpha|}{|z^\beta|\log|z^\beta|}\right) \quad \text{for} \quad 1 \le \alpha \ne \beta \le k.
\end{aligned}$$

To prove that $L_v\psi'' = A_s \cup \psi$ is L^2-integrable means to verify that

$$\int_{X'_s} (A_s \cup \psi) \wedge \overline{(A_s \cup \psi)}$$

is finite because the form is primitive by Lemma 6.1. For this we first note that the sum in the expression of $(A_s \cup \psi)_{\overline{\beta}_n\alpha_1\ldots\alpha_{n-1}}$ reduces to

$$A^{\alpha_n}_{s\,\overline{\beta}_n}\psi_{\alpha_n\alpha_1\ldots\alpha_{n-1}}.$$

The only really critical term in $(A_s \cup \psi)_{\overline{\beta}_n\alpha_1\ldots\alpha_{n-1}}$ occurs if $\alpha_n \in \{k+1,\ldots,n\}$, $\beta_n \in \{1,\ldots,k\}$ and β_n is among the $\alpha_1,\ldots,\alpha_{n-1}$, because in this case the order in z^{β_n} is

$$\frac{1}{|z^{\beta_n}|^2\log|z^{\beta_n}|}.$$

But then, in the above integral, this term can only be paired with another term that contains neither dz^{β_n} nor $d\overline{z}^{\beta_n}$. So we see that the product $(A_s \cup \psi) \wedge \overline{(A_s \cup \psi)}$ remains integrable. □

We need the following lemma:

Lemma 6.3. *The Lie derivative of the volume element $dV = \omega_s^n/n!$ along the horizontal lift v vanishes: that is,*

$$L_v(dV) = 0.$$

Proof. It suffices to show that the $(1,1)$ component of $L_v(g_{\alpha\overline{\beta}})$ vanishes, which implies $L_v(\det(g_{\alpha\overline{\beta}})) = 0$. We have

$$L_v(g_{\alpha\overline{\beta}})_{\alpha\overline{\beta}} = g_{\alpha\overline{\beta},s} + a^{\gamma}_{s}g_{\alpha\overline{\beta};\gamma} + a^{\gamma}_{s;\alpha}g_{\gamma\overline{\beta}} = -a_{s\overline{\beta};\alpha} + a^{\gamma}_{s;\alpha}g_{\gamma\overline{\beta}} = 0.$$ □

6.4. *Main part of the computation*

We start computing the curvature by computing the first-order variation. Using Lie derivatives, the pointwise inner products can be broken up:

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Proposition 8.

$$\frac{\partial}{\partial s}\langle \psi^k, \psi^l \rangle = \langle L_v \psi^k, \psi^l \rangle,$$

where $\partial/\partial s$ denotes a tangent vector on the base S and v its horizontal lift and analogous for $\partial/\partial \overline{s}$.

Proof. We first apply Lemma 3.10 and get that

$$\frac{\partial}{\partial s}\langle \psi^k, \psi^l \rangle(s) = \int_{X_s'} L_v(\psi^k \cdot \psi^{\bar{l}}\, dV) = \int_{X_s'} L_v(\psi^k \cdot \psi^{\bar{l}})\, dV$$

by Lemma 6.3. Now it follows by a direct computation (see [17, Prop.1]) that

$$L_v(\psi^k \cdot \psi^{\bar{l}}) = L_v \psi^k \cdot \psi^{\bar{l}} + \psi^k \cdot L_v \psi^{\bar{l}}$$

so that

$$\frac{\partial}{\partial s}\langle \psi^k, \psi^l \rangle = \langle L_v \psi^k, \psi^l \rangle + \langle \psi^k, L_{\overline{v}} \psi^l \rangle = \langle L_v \psi^k, \psi^l \rangle$$

because $L_{\overline{v}} \psi^l = 0$. □

The above proposition is the primary reason for the use of Lie derivatives. For later computations, we need to compare Laplacians:

Lemma 6.4. *We have the following relation on the space $A^{p,q}_{(2)}(X_s', L_s)$:*

$$\Box_\partial - \Box_{\bar{\partial}} = (n - p - q) \cdot \mathrm{id}\,. \tag{6.7}$$

In particular, the harmonic forms $\psi \in A^{n,0}_{(2)}(X_s', L_s)$ are also harmonic with respect to ∂, which is the $(1,0)$- part of the hermitian connection on $A^{n,0}_{(2)}(X_s', L_s)$.

Proof. The Bochner-Kodaira-Nakano identity says (on the fibre X_s')

$$\Box_{\bar{\partial}} - \Box_\partial = \left[\sqrt{-1}\Theta(L_s), \Lambda\right].$$

But by definition, we have $\omega_{X_s} = \sqrt{-1}\Theta(L_s)$. Furthermore, it holds (see [7, Cor.VI.5.9])

$$[L_\omega, \Lambda_\omega]u = (p + q - n)\, u \quad \text{for } u \in \mathscr{A}^{p,q}(X_s', L_s).$$ □

Next, we start to compute the second-order derivative of $H^{\bar{l}k}$ and begin with

$$\frac{\partial}{\partial s} H^{\bar{l}k} = \langle L_v \psi^k, \psi^l \rangle.$$

We obtain

$$\begin{aligned}
\partial_{\overline{s}}\partial_s \langle \psi^k, \psi^l \rangle &= \langle L_{\overline{v}} L_v \psi^k, \psi^l \rangle + \langle L_v \psi^k, L_v \psi^l \rangle \\
&= \langle (L_{[\overline{v},v]} + \Theta(L|_{\mathscr{X}'})_{\overline{v}v})\psi^k, \psi^l \rangle + \langle L_v L_{\overline{v}} \psi^k, \psi^l \rangle + \langle L_v \psi^k, L_v \psi^l \rangle \\
&= \langle (L_{[\overline{v},v]} + \Theta(L|_{\mathscr{X}'})_{\overline{v}v})\psi^k, \psi^l \rangle + \partial_s \langle L_{\overline{v}} \psi^k, \psi^l \rangle - \langle L_{\overline{v}} \psi^k, L_{\overline{v}} \psi^l \rangle + \langle L_v \psi^k, L_v \psi^l \rangle.
\end{aligned}$$

Because of $L_{\overline{v}} \psi^k \equiv 0$, as we just saw, we get

$$\partial_{\overline{s}}\partial_s \langle \psi^k, \psi^l \rangle = \langle (L_{[\overline{v},v]} + \Theta(L|_{\mathscr{X}'})_{\overline{v}v})\psi^k, \psi^l \rangle + \langle L_v \psi^k, L_v \psi^l \rangle. \tag{6.8}$$

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We will see below that the smooth $(n, 0)$-form $(L_{[\overline{v},v]} + \Theta(L|_{\mathcal{X}'})_{\overline{v}v})\psi^k$ is indeed square integrable, which justifies that $L_{\overline{v}}L_v\psi^k$ is square integrable, too.

Now we treat each term on the right-hand side of equation (6.8) separately. For the first summand, we have

Lemma 6.5.

$$L_{[\overline{v},v]} + \Theta(L_{\mathcal{X}'})_{\overline{v}v} = [-\varphi^{;\alpha}\partial_\alpha + \varphi^{;\overline{\beta}}\partial_{\overline{\beta}}, \_] - \varphi \cdot \mathrm{id}, \tag{6.9}$$

where the bracket $[w, \_]$ *stands for a Lie derivative along the vector field* w.

Proof. We first compute the vector field $[\overline{v}, v]$:

$$\begin{aligned}[\overline{v}, v] &= [\partial_{\overline{s}} + a_{\overline{s}}^{\overline{\beta}}\partial_{\overline{s}}, \partial_s + a_s^\alpha\partial_\alpha] \\ &= \left(\partial_{\overline{s}}(a_s^\alpha) + a_{\overline{s}}^{\overline{\beta}} a^\alpha_{a|\overline{\beta}}\right)\partial_\alpha - \left(\partial_s(a_{\overline{s}}^{\overline{\beta}}) + a_s^\alpha a^{\overline{\beta}}_{\overline{s}|\alpha}\right)\partial_{\overline{\beta}}.\end{aligned}$$

Now we have

$$\begin{aligned}\partial_{\overline{s}}(a_s^\alpha) &= -\partial_{\overline{s}}(g^{\overline{\beta}\alpha}g_{s\overline{\beta}}) = g^{\overline{\beta}\sigma}g_{\sigma\overline{s}|\overline{\tau}}g^{\overline{\tau}\alpha}g_{s\overline{\beta}} - g^{\overline{\beta}\alpha}g_{s\overline{\beta}|s} \\ &= g^{\overline{\beta}\sigma}a_{\overline{s}\sigma;\overline{\tau}}g^{\overline{\tau}\alpha}a_{s\overline{\beta}} - g^{\overline{\beta}\alpha}g_{s\overline{s};\overline{\beta}}.\end{aligned}$$

Because of $\varphi = g_{s\overline{s}} - g_{\alpha\overline{s}}g_{s\overline{\beta}}g^{\overline{\beta}\alpha}$, the coefficient of ∂_α is $g^{\overline{\beta}\alpha}\varphi_{;\overline{\beta}} = \varphi^{;\alpha}$. In the same way, we get the coefficient of $\partial_{\overline{\beta}}$. Next, we need to compute the contribution of the connection on $L|_{\mathcal{X}'}$. Because of $\sqrt{-1}[\partial, \bar{\partial}] = \sqrt{-1}\Theta(L)|_{\mathcal{X}'} = \omega_{\mathcal{X}'}$, we have

$$\begin{aligned}\Theta(L|_{\mathcal{X}'})_{\overline{v}v} &= -\Theta(L|_{\mathcal{X}'})_{v\overline{v}} \\ &= -\left(g_{s\overline{s}} + a_{\overline{s}}^{\overline{\beta}}g_{s\overline{\beta}} + a_s^\alpha g_{\alpha\overline{s}} + a_{\overline{s}}^{\overline{\beta}}a_s^\alpha g_{\alpha\overline{\beta}}\right) \\ &= -\varphi.\end{aligned}$$

□

Lemma 6.6.

$$\langle (L_{[\overline{v},v]} + \Theta(L|_{\mathcal{X}'})_{\overline{v}v})\psi^k, \psi^l\rangle = -\langle \varphi \cdot \psi^k, \psi^l\rangle = -\int_{X'_s} \varphi \cdot (\psi^k \cdot \psi^{\overline{l}})\, dV. \tag{6.10}$$

Proof. The ∂-closedness of ψ^k means

$$\psi^k_{;\alpha} = \sum_{j=1}^n \psi^k_{\alpha_1\ldots\underset{j}{\alpha}\ldots\alpha_n;\alpha_j}.$$

Thus

$$\begin{aligned}[\varphi^{;\alpha}\partial_\alpha, \psi^k_{A_n}]' &= \varphi^{;\alpha}\psi^k_{;\alpha} + \sum_{j=1}^n \varphi^{;\alpha}_{;\alpha_j}\psi^k_{\alpha_1\ldots\underset{j}{\alpha}\ldots\alpha_n} \\ &= \sum_{j=1}^n (\varphi^{;\alpha}\psi^k_{\alpha_1\ldots\underset{j}{\alpha}\ldots\alpha_n})_{;\alpha_j} \\ &= \partial\left(\varphi^{;\alpha}\partial_\alpha \cup \psi^k\right).\end{aligned}$$

It is clear that $(\varphi^{;\alpha}\partial_\alpha \cup \psi^k)$ is square integrable, because $\varphi|_{X'_s}$ lies in $\mathscr{C}^{k,\alpha}(X'_s)$. Moreover, it guarantees that this form lies in the domain of ∂. This leads to

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$$\begin{aligned}\langle[\varphi^{;\alpha}\partial_\alpha,\psi^k_{A_n}],\psi^l\rangle &= \langle[\varphi^{;\alpha}\partial_\alpha,\psi^k_{A_n}]',\psi^l\rangle\\ &= \langle\partial\Big(\varphi^{;\alpha}\partial_\alpha\cup\psi^k\Big),\psi^l\rangle = \langle\varphi^{;\alpha}\partial_\alpha\cup\psi^k,\partial^*\psi^l\rangle = 0.\end{aligned}$$

Note that by Gaffney's theorem, Proposition 5, the formal adjoint of ∂ is equal to the adjoint operator.

In the same way, we get

$$\langle[\varphi^{;\overline{\beta}}\partial_{\overline{\beta}},\psi^k_{A_n}],\psi^l\rangle = 0.$$

□

The following proposition contains important identities that allow us to obtain an intrinsic expression for the curvature:

Proposition 9.

$$\bar\partial(L_v\psi^k)' = \partial(A_s\cup\psi^k), \tag{6.11}$$

$$\bar\partial^*(L_v\psi^k)' = 0, \tag{6.12}$$

$$\partial^*(A_s\cup\psi^k) = 0. \tag{6.13}$$

We note that here the operators $\partial,\bar\partial,\partial^*$ and $\bar\partial^*$ mean the fibrewise operators, because we are always dealing with relative forms. For a proof, we refer to [17, Appendix A]. We see from the proof of Lemma 6.2 that $\bar\partial(L_v\psi^k)'$ is again square integrable.

Now we look at the second term in equation (6.8) and decompose it into its two types

$$\begin{aligned}\langle L_v\psi^k,L_v\psi^l\rangle &= \langle(L_v\psi^k)',(L_v\psi^l)'\rangle - \langle(L_v\psi^k)'',(L_v\psi^l)''\rangle\\ &= \langle(L_v\psi^k)',(L_v\psi^l)'\rangle - \langle A_s\cup\psi^k,A_s\cup\psi^l\rangle\end{aligned}$$

because of equation (6.5).

Now let G_∂ and $G_{\bar\partial}$ be the Green operators on the spaces $A^{p,q}_{(2)}(X_s', L|_{X_s'})$ with respect to $\Box_\partial$ and $\Box_{\bar\partial}$, respectively. According to Lemma 6.4, they coincide for $p+q=n$. We use normal coordinates (of the second kind) at a given point $s_0\in S$. The condition $(\partial/\partial s)H^{\bar lk}|_{s_0}=0$ for all k,l means for $s=s_0$ the harmonic projection

$$H((L_v\psi^k)') = 0$$

vanishes for all k. Thus, using the identity $\mathrm{id} = H + G_{\bar\partial}\Box_{\bar\partial}$, we can write

$$(L_v\psi^k)' = G_{\bar\partial}\Box_{\bar\partial}(L_v\psi^k)' = G_{\bar\partial}\bar\partial^*\bar\partial(L_v\psi^k)' = \bar\partial^*G_{\bar\partial}\partial(A_s\cup\psi^k)$$

by equations (6.12) and (6.11). Because the form $\bar\partial(L_v\psi^k)' = \partial(A_s\cup\psi^k)$ is of type $(n,1)$, we have $G_{\bar\partial} = (\Box_\partial+1)^{-1}$ on such forms by Lemma 6.4. We proceed by

$$\begin{aligned}\langle(L_v\psi^k)',(L_v\psi^l)'\rangle &= \langle\bar\partial^*G_{\bar\partial}\partial(A_s\cup\psi^k),(L_v\psi^l)'\rangle\\ &= \langle G_{\bar\partial}\partial(A_s\cup\psi^k),\partial(A_s\cup\psi^l)\rangle\\ &= \langle(\Box_\partial+1)^{-1}\partial(A_s\cup\psi^k),\partial(A_s\cup\psi^l)\rangle\\ &= \langle\partial^*(\Box_\partial+1)^{-1}\partial(A_s\cup\psi^k),A_s\cup\psi^l\rangle.\end{aligned}$$

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Again, we used Gaffney's theorem. Now using equation (6.13) gives

$$\begin{aligned}\langle (L_v\psi^k)', (L_v\psi^l)'\rangle &= \langle (\Box_\partial + 1)^{-1}\Box_\partial (A_s \cup \psi^k), A_s \cup \psi^l\rangle \\ &= \langle (\Box_\partial + 1)^{-1}(\Box_\partial + 1 - 1)(A_s \cup \psi^k), A_s \cup \psi^l\rangle \\ &= \langle A_s \cup \psi^k, A_s \cup \psi^l\rangle - \langle (\Box_\partial + 1)^{-1}(A_s \cup \psi^k), A_s \cup \psi^l\rangle.\end{aligned}$$

Altogether, we have

Lemma 6.7.

$$\langle L_v\psi^k, L_v\psi^l\rangle = -\int_{X'_s} (\Box + 1)^{-1}(A_s \cup \psi^k)\cdot (A_{\bar{s}} \cup \psi^{\bar{l}})\, g\, dV. \tag{6.14}$$

(We write $\Box = \Box_\partial = \Box_{\bar\partial}$ *when applied to* $(n-1,1)$*-forms.)*

Now our main result Theorem 2.1 follows from equations (6.8), (6.10), (6.14) and the fact that $R_{i\bar{j}}^{\bar{l}k}(s_0) = -\partial_{\bar{j}}\partial_i H^{\bar{l}k}(s_0)$ in normal coordinates at a point $s_0 \in S$.

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References

[1] B. Berndtsson, 'Curvature of vector bundles associated to holomorphic fibrations', *Ann. Math.* **169** (2009), 531–560.
[2] S. Boucksom, 'On the volume of a line bundle', *Int. J. of Math.* **13**(10) (2002), 1043–1063.
[3] B. Berndtsson, M. Păun, 'Bergman kernels and the pseudoeffectivity of relative canonical bundles', *Duke Math. Journal* **145**(2) (2008), 341–378 (2008).
[4] B. Berndtsson, M. Păun, and X. Wang, 'Algebraic fiber spaces and curvature of higher direct images', *J. of the Inst. of Math. of Jussieu*, **21**(3) (2022), 973–1028.
[5] J.-L. Brylinski, S. Zucker, *An Overview of Recent Advances in Hodge Theory*, Complex Manifolds, Springer (1998).
[6] S.Y. Cheng, S.T. Yau, 'On the existence of a complete Kähler metric in non-compact complex manifolds and the regularity of Fefferman's equation', *Comm. Pure Appl. Math.* **33** (1980), 507–544.
[7] J.-P. Demailly, *Complex Analytic and Differential Geometry*, https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf, Grenoble (2012).
[8] A. Fujiki, 'An L^2-Dolbeault lemma and its applications', *Publ. RIMS, Kyoto Univ.* **20** (1984), 977–1005.
[9] M.P. Gaffney, 'A special Stokes's theorem for complete Riemannian manifolds', *Ann. of Math.* 2 **60** (1954), 140–145.
[10] W. Greub, S. Halperin, and R. Vanstone, *Connections, Curvature and Cohomology, Vol. 1*, Pure and Applied Mathematics **47** (1972), Academic Press, New York-London.
[11] H. Guenancia, 'Families of conic Kähler-Einstein metrics', *Math. Ann.* **376**(1-2) (2020), 1–37.
[12] Y. Kawamata, 'On deformations of compactifiable complex manifolds', *Math. Ann.* Vol. **235** (1978), 247–265.
[13] S. Kebekus, 'Pull-back morphisms for reflexive differential forms', *Advances in Mathematics* **245** (2013), 78–112.
[14] R. Kobayashi, 'Kähler-Einstein metric on an open algebraic manifold', *Osaka J. Math.* **21** (1984), 399–418.
[15] K. Kodaira, *Complex Manifolds and Deformation of Complex Structures*, Grundlehren der math. Wiss. **283** (1986), Springer.
[16] X. Ma, G. Marinescu, *Holomorphic Morse Inequalities and Bergman Kernels*, Progress in Mathematics, **254** (2007), Birkhäuser.
[17] P. Naumann, 'Curvature of higher direct images', *Ann. de la Faculté des Sciences de Toulouse*, Series 6: **20** (2021), 171–201.
[18] G. Schumacher, 'Moduli of framed manifolds', *Invent. Math.* **134** (1998), 229–249.
[19] G. Schumacher, 'Positivity of relative canonical bundles and applications', *Invent. Math.* **190** (2012), 1–56.
[20] K.Th. Sturm, 'Heat kernel bounds on manifolds', *Math. Ann.* **292** (1992), 149–162.
[21] G. Tian, S.T. Yau, Kähler-Einstein metrics on complete Kähler manifolds and their applications to algebraic geometry, Adv. Ser. in Math. Phys. **1** (1986), in Mathematical Aspects of String Theory, S.T. Yau ed., San Diego, 574–628.
[22] W.-K. To, S.-K. Yeung, 'Hyperbolicity of moduli spaces of log-canonically polarized manifolds', *J. Geom. Analysis*, No. **3** (2021), 2941–2969.

https://doi.org/10.1017/fms.2022.79 Published online by Cambridge University Press

[23] S.T. Yau, ‘A general Schwarz lemma for Kähler manifolds’, *Am. J. Math.* **100** (1978), pp. 197–203.
[24] S. Zucker, Hodge theory with degenerating coefficients: L_2-cohomology in the Poincaré metric, *Ann. of Math*. **109** (1979), 415–476.
[25] S. Zucker, ‘L_2-cohomology of warped products and arithmetic groups,’ *Invent. Math*. **70** (1982), 169–218.