CLIFFORD DIVISION ALGEBRAS AND ANISOTROPIC QUADRATIC FORMS: TWO COUNTEREXAMPLES

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In a recent paper [**3**], D. W. Lewis proposed the following conjecture. (The notation is the same as that in [**2**] and [**3**].)

CONJECTURE. *Let F be a field of characteristic not 2 and let $a_1, b_1, \ldots, a_n, b_n \in F^\times$. The tensor product of quaternion algebras*

$$\left(\frac{a_1 \;\; b_1}{F}\right) \otimes_F \ldots \otimes_F \left(\frac{a_n \;\; b_n}{F}\right)$$

is a division algebra if and only if the quadratic form over F

$$\perp_{i=1}^{n} (-1)^{i+1} \langle a_i, b_i, -a_i b_i \rangle$$

is anisotropic.

This equivalence indeed holds for $n = 1$ as is well known [**2**, Theorem 2.7], and Albert [**1**] (see also [**4**, §15.7]) has shown that it also holds for $n = 2$. The aim of this note is to provide counterexamples to both of the implications for $n \geqslant 3$.

Let k be a field of characteristic different from 2, and let $x_1, \ldots, x_{n-1}, y_1, \ldots, y_n$ be independent indeterminates over k (with $n \geqslant 3$). Let also $f(x_1, x_2) \in k(x_1, x_2)$ and

$$F = k(x_1, \ldots, x_{n-1}, y_1, \ldots, y_n).$$

THEOREM. (1) *The tensor product of quaternion algebras*

$$T = \left(\frac{x_1 \;\; y_1}{F}\right) \otimes \ldots \otimes \left(\frac{x_{n-1} \;\; y_{n-1}}{F}\right) \otimes \left(\frac{f(x_1, x_2) \;\; y_n}{F}\right)$$

is a division algebra if and only if $f(x_1, x_2)$ is not a square in $k(\sqrt{x_1}, \sqrt{x_2})$.

(2) *The quadratic form over F*

$$Q = \langle x_1, y_1, -x_1 y_1 \rangle \perp \ldots \perp (-1)^n \times \langle x_{n-1}, y_{n-1}, -x_{n-1} y_{n-1} \rangle \perp (-1)^{n+1} \langle f(x_1, x_2), y_n, -f(x_1, x_2) y_n \rangle$$

is anisotropic if and only if $(-1)^n f(x_1, x_2)$ is not represented by the quadratic form $\langle x_1, -x_2 \rangle$ over $k(x_1, x_2)$ and $f(x_1, x_2)$ is not a square in $k(x_1, x_2)$.

The proof will follow by repeated use of the following results.

LEMMA. *Let K be a field of characteristic different from 2 and let t be an indeterminate over K.*

(1) *If A is a central simple algebra over K and $c \in K^\times$, then $A \otimes_K \left(\frac{c \;\; t}{K(t)}\right)$ is a division algebra if and only if $A \otimes_K K(\sqrt{c})$ is a division algebra.*

(2) *If q_1 and q_2 are quadratic forms over K, then $q_1 \perp \langle t\rangle q_2$ is anisotropic over $K(t)$ if and only if q_1 and q_2 are anisotropic over K.*

Proof. (1) See [**5**, Proposition 2.4]; (2) see [**2**, p. 273].

Proof of the theorem. (1) We apply part (1) of the lemma $(n-1)$ times, taking successively $t=y_1$, $t=y_2, \ldots, t=y_{n-1}$. It follows that T is a division algebra if and only if

$$\left(\frac{f(x_1,x_2), \; y_n}{k(x_1,x_2,\ldots,x_{n-1},y_n)}\right) \otimes k(\sqrt{x_1}, \sqrt{x_2}, \ldots, \sqrt{x_{n-1}}, y_n)$$

is a division algebra. This last condition is equivalent to the following: the quadratic form

$$\langle 1, -f(x_1,x_2)\rangle \perp -\langle y_n\rangle\langle 1, -f(x_1,x_2)\rangle$$

is anisotropic over $k(\sqrt{x_1}, \sqrt{x_2}, \ldots, \sqrt{x_{n-1}}, y_n)$. Applying then the second part of the lemma with successively $t=y_n$, $\sqrt{x_{n-1}}, \ldots, \sqrt{x_3}$, we see that this condition holds if and only if $\langle 1, -f(x_1,x_2)\rangle$ is anisotropic over $k(\sqrt{x_1}, \sqrt{x_2})$, i.e. $f(x_1,x_2)$ is not a square in $k(\sqrt{x_1}, \sqrt{x_2})$.

(2) readily follows from the second part of the lemma, applied successively with $t=y_1, y_2, \ldots, y_n, x_3, x_4, \ldots, x_{n-1}$.

Now, for $f(x_1,x_2)=(-1)^n(x_1-x_2)$, the theorem shows that the tensor product T is a division algebra, while the corresponding quadratic form Q is isotropic.

Conversely, for $f(x_1,x_2)=x_1x_2$, the tensor product T is not a division algebra, but the corresponding quadratic form Q is anisotropic, since part (2) of the lemma, with $t=x_2$, shows that $\langle x_1, -x_2, (-1)^n x_1x_2\rangle$ is anisotropic over $k(x_1,x_2)$.

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