THE INTERACTION OF A CONTACT DISCONTINUITY WITH SOUND WAVES ACCORDING TO THE LINEARISED NAVIER-STOKES EQUATIONS

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Abstract

The resolution of a small initial discontinuity in a gas is examined using the linearised Navier-Stokes equations. The smoothing of the resultant contact surface and sound waves due to dissipation results in small flows which interact. The problem is solved for arbitrary Prandtl number by using a Fourier transform in space and a Laplace transform in time. The Fourier transform is inverted exactly and the density perturbation is found as two asymptotic series valid for small dissipation near the contact surface and the sound waves respectively. The modifications to the structures of the contact surface and the sound waves are exhibited.

1. Introduction

In the Riemann problem for a gas two one-dimensional half-spaces with different densities and pressures are separated by a membrane at $x = 0$, say, which is instantaneously ruptured. According to ideal gas theory the resultant flow will consist of a shock and an expansion wave which move in opposite directions, and a contact surface which moves with the gas originally adjacent to the membrane. In a real gas the presence of heat conductivity and viscosity will result in two additional effects. First, the discontinuities are resolved into narrow regions of rapid but continuous change. In particular the temperature jump across the contact surface is smeared by conduction thus inducing a small scale flow which will interact with the shock wave to change its position and structure. This is the second effect.

Studies have been made of the resolution of an initial discontinuity by Chu (2), (3) and Bienkowski (1). Chu uses the Krook model of the Boltzmann equation for times large enough for the shock to be well developed. Bienkowski studies the initial effects by using a series solution for both the Krook model and exact Boltzmann equation with Maxwell molecules. To find the solution for larger times Bienkowski employs the linearised Navier-Stokes equations. In this model the shock and expansion waves are replaced by sound waves of equal strength. This part is specialised to a Prandtl number of $\frac{3}{4}$ and exhibits the smoothed out contact surface and sound waves, but the interaction between them is not found. Goldsworthy (4) studied the interaction of a large contact discontinuity and a shock wave. The contact surface structure is found by using a thin layer approximation. The perturbations produced by the flow inside the contact layer are assumed to be propagated outside it by a wave equation, and are matched to an unstructured shock.

This paper improves on Bienkowski's solution of the linearised Navier-Stokes equations by letting the Prandtl number be arbitrary and by finding the effect of the smoothing of the contact discontinuity on the sound wave approximately.

2. Formulation of the problem

The theory of the linearised Navier-Stokes equations was given by Lagerstrom, Cole and Trilling (5), to which reference should be made for details. We may assume that the dynamic viscosity and heat conductivity coefficients are constants ν_0 and κ_0. If pressure, density and velocity are P, ρ and u we write

$$P = P_0(1+p), \quad \rho = \rho_0(1+s), \quad u = u_0+v, \tag{2.1}$$

where P_0, ρ_0 and u_0 are constant values and p, s and v are small perturbations. If $u_0 = 0$, the one dimensional linearised Navier-Stokes equations are

$$\left.\begin{aligned} v_t+(a^2/\gamma)p_x-(4\nu_0/3)u_{xx} &= 0,\\ s_t+v_x &= 0,\\ \text{and} \quad p_t-\gamma s_t-(\kappa_0/\rho_0 C_v)(p_{xx}-s_{xx}) &= 0 \end{aligned}\right\} \tag{2.2}$$

where t and x are time and distance coordinates and C_v is the specific heat at constant volume. This set of equations is to be solved subject to the initial conditions

$$s = \tfrac{1}{2}s_0 \operatorname{sgn}(x), \quad p = \tfrac{1}{2}p_0 \operatorname{sgn}(x), \quad v = 0 \text{ at } t = 0. \tag{2.3}$$

We write

$$\kappa_0/\rho_0 C_v = \kappa; \; 4\nu_0/3 = 4\kappa \,\mathrm{Pr}/3\gamma = \sigma\kappa; \tag{2.4}$$

$$\chi = x/L; \; \tau = at/L; \; aw = v; \tag{2.5}$$

and

$$m = \kappa/aL. \tag{2.6}$$

Pr is the Prandtl number, and L is a length scale which is not specified at this stage. The non-dimensional equations are then

$$\left.\begin{aligned} w_\tau+p_\chi/\gamma-m\sigma w_{\chi\chi} &= 0,\\ s_\tau+w_\chi &= 0,\\ \text{and} \quad p_\tau-\gamma s_\tau+m(s_{\chi\chi}-p_{\chi\chi}) &= 0. \end{aligned}\right\} \tag{2.7}$$

The equations are solved by making a Laplace transform in time and a Fourier transform in space:

$$\bar{f}(\beta) = \int_0^\infty e^{-\beta\tau} f(\tau)\,d\tau, \tag{2.8}$$

and

$$\hat{g}(\alpha) = \int_{-\infty}^{\infty} g(\chi)e^{i\alpha\chi}d\chi.$$

Solving the resultant algebraic equations for the twice transformed density perturbation $\hat{\hat{s}}$ gives

$$[\alpha^4 m(m\sigma\beta + 1/\gamma) + \alpha^2\beta\{1 + \beta m(1+\sigma)\} + \beta^3]\hat{\hat{s}}$$
$$= \beta^2 s_0/\alpha + \alpha\{(s_0 - p_0/\gamma) + \beta m(1+\sigma)s_0\} + \alpha^3\sigma m^3 s_0. \quad (2.9)$$

3. The poles of $\hat{\hat{s}}$

Equation (2.9) shows that $\hat{\hat{s}}$ has a pole when $\alpha = 0$ and when

$$\alpha^2 = \beta\{1 - \beta m(1+\sigma) + r\}/\{2m(m\sigma\beta + 1/\gamma)\}, \quad (3.1)$$

where

$$r = [1 + 2\beta m(1 + \sigma - 2/\gamma) + \beta^2 m^2(1-\alpha)^2]. \quad (3.2)$$

Branches of $\alpha^2(\beta)$ will occur when $r = 0$; that is at

$$\beta = \{-(1+\sigma-2/\gamma) \pm 2\sqrt{[(1-1/\gamma)(\sigma-1/\gamma)]}\}/\{m(1-\sigma)^2\}. \quad (3.3)$$

Since $\sigma = 4\,\mathrm{Pr}/3\gamma$ these roots are complex provided that $\mathrm{Pr} < \frac{3}{4}$, which is the case for most gases, and their real parts are negative in this case if

$$\sigma > (2-\gamma)/\gamma. \quad (3.4)$$

Equation (3.2) shows that the product of the roots is positive, so that the roots, if real, are negative whenever (3.4) holds. The inequality (3.4) is satisfied for most gases, and in particular for monatomic and diatomic gases. Both roots $\alpha^2(\beta)$ vanish at the origin, the one with the plus sign attached to r being $O(\beta^2)$ there, and the other $O(\beta)$. The denominator of (3.1) vanishes when

$$\beta = -1/(m\gamma\sigma) = \beta_0.$$

Examination of (3.1) shows, however, that the numerator of one root $\alpha^2(\beta)$ also vanishes at β_0. If $4(\gamma-1)\,\mathrm{Pr} > 3\gamma$ then $\{-1+(1+\sigma)/\gamma\sigma\}$ is positive, and the root $\alpha^2(\beta)$ with the minus sign attached to r has vanishing numerator and regular behaviour at β_0, while the other root has a simple pole there. If the inequality is reversed the roots have opposite properties at β_0. The branch lines of both roots $\alpha^2(\beta)$ stretch from the left of the points (3.3) to the point at infinity along lines parallel to the real axis. It is easily verified that for large $|\beta|$ the roots are asymptotically like $-\beta/m$ and $-\beta/m\sigma$, so that they have the same argument at infinity.

To define the functions $\alpha(\beta)$, write

$$\alpha_1 = \{\beta[-1-\beta m(1+\sigma)+r]/[2m(\beta m\sigma + 1/\gamma)]\}^{\frac{1}{2}}. \quad (3.5)$$

α_2 is the same expression but with a negative sign in front of r; $\alpha_3 = -\alpha_1$ and $\alpha_4 = -\alpha_2$. The definition is completed by requiring that α_1 and α_2 have argument $\pi/2$ for real, large, positive β. For each root α_j, $|\alpha_j| = O(|\beta^{\frac{1}{2}}|)$ when $|\beta|$ is large. The remarks on α^2 show that each α_j has branch points at

(3.3) and corresponding branch lines. α_3 and α_4 have additional branch points at the origin. If $\Pr < 3\gamma/4(\gamma-1)$, α_2 and α_4 have branch points at $\beta = 0$ and at $\beta = -1/(m\gamma\sigma)$ which are joined by a branch line. All roots have the branch points and corresponding branch lines of $\alpha^2(\beta)$. If the inequality is reversed, α_1 and α_3 have branch points at $\beta = -1/(m\gamma\sigma)$, the origin is a branch point of α_2 and α_4 and again all the roots have the branch points of $\alpha^2(\beta)$. The branch lines in all cases stretch from the left of the corresponding branch point along lines parallel to the real axis up to the point at infinity.

4. Inversion of the Fourier transform

For Im (α) negative and χ positive, Re $(-i\alpha\chi)$ is negative so that

$$\bar{\bar{s}}(\alpha, \beta) \exp(-i\alpha\chi)$$

is exponentially small along the semi-circle $|\alpha| = R$ in Im $(\alpha)>0$ when R is large. Consequently

$$\bar{s}(\chi, \beta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (\chi, \beta) \exp(-i\alpha\chi) d\alpha \tag{4.1}$$

is given by the residues of the integrand in Im $(\alpha)<0$. A similar argument applies when χ is negative, the relevant residues being those in Im $(\alpha) \geqq 0$.

If the Laplace transform is inverted by integrating along a line Re $(\beta) \gg 0$ we find that the arguments of α_3 and α_4 lie between $5\pi/4$ and $7\pi/4$ on this line, since $\alpha_3 \sim \alpha_4 \sim i\beta^{\frac{1}{2}}$ for large $|\beta|$. Thus α_3 and α_4 lie in Im $(\alpha)<0$: α_1 and α_2 must then lie in Im $(\alpha)>0$, which shows that the residues at $\alpha = 0$, α_3 and α_4 are relevant for $\chi>0$. A straightforward calculation then shows that for $\chi>0$,

$$\bar{s}(\chi, \beta) = \tfrac{1}{2}s_0/\beta + \exp(-i\alpha_3\chi)A(+) - \exp(-i\alpha_4\chi)A(-), \tag{4.2}$$

where

$$A(\pm) = \frac{1}{4\beta r}[s_0\{1+\beta m(1+\sigma)\pm r\} - 2(s_0 - p_0/\gamma) + 2\beta m(1+\sigma)s_0 + \beta m\gamma\sigma s_0\{1+\beta m(1+\sigma)\mp r\}/(m\gamma\sigma\beta+1)]. \tag{4.3}$$

A similar result holds for $\chi<0$.

5. Inversion of the Laplace transform

The solution $s(\chi, \tau)$ must be anti-symmetric in χ because the equations (2.7) are linear and the initial conditions (2.3) are anti-symmetric. The solution for $\chi>0$ therefore gives that for $\chi<0$ by inspection, and only $\chi>0$ will be examined.

For $\chi>0$ the right hand side of equation (4.2) tends to $\frac{1}{2}s_0/\beta$ for large χ, so that $s(\chi, \tau)$ is then approximately

$$\tfrac{1}{2}s_0, \tag{5.1}$$

which verifies that equations (4.2), (4.3) and (3.2) yield the correct answers for large $|\chi|$.

It will now be shown that inversion of $A(-) \exp(-i\alpha_4\chi)$ gives the contact surface and of $A(+) \exp(-i\alpha_3\chi)$ gives the sound wave. It is not possible to invert either term exactly, and an asymptotic solution valid for $m \ll 1$ and $\tau \gg 1$ will be sought.

To invert $A(-)\exp(-i\alpha_4\chi)$ take a contour along $\mathrm{Re}(\beta) = -M<0$, indenting it round the branch lines and the pole at the origin. M is a suitably large constant. Since $A(-)\exp(-i\alpha_4\chi)$ is at most of order $\exp(\chi M^{\frac{1}{2}})$ on $\mathrm{Re}(\beta) = -M$, the integral

$$\frac{1}{2\pi i}\int A(-)\exp(-i\alpha_4\chi)\exp(\beta\tau)d\beta$$

is of order $\exp(-M\tau)$ which is negligibly small if $M\tau \gg 1$. The integral round the branches terminating at the points (3.2) is at most of order

$$\exp(-\tau/m)\exp(\chi),$$

which is negligible if $\chi \ll \tau/m$. The largest contribution must therefore come from the integral round the branch line to the origin. This branch line is enclosed by the contour C, consisting of two lines

$$\beta = h \pm i\delta, \quad -M<h<-\rho, \quad \delta>0; \tag{5.2}$$

and the circle

$$\beta = \rho e^{i\theta}, \qquad -\pi<\theta<\pi. \tag{5.3}$$

Note that taking the integration to $h = \infty$ introduces only an exponentially small error, even if α_4 has two branch points on the real axis and the branch line terminates at β_0. Thus the value of the Prandtl number does not significantly affect the answer.

The argument of α_4 is $3\pi/2$ for large real $\beta>0$, and this is unchanged on the positive real axis because α_4, $1/\alpha_4$ and r are non-vanishing there. Thus α_4 is real on either side of the branch line to the origin, and $|\exp(-i\alpha_4\chi)| = 1$ there. It follows that far from the origin the integral along C is exponentially small like $\exp(-\tau\beta)$. $A(-)$ and α_4 are therefore expanded as power series in βm valid near the origin.

Some algebra shows that the inversion of $A(-)\exp(-i\alpha_4\chi)$ is

$$-\frac{1}{2\pi i}[(s_0-p_0/\gamma)+\{(s_0-p_0/\gamma)(1+\sigma-2/\gamma)-s_0(\gamma-1)(\sigma-1/\gamma)\}m(\partial/\partial\tau)$$
$$+O(m^2\partial^2/\partial\tau^2)]\times\int_C \tfrac{1}{2}\beta^{-1}\exp[-i\alpha_4\chi+\beta\tau]d\beta. \tag{5.4}$$

For small $|\beta|$, $\alpha_4 \sim \exp(-i\pi/2)(\gamma\beta/m)^{\frac{1}{2}}$, and it follows from this that the integral round the circle (5.3) has limit $2\pi i$ as ρ vanishes. Further, the integral along the lines $\beta = h+i\delta$ and $\beta = h-i\delta$ has limit as $\delta\to 0$ and $\rho\to 0$

$$-\tfrac{1}{2}i\int_0^\infty \sin[\chi\sqrt{(\gamma\eta/m)}]\exp(-\eta\tau)\eta^{-1}d\eta = -\pi i\,\mathrm{erf}\{\chi\sqrt{(\gamma/m\tau)}\}. \tag{5.5}$$

Substituting these results in (5.4) gives

$$\tfrac{1}{2}[(s_0-p_0/\gamma)+\{(s_0-p_0/\gamma)(1+\sigma-2/\gamma)-s_0(\gamma-1)(\sigma-1/\gamma)\}m(\partial/\partial\tau)]\times [\operatorname{erf}\{\chi\sqrt{(\gamma/m\tau)}\}-1]. \quad (5.6)$$

Examination of the derivatives shows that each is negligible compared with previous ones if $\chi \ll m^{\frac{1}{2}}\tau$, which gives a limit to the range of validity of (5.6).

The inversion of $A(+)\exp(-i\alpha_3\chi)$ is simplified if at the outset it is assumed that $\tau-\chi$ is small. More exactly write

$$\tau-\chi = \varepsilon\tau. \quad (5.7)$$

Limits on the magnitude of ε will be set by requiring that the errors of the inversion are small. $A(+)\exp(-i\alpha_3\chi)$ does not have a branch point at the origin and the contour D of the inversion integral is the imaginary axis indented to the left of the origin. More exactly D is the limit as $\delta\to 0$ and $L\to\infty$ of the two lines

$$\beta = -\delta\pm iy, \quad 0<y<L; \quad (5.8)$$

and the circular arc

$$\beta = \delta e^{i\theta}, \qquad -\pi<\theta<\pi. \quad (5.9)$$

To estimate the errors of inversion introduce two additional contours. The first is D_a, consisting of the two circular arcs $|\beta| = L, \frac{\pi}{2} \leqq \theta \leqq \Theta$ and $|\beta| = L$, $-\Theta \leqq \theta \leqq -\pi/2$. Θ is restricted to the sector $3\pi/4 \geqq \Theta \geqq \pi/2$. The second contour is D_r consisting fo the two radii $\beta = \rho\exp(\pm i\Theta)$, $\delta_1<\rho<L$ and the circular arc $\beta = \delta_1\exp(i\theta)$, $-\pi<\theta<\pi$, where $\delta_1<\delta$. The integral over D_a is exponentially small for sufficiently large L, because $|\alpha_3| = O(|\beta^{\frac{1}{2}}|)$ for large $|\beta|$. Further $A(+)\exp(\tau\beta-i\alpha_3\chi)$ is analytic in the region enclosed by D, D_a and D_r, and it follows that the integrals along D_r and D are equal in the limit of large L. As a further aid to estimating errors of inversion, D_r is divided into two parts: on one, called λ, $|\beta m| = o(1)$; the remainder is called Λ. The integrand is then written as an asymptotic expansion in βm on λ, plus an error term of known magnitude; and the integral on Λ is estimated directly. The difference between the integrals of the expansion along λ and C is its integral along Λ which is again estimated directly. The integral along C is then written in the form of equation (5.4).

The expansion of $A(+)$ is

$$A(+) = P+E = \beta^{-1}\sum_{r=1}^{n} a_r(\beta m)^r+O(\beta^n m^{n+1}),$$

where

$$a_0 = -\tfrac{1}{2}p_0/\gamma,$$

$$a_1 = -\tfrac{1}{2}\{(s_0-p_0/\gamma)(1+\sigma-2/\gamma)-s_0(1+\sigma-1/\gamma)\}, \quad (5.10)$$

and

$$a_2 = -\tfrac{1}{2}\{(s_0-p_0/\gamma)[3(1-\sigma-2/\gamma)^2-(1-\sigma)^2] - s_0[(1-\sigma)(\sigma-1+2/\gamma)+3(1+\sigma-1/\gamma)(1+\sigma-3/\gamma)]\}.$$

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Also

$$\exp[-i\alpha_3\chi] = \exp[-\chi\beta+\tfrac{1}{2}m\chi(1+\sigma-1/\gamma)\beta^2](1+\chi b_3 m^2\beta^3+\chi b_4 m^3\beta^4+\ldots)$$

where

$$b_3 = \tfrac{1}{8}[4\gamma^2\sigma^2+4(1-\gamma)(\alpha-1/\gamma)(1+\sigma+\gamma\sigma-2/\gamma)-(1+\sigma-1/\gamma)^2] \quad (5.11)$$

and

$$b_4 = -\tfrac{1}{16}[8\gamma^3\sigma^3-4\gamma^2\sigma^2(1+\sigma-1/\gamma)$$
$$+4(\sigma-1/\gamma)(1-\gamma)(1+\sigma+\gamma\sigma-2/\gamma)(2\gamma\sigma+1/\gamma-1-\sigma)+(1+\sigma-1/\gamma)^3].$$

The integral of $A(+)\exp(\beta\tau-i\alpha_3\chi)$ along Λ is exponentially small provided that $|\beta| \gg 1$ on Λ, because in that case $|\alpha| = O(|\beta|^{\frac{1}{2}})$ and the integrand is dominated by $\exp(\beta\tau)$.

The integral of $E\exp(\tau\beta-i\alpha_3\chi)$ on λ can be split into two:

$$\int_\lambda \exp\{(\tau-\chi)\beta\}Ed\beta+\int_\lambda \exp(\tau\beta)\{\exp(-i\alpha_3\chi)-\exp(-\chi\beta)\}Ed\beta. \quad (5.12)$$

Integrating the first of these by parts yields, along with (5.10),

$$[\exp\{(\tau-\chi)\beta\}\,.\,nm^{n+1}\beta^n/(\tau-\chi)]_\lambda-\{nm^{n+1}/(\tau-\chi)\}\int_\lambda \exp\{(\tau-\chi)\beta\}\beta^{n-1}d\beta. \quad (5.13)$$

The integrated term vanishes at $\beta = 0$ and is exponentially small at the other limit provided that $|(\tau-\chi)\beta| \gg 1$. If the largest value of $|\beta|$ on λ is $O(m^{\zeta-1})$, then $\tau-\chi = \varepsilon\tau$ leads to

$$\varepsilon \gg m^{\zeta-1}/\tau. \quad (5.14)$$

Repeated integration by parts then gives a final estimate $O\{n!\,m^{n+1}(\tau-\chi)^{-(n+1)}\}$, which is small provided that

$$\varepsilon \gg m/\tau. \quad (5.15)$$

It may be noted in passing that the factorial in the estimate shows that the estimate is not uniform in n. The second integral in (5.12) can be estimated in the same manner because $\{\exp(-i\alpha_3\chi+\chi\beta)-1\}$ is uniformly $O(\chi m\beta^2)$ on λ, and it is small provided that

$$\varepsilon^{n+3} \gg (m/\tau)^{n+2}. \quad (5.16)$$

Finally if $\exp(-i\alpha_3\chi)$ is replaced by a truncated expansion in the form of (5.11), an error term results whose largest term is

$$\int_\lambda \exp\{(\tau-\chi)\beta+\tfrac{1}{2}\chi m(1+\sigma-1/\gamma)\beta^2\}O(\chi^k m^{2k}\beta^{3k-1})d\beta$$

which is small provided that

$$\varepsilon \gg (m/\tau)^{\frac{2}{3}}. \quad (5.17)$$

We are left with

$$\int_{\lambda} P\{1+\chi m^2\beta^2(b_3\beta+b_4 m\beta^2+\ldots)\} \exp\{(\tau-\chi)\beta+\tfrac{1}{2}\chi m(1+\sigma-1/\gamma)\beta^2\}d\beta, \quad (5.18)$$

which will differ from the same integral over D by an exponentially small quantity if (5.14) holds. The integral (5.18) may be written in the form of (5.4), the integral now being

$$\int_{c} \exp[(\tau-\chi)\beta+\tfrac{1}{2}\chi m(1+\sigma-1/\gamma)\beta^2]\beta^{-1}d\beta.$$

The integral of this round the circle gives 1 independently of δ, and in the limit as $\delta\to 0$ the integral along the line $\beta = -\delta+iy$ gives

$$\pi i \operatorname{erf}[(\tau-\chi)\{2\chi m(1+\sigma-1/\gamma)\}^{-\frac{1}{2}}].$$

These results together with (5.18) give

$$\{\tfrac{1}{2}p_0/\gamma - ma_1(\partial/\partial\tau)+m^2a_2(\partial/\partial\tau)^2-a_0b_3\chi m^2(\partial/\partial\tau)^3$$
$$+(a_0b_4+a_1b_3)\chi m^3(\partial/\partial\tau)^4-\ldots\}\times\{1+\operatorname{erf}[(\tau-\chi)\{2\chi m(1+\sigma-1/\gamma)\}^{-\frac{1}{2}}]\} \quad (5.19)$$

The condition that each term in this series should be negligible compared with previous ones imposes further restrictions on ε. It is readily verified that if

$$\varepsilon = O[(m/\tau)^{\frac{1}{2}}] \text{ at most} \quad (5.20)$$

then

$$m^r(\partial/\partial\tau)^r \operatorname{erf}[(\tau-\chi)\{2\chi m(1+\sigma-1/\gamma)\}^{-\frac{1}{2}}] = O[(m/\tau)^{r/2}], \quad r \text{ odd},$$
$$= O[\varepsilon(m/\tau)^{(r-1)/2}], \quad r \text{ even}. \quad (5.21)$$

A similar result holds for the series derived from $\exp(-i\alpha_3\chi)$. A consequence of (5.21) is that because $\chi/\tau = O(1)$ near the sound wave, $m(\partial/\partial\tau)$ and $\chi m^2(\partial/\partial\tau)^3$ are of order $(m/\tau)^{\frac{1}{2}}$; and $m^2(\partial/\partial\tau)^2$ and $\chi m^3(\partial/\partial\tau)^4$ are of order $\varepsilon(m/\tau)^{\frac{1}{2}}$.

6. Results and Discussion

The solution for $\chi>0$ is given by adding (5.1), (5.6) and (5.19). The constant terms cancel out and the solution consists of two error functions and the series of their time derivatives. Because the solution is antisymmetric in χ, (5.6) is seen to represent a smoothed out discontinuity centred at the origin, namely the contact surface. Expression (5.19) on the other hand is a wave travelling with velocity $\chi/\tau = a$. On a curve $\chi = v\tau$ the argument of the error function increases like $\tau^{\frac{1}{2}}$ so that the width of the sound wave increases at this rate. The arguments of the error functions in (5.6) and (5.19) become unbounded when $m = 0$, unless $\chi = 0$ or $\chi = \tau$ respectively, so that in the absence of dissipation discontinuities are recovered.

The contact discontinuity of non-dissipative gas dynamics vanishes if $\gamma s_0 = p_0$; the largest order term in (5.6) due to the interaction is therefore

$$\tfrac{1}{4}s_0(\gamma-1)(\sigma-1/\gamma)\chi(m\gamma\tau^{-3})^{\frac{1}{2}}\exp(-\chi^2\gamma/m\tau).$$

Similarly the effect of the interaction on the sound wave is approximately

$$\tfrac{1}{2}(s_0-p_0/\gamma)\{(1+\sigma-2/\gamma)[m(\partial/\partial\tau)-\chi m^3 b_3(\partial/\partial\tau)^4]$$
$$-\tfrac{1}{2}m^2[3(1-\sigma-2/\gamma)^2-(1-\sigma)^2](\partial/\partial\tau)^2\}\operatorname{erf}[(\chi-\tau)\{2\chi m(1+\sigma-1/\gamma)\}^{-\frac{1}{2}}]. \quad (6.1)$$

If the sound wave position is defined to coincide with a maximum of $(\partial/\partial\chi)s(\chi,\tau)$ no shift is found since the relevant maximum always occurs at $\chi=\tau$ i.e. at $x = at$. If the position of the sound wave is defined to occur where the term involving the error function in (5.19) has value $\frac{1}{4}p_0/\gamma$, it follows, since (6.1) is of order

$$(m/\tau)^{\frac{1}{2}}, \quad (6.2)$$

that there will be a shift of this order in the wave position. This differs from Goldsworthy's result that the shock speed is attenuated by an amount proportional to $\tau^{-\frac{1}{2}}$, but it is fundamental to linear theory to incorporate the fixed wave speed a.

A measure of the shift can be obtained if $\varepsilon(\tau/m)^{\frac{1}{2}} = o(1)$, for then the sound wave is, from (6.1),

$$\tfrac{1}{2}(p_0/\gamma)[1+(2a_1 m\gamma/p_0)(\partial/\partial\tau)]\operatorname{erf}[(\chi-\tau)\{2\chi m(1+\sigma-1/\gamma)\}^{-\frac{1}{2}}]. \quad (6.3)$$

Suppose χ is fixed, and that the value of τ that makes (6.3) have value $\frac{1}{4}p_0/\gamma$ is τ^*. The unperturbed solution may be written approximately

$$\tfrac{1}{2}(p_0/\gamma)\operatorname{erf}[(\chi-\tau)\{2\chi m(1+\sigma-1/\gamma)\}^{-\frac{1}{2}}]$$
$$=\tfrac{1}{2}(p_0/\gamma)[1+(\tau-\tau^*)(\partial/\partial\tau)]\operatorname{erf}[(\chi-\tau^*)\{2\chi m(1+\sigma-1/\gamma)\}^{-\frac{1}{2}}\}]. \quad (6.4)$$

Expressions (6.3) and (6.4) have the same value if

$$\tau-\tau^* = 2a_1 m\gamma/p_0 = m(\gamma s_0/p_0-1)(1+\sigma-2/\gamma). \quad (6.5)$$

In terms of the original coordinates the restrictions on ε are

$$(\kappa/at^2)^{\frac{1}{2}} \ll \varepsilon = O[(\kappa/at^2)^{\frac{1}{4}}]. \quad (6.6)$$

Here $z = O(y)$ includes the possibility $z = o(y)$.

It will be noticed that the asymptotic expansion (5.6) can be found just as well if $m = O(1)$, but that the validity of (5.18) requires $m = o(1)$. A suitable choice for the length scale is therefore $L = at$, the distance travelled by the sound wave. The assumption that m is small then means that κ/a^2t is small; in other words the sound wave has travelled far enough for only a small amount of dissipation to occur.

The length scale does not involve any objective length because none occurs in the problem, but its apparent arbitrariness is unimportant, because when

(5.6) and (5.19) are put back in the original co-ordinates L is absent. For example

$$\chi(\gamma/m\tau)^{\frac{1}{2}} = x(\gamma/\kappa t)^{\frac{1}{2}},$$

and

$$(\tau-\chi)\{2\chi m(1+\sigma-1/\gamma)\}^{-\frac{1}{2}} = (at-x)\{(2x\kappa/a)(1+\sigma-1/\gamma)^{-\frac{1}{2}}.$$

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